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Fuzzy relation equations in pseudo BL-algebras

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Abstract

Bandler and Kohout investigated the solvability of fuzzy relation equations with inf-implication compositions in complete lattices. Perfilieva and Noskova investigated the solvability of fuzzy relation equations with inf-implication compositions in BL-algebras. In this paper, we investigate various solutions of fuzzy relation equations with inf-implication compositions in pseudo BL-algebras.

Keywords: Pseudo BL-algebras, inf-implication compositions, fuzzy relation equations

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1. Introduction

Sanchez [1] introduced the theory of fuzzy relation equations with various types of compositions: max-min, min-max, and min-α. Fuzzy relation equations with new types of compositions (continuous t-norm and residuated lattice) have been developed [2-5]. In particular, Bandler and Kohout [6] investigated the solvability of fuzzy relation equations with inf-implication compositions in complete lattices. Perfilieva and Noskova investigated the solvability of fuzzy relation equations with inf-implication compositions in BL-algebras. In contrast, noncommutative structures play an important role in metric spaces and algebraic structures (groups, rings, quantales, and pseudo BL-algebras) [7-15]. Georgescu and Iorgulescu [12] introduced pseudo MV-algebras as the generalization of MV-algebras. Georgescu and Leustean [11] introduced generalized residuated lattice as a noncommutative structure.

In this paper, we investigate various solutions of fuzzy relation equations with inf-implication compositions $A_i \Rightarrow R = B_i$ and $A_i \to R = B_i$ in pseudo BL-algebras.

2. Preliminaries

Definition 2.1. [11] A structure $(L, \vee, \wedge, \odot, \to, \Rightarrow, \top, \bot)$ is called a*pseudo BL-algebra* if it satisfies the following conditions:

(A1) $(L, \vee, \wedge, \top, \bot)$ is bounded where $\top$ is the universal upper bound and $\bot$ denotes the universal lower bound;

(A2) $(L, \odot, \top)$ is a monoid;

(A3) it satisfies a residuation, i.e.,

$$a \odot b \le c \text{ iff } a \le b \to c \text{ iff } b \le a \Rightarrow c.$$

(A4) $a \wedge b = (a \to b) \odot a = a \odot (a \Rightarrow b)$.

(A5) $(a \to b) \vee (b \to a) = \top$ and $(a \Rightarrow b) \vee (b \Rightarrow a) = \top$.

We denote $a^0 = a \to \bot$ and $a^* = a \Rightarrow \bot$.

A pseudo BL-chain is a linear pseudo BL-algebra, i.e., a pseudo BL-algebra such that its lattice order is total.

In this paper, we assume that $(L, \wedge, \vee, \odot, \to, \Rightarrow, \bot, \top)$ is a pseudo BL-algebra.

Lemma 2.2. [11] For each $x, y, z, x_i, y_i \in L$, we have the following properties:

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \to y \leq x \to z$, and $z \to x \leq y \to x$ for $\to \in \{\to, \Rightarrow\}$.

(2) $x \odot y \leq x \wedge y \leq x \vee y$.

(3) $(x \odot y) \to z = x \to (y \to z)$ and $(x \odot y) \Rightarrow z = y \Rightarrow (x \Rightarrow z)$.

(4) $x \to (y \Rightarrow z) = y \Rightarrow (x \to z)$ and $x \Rightarrow (y \to z) = y \to (x \Rightarrow z)$.

(5) $x \odot (x \Rightarrow y) \leq y$ and $(x \to y) \odot x \leq y$.

(6) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$ and $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$.

(7) $x \to y = \top$ iff $x \leq y$ iff $x \Rightarrow y = \top$.

3. Fuzzy Relation Equations in Pseudo BL-Algebras

Theorem 3.1. Let $\mathbf{a} = (a_1, a_2, ..., a_n) \in L^n$ and $b \in L$. We define two equations with respect to an unknown $\mathbf{x} = (x_1, ..., x_n) \in L^n$ as

$$\bigwedge_{j=1}^{n} (a_j \to x_j) = b, \tag{I}$$

$$\bigwedge_{j=1}^{n} (a_j \Rightarrow x_j) = b, \tag{II}$$

Then, (1) (I) is solvable iff it has the least solution $\mathbf{y} = (y_1, ..., y_n) \in L^n$ such that $y_j = b \odot a_j, j = 1, ..., n$.

(2) (II) is solvable iff it has the least solution $\mathbf{x} = (x_1, ..., x_n) \in L^n$ such that $x_j = a_j \odot b, j = 1, ..., n$.

(3) If (I) is solvable, then $b \geq \bigwedge_{j=1}^{n} a_j^0$.

(4) If (II) is solvable, then $b \geq \bigwedge_{j=1}^{n} a_j^*$.

Proof. (1) ($\Rightarrow$) Let $\mathbf{x} = (x_1, ..., x_n)$ be a solution of (I). Since $b = \bigwedge_{j=1}^{n}(a_j \to x_j) \leq a_j \to x_j$, $b \odot a_j \leq x_j$. Moreover, $b \leq \bigwedge_{j=1}^{n}(a_j \to b \odot a_j) \leq \bigwedge_{j=1}^{n}(a_j \to x_j) = b$. Therefore, $\bigwedge_{j=1}^{n}(a_j \to b \odot a_j) = b$. Thus, $\mathbf{y} = (b \odot a_1, ..., b \odot a_n)$ is the least solution.

($\Leftarrow$) It is trivial.

(3) Let $\mathbf{x} = (x_1, ..., x_n)$ denote a solution of (I). Then, $b = \bigwedge_{j=1}^{n}(a_j \to x_j) \geq \bigwedge_{j=1}^{n}(a_j \to \bot) = \bigwedge_{j=1}^{n}(a_j)^0$.

(2) and (4) are similarly proved as (1) and (3), respectively.

Theorem 3.2. Let L denote a pseudo BL-chain in equations (I) and (II) of Theorem 3.1.

(1) If $b < \top$ and $b = \bigwedge_{j=1}^{n} a_j^*$ with $B = \{a_{j_k} \mid 1 \leq k \leq m, b = (a_{j_k})^*\}$, then $X = \{\mathbf{x_{j_k}} = (\top, ..., \overbrace{\bot}^{j_k}, ..., \top) \mid 1 \leq k \leq m\}$ is a maximal solution of (II). Moreover, if $\mathbf{x}$ is a solution of (II), there exists $k \in \{j_k \mid 1 \leq k \leq m\}$ such that

$$x_{j_k} = 0, j = k, \ x_j \geq a_j \odot b, \ j \neq k$$

where there exists $\mathbf{x_{j_k}} \in X$ such that $\mathbf{x} \leq \mathbf{x_{j_k}}$.

(2) If $b < \top$ and $b = \bigwedge_{j=1}^{n} a_j^0$ with $B = \{a_{j_k} \mid 1 \leq k \leq m, b = (a_{j_k})^0\}$, then $X = \{\mathbf{x_{j_k}} = (\top, ..., \overbrace{\bot}^{j_k}, ..., \top) \mid 1 \leq k \leq m\}$ is a maximal solution of (I). Moreover, if $\mathbf{x}$ is a solution of (I), there exists $k \in \{j_k \mid 1 \leq k \leq m\}$ such that

$$x_{j_k} = 0, j = k, \ x_j \geq b \odot a_j, \ j \neq k$$

where there exists $\mathbf{x_{j_k}} \in X$ such that $\mathbf{x} \leq \mathbf{x_{j_k}}$.

Proof. (1) ($\Rightarrow$) $\mathbf{x_{j_k}} = (\top, ..., \overbrace{\bot}^{j_k}, ..., \top)$ is a solution of (II) because

$$\bigwedge_{j=1}^{n}(a_j \Rightarrow x_j) = a_{j_k} \Rightarrow \bot = a_{j_k}^* = b.$$

Let $\mathbf{x} \geq \mathbf{x_{j_k}}$ be a solution of (II). Then, $\mathbf{x} = (\top, ..., \overbrace{x_{j_k}}^{j_k}, ..., \top)$ with $x_{j_k} \geq a_{j_k} \odot b$ and

$$\bigwedge_{j=1}^{n}(a_j \Rightarrow x_j) = a_{j_k} \Rightarrow x_{j_k} = b.$$

Since $b < 1$, $a_{j_k} \not\leq x_{j_k}$. Since L is linear, $a_{j_k} > x_{j_k}$. Since $b = a_{j_k} \Rightarrow x_{j_k} = a_{j_k}^*$, we have

$$\begin{aligned} x_{j_k} &= a_{j_k} \wedge x_{j_k} = a_{j_k} \odot (a_{j_k} \Rightarrow x_{j_k}) \\ &= a_{j_k} \odot b = a_{j_k} \odot (a_{j_k} \Rightarrow \bot) = \bot. \end{aligned}$$

Thus, $\mathbf{x} = \mathbf{x_{j_k}}$. $\mathbf{x_{j_k}} = (\top, ..., \overbrace{\bot}^{j_k}, ..., \top)$ is a maximal solution of (II).

Let $\mathbf{x} = (x_1, ..., x_n)$ be a solution of (II). Since $\bigwedge_{j=1}^{n}(a_j \Rightarrow x_j) = b$, by the linearity of L, there exists a family $K = \{j_k \mid a_{j_k} \in B, a_{j_k} \Rightarrow \bot = b, 1 \leq k \leq m\}$ such that

$$\bigwedge_{j=1}^{n}(a_j \Rightarrow x_j) = \bigwedge_{k=1}^{m}(a_{j_k} \Rightarrow x_{j_k}) = b$$

, because by linearity of L, $a_{j_k} \notin B, (a_j)^* > b$ implies that $\bigwedge_{a_{j_k} \notin B}(a_j \Rightarrow x_j) \geq \bigwedge_{a_{j_k} \notin B}(a_j \Rightarrow \bot) > b$.

For $k \in K$, since $a_k \Rightarrow \bot = a_k \Rightarrow x_k = b \neq \top$ and L is linear, $a_k > x_k$ and $a_k \odot b = a_k \odot (a_k \Rightarrow x_k) = a_k \odot (a_k \Rightarrow \bot) = \bot = a_k \wedge x_k = x_k$. Then, $\mathbf{x} = (x_1, ..., \overbrace{\bot}^{k}, ..., x_n) \leq (\top, ..., \overbrace{\bot}^{k}, ..., \top)$.

($\Leftarrow$) It is trivial.

(2) It is similarly proved as (1).

Example 3.3. Let $K = \{(x, y) \in R^2 \mid x > 0\}$ denote a set, and we define an operation $\otimes : K \times K \to K$ as follows:

$$(x_1, y_1) \otimes (x_2, y_2) = (x_1 x_2, x_1 y_2 + y_1).$$

Then, $(K, \otimes)$ is a group with $e = (1, 0)$, $(x, y)^{-1} = (\frac{1}{x}, -\frac{y}{x})$.

We have a positive cone $P = \{(a, b) \in R^2 \mid a = 1, b \geq 0$,or $a > 1\}$ because $P \cap P^{-1} = \{(1, 0)\}$, $P \odot P \subset P$, $(a, b)^{-1} \odot P \odot (a, b) = P$, and $P \cup P^{-1} = K$. For $(x_1, y_1), (x_2, y_2) \in K$, we define

$$\begin{aligned} (x_1, y_1) \leq (x_2, y_2) \quad & \Leftrightarrow (x_1, y_1)^{-1} \odot (x_2, y_2) \in P, \\ & (x_2, y_2) \odot (x_1, y_1)^{-1} \in P \\ & \Leftrightarrow x_1 < x_2 \ \text{ or } \ x_1 = x_2, y_1 \leq y_2. \end{aligned}$$

Then, $(K, \leq \otimes)$ is a lattice-group with totally order $\leq$. (ref. [1])

The structure $(L, \odot, \Rightarrow, \to, (\frac{1}{2}, 1), (1, 0))$ is a Pseudo BL-chain where $\bot = (\frac{1}{2}, 1)$ is the least element and $\top = (1, 0)$ is the greatest element from the following statements:

$$\begin{aligned} (x_1, y_1) \odot (x_2, y_2) \quad & = (x_1, y_1) \otimes (x_2, y_2) \vee (\tfrac{1}{2}, 1) \\ & = (x_1 x_2, x_1 y_2 + y_1) \vee (\tfrac{1}{2}, 1), \\ (x_1, y_1) \Rightarrow (x_2, y_2) \quad & = ((x_1, y_1)^{-1} \otimes (x_2, y_2)) \wedge (1, 0) \\ & = (\tfrac{x_2}{x_1}, \tfrac{y_2 - y_1}{x_1}) \wedge (1, 0), \\ (x_1, y_1) \to (x_2, y_2) \quad & = ((x_2, y_2) \otimes (x_1, y_1)^{-1}) \wedge (1, 0) \\ & = (\tfrac{x_2}{x_1}, -\tfrac{x_2 y_1}{x_1} + y_2) \wedge (1, 0). \end{aligned}$$

Furthermore, we have $(x, y) = (x, y)^{*\circ} = (x, y)^{\circ *}$ from:

$$(x, y)^* = (x, y) \Rightarrow (\frac{1}{2}, 1) = (\frac{1}{2x}, \frac{1-y}{x}),$$

$$(x, y)^{*\circ} = (\frac{1}{2x}, \frac{1-y}{x}) \to (\frac{1}{2}, 1) = (x, y).$$

(1) An equation is defined as

$$\begin{aligned} & ((\tfrac{1}{2}, 2) \to (x_1, y_1)) \wedge ((\tfrac{2}{3}, \tfrac{5}{3}) \to (x_2, y_2)) \\ & \wedge ((\tfrac{2}{3}, \tfrac{5}{3}) \to (x_3, y_3)) = (\tfrac{3}{5}, 3). \end{aligned}$$

Since $(\frac{1}{2}, 2)^0 \wedge (\frac{2}{3}, \frac{5}{3})^0 = (\frac{3}{4}, -\frac{1}{4}) > (\frac{3}{5}, 3)$ by Theorem 3.1(3), it is not solvable.

(2) An equation is defined as

$$\begin{aligned} & ((\tfrac{1}{2}, 2) \to (x_1, y_1)) \wedge ((\tfrac{2}{3}, \tfrac{5}{3}) \to (x_2, y_2)) \\ & \wedge ((\tfrac{2}{3}, \tfrac{5}{3}) \to (x_3, y_3)) = (\tfrac{3}{4}, -\tfrac{1}{4}). \end{aligned}$$

Since $(\frac{1}{2}, 2)^0 \wedge (\frac{2}{3}, \frac{5}{3})^0 = (\frac{3}{4}, -\frac{1}{4})$,

$$\begin{aligned} X = \{ & \mathbf{x} = ((x_1, y_1), (x_2, y_2), \bot) \\ & \text{or } \mathbf{x} = ((x_1, y_1), \bot, (x_3, y_3)) \\ & \mid (x_1, y_1), (x_2, y_2), (x_3, y_3) \geq \bot\} \end{aligned}$$

is a solution set of (I).

$M = \{(\top, \top, \bot), (\top, \bot, \top)\}$ is a maximal solution family of (I).

(3) An equation is defined as

$$\begin{aligned} & ((\tfrac{1}{2}, 2) \Rightarrow (x_1, y_1)) \wedge ((\tfrac{2}{3}, \tfrac{1}{3}) \Rightarrow (x_2, y_2)) \\ & \wedge ((\tfrac{2}{3}, \tfrac{5}{3}) \Rightarrow (x_3, y_3)) = (\tfrac{3}{5}, -1). \end{aligned}$$

Since $(\frac{1}{2}, 2)^* \wedge (\frac{2}{3}, \frac{1}{3})^* \wedge (\frac{2}{3}, \frac{5}{3})^* = (\frac{3}{4}, -1) > (\frac{3}{5}, -1)$ by Theorem 3.1(3), it is not solvable.

(4) An equation is defined as

$$\begin{aligned} & ((\tfrac{1}{2}, 2) \Rightarrow (x_1, y_1)) \wedge ((\tfrac{2}{3}, \tfrac{1}{3}) \Rightarrow (x_2, y_2)) \\ & \wedge ((\tfrac{2}{3}, \tfrac{5}{3}) \Rightarrow (x_3, y_3)) = (\tfrac{3}{4}, -1). \end{aligned}$$

Since $(\frac{1}{2}, 2)^* \wedge (\frac{2}{3}, \frac{1}{3})^* \wedge (\frac{2}{3}, \frac{5}{3})^* = (\frac{3}{4}, -1)$, $X = \{\mathbf{x} = ((x_1, y_1), (x_2, y_2), \bot) \mid (x_1, y_1), (x_2, y_2) \geq \bot\}$ is a solution family of (II). $(\top, \top, \bot)$ is a maximal solution of (II).

Definition 3.4. Let L denote a pseudo BL-chain. L satisfies the right conditional cancellation law if

$$\top < a \odot x \leq a \odot y \Rightarrow x \leq y.$$

L satisfies the left conditional cancellation law if

$$\top < x \odot a \le y \odot a \Rightarrow x \le y.$$

Theorem 3.5. Let L denote a pseudo BL-chain in two equations (I) and (II) of Theorem 3.1.

Then, (1) If L satisfies the right conditional cancellation law $b < \top$ and $b > \bigwedge_{j=1}^{n} a_j^*$ with $B = \{a_{j_k} \mid 1 \le k \le m, b > (a_{j_k})^*\}$, then $X = \{\mathbf{x_{j_k}} = (\top, ..., \overbrace{a_{j_k} \odot b}^{j_k}, ..., \top) \mid 1 \le k \le m\}$ is a maximal solution family of (II). Moreover, if $\mathbf{x}$ is a solution of (II), there exists a family $K = \{j_k \mid a_{j_k} \in B, a_{j_k} \Rightarrow x_{j_k} = b, 1 \le k \le m\}$ such that

$$x_k = a_k \odot b, k \in K,\ x_j \ge a_j \odot b,\ j \notin K$$

where there exists $\mathbf{x_{j_k}} \in X$ such that $\mathbf{x} \le \mathbf{x_{j_k}}$.

(2) If L satisfies the left conditional cancellation law $b < \top$ and $b > \bigwedge_{j=1}^{n} a_j^0$ with $B = \{a_{j_k} \mid 1 \le k \le m, b = (a_{j_k})^0\}$, then $X = \{\mathbf{x_{j_k}} = (\top, ..., \overbrace{b \odot a_{j_k}}^{j_k}, ..., \top) \mid 1 \le k \le m\}$ is a maximal solution of (I). Moreover, if $\mathbf{x}$ is a solution of (I), there exists $k \in \{j_k \mid 1 \le k \le m\}$ such that

$$x_k = b \odot a_k, j = k,\ x_j \ge b \odot a_j,\ j \ne k$$

where there exists $\mathbf{x_{j_k}} \in X$ such that $\mathbf{x} \le \mathbf{x_{j_k}}$.

Proof. (1) ($\Rightarrow$) $\mathbf{x_{j_k}} = (\top, ..., \overbrace{a_{j_k} \odot b}^{j_k}, ..., \top)$ is a solution of (II) because

$$\begin{aligned} &\textstyle\bigwedge_{j=1}^{n}(a_j \Rightarrow x_j) = a_{j_k} \Rightarrow a_{j_k} \odot b \\ &= \bigvee\{y \mid a_{j_k} \odot y \le a_{j_k} \odot b\} = \bigvee\{y \mid y \le b\} = b. \end{aligned}$$

Let $\mathbf{x} \ge \mathbf{x_{j_k}}$ denote a solution of (II). Then, $\mathbf{x} = (\top, ..., \overbrace{x_{j_k}}^{j_k}, ..., \top)$ with $x_{j_k} \ge a_{j_k} \odot b$ and

$$\bigwedge_{j=1}^{n}(a_j \Rightarrow x_j) = a_{j_k} \Rightarrow x_{j_k} = b.$$

Since $b < 1$, $a_{j_k} \not\le x_{j_k}$. Since L is linear, $a_{j_k} > x_{j_k}$. Thus,

$$x_{j_k} = a_{j_k} \wedge x_{j_k} = a_{j_k} \odot (a_{j_k} \Rightarrow x_{j_k}) = a_{j_k} \odot b.$$

Therefore, $\mathbf{x} = \mathbf{x_{j_k}}$. $\mathbf{x_{j_k}} = (\top, ..., \overbrace{a_{j_k} \odot b}^{j_k}, ..., \top)$ is a maximal solution of (II).

Let $\mathbf{x} = (x_1, ..., x_n)$ denote a solution of (II). Since

$$\bigwedge_{j=1}^{n}(a_j \Rightarrow x_j) = b,$$

by the linearity of L, there exists a family $K = \{j_k \mid a_{j_k} \in B, a_{j_k} \Rightarrow x_{j_k} = b, 1 \le k \le m\}$ such that

$$\bigwedge_{j=1}^{n}(a_j \Rightarrow x_j) = \bigwedge_{k=1}^{m}(a_{j_k} \Rightarrow x_{j_k}) = b$$

because $a_{j_k} \notin B, (a_j)^0 \ge b$ implies that $\bigwedge_{a_{j_k} \notin B}(a_j \Rightarrow x_j) \ge \bigwedge_{a_{j_k} \notin B}(a_j \Rightarrow \bot) \ge b$.

For $k \in K$, since $a_k \Rightarrow x_k = b \ne \top$ and L is linear, $a_k > x_k$ and $a_k \odot b = a_k \odot (a_k \Rightarrow x_k) = a_k \wedge x_k = x_k$. For $j \notin K$, since $a_j \Rightarrow x_j \ge b$, $x_j \ge a_j \odot b$. Hence,

$$x_k = a_k \odot b, k \in K,\ x_j \ge a_j \odot b,\ j \notin K$$

($\Leftarrow$) It is trivial.

(2) It is similarly proved as (1).

Example 3.6. The structure $(L, \odot, \Rightarrow, \rightarrow, (\frac{1}{2}, 1), (1, 0))$ is defined as that in Example 3.3. Then, L satisfies the right conditional cancellation law because

$$\begin{aligned} &\bot < (a, b) \odot (x_1, y_1) \le (a, b) \odot (x_2, y_2) \\ &(\Leftrightarrow)\bot < (ax_1, ay_1 + b) \le (ax_2, ay_2 + b) \\ &(\Rightarrow)ax_1 = ax_2, ay_1 + b \le ay_1 + b, \text{ or } ax_1 < ax_2 \\ &(\Rightarrow)x_1 = x_2, y_1 \le y_1, \text{ or } x_1 < x_2 \\ &(\Rightarrow)(x_1, y_1) \le (x_2, y_2). \end{aligned}$$

Similarly, L satisfies the left conditional cancellation law.

(1) An equation is defined as

$$\begin{aligned} &((\tfrac{1}{2}, 2) \Rightarrow (x_1, y_1)) \wedge ((\tfrac{2}{3}, \tfrac{1}{3}) \Rightarrow (x_2, y_2)) \\ &\wedge((\tfrac{2}{3}, \tfrac{5}{3}) \Rightarrow (x_3, y_3)) = (\tfrac{3}{4}, -\tfrac{1}{4}). \end{aligned}$$

Since $(\frac{1}{2}, 2)^* \wedge (\frac{2}{3}, \frac{1}{3})^* \wedge (\frac{2}{3}, \frac{5}{3})^* = (\frac{3}{4}, -1) < (\frac{3}{4}, -\frac{1}{4})$, $B = \{(\frac{2}{3}, \frac{5}{3}) \mid (\frac{2}{3}, \frac{5}{3})^* < (\frac{3}{4}, -\frac{1}{4})\}$ and $\mathbf{x} = (\top, \top, (\frac{1}{2}, \frac{3}{2})$ is a maximal solution of (II) because $(\frac{2}{3}, \frac{5}{3}) \odot (\frac{3}{4}, -\frac{1}{4}) = (\frac{1}{2}, \frac{3}{2})$.

$X = \{\mathbf{x} = ((x_1, y_1), (x_2, y_2), \bot) \mid (x_1, y_1), (x_2, y_2) \ge \bot\}$ is a solution set of (II).

(2) An equation is defined as

$$\begin{aligned} &((\tfrac{1}{2}, 2) \Rightarrow (x_1, y_1)) \wedge ((\tfrac{2}{3}, \tfrac{7}{3}) \Rightarrow (x_2, y_2)) \\ &\wedge((\tfrac{2}{3}, \tfrac{5}{3}) \Rightarrow (x_3, y_3)) = (\tfrac{3}{4}, -\tfrac{1}{4}). \end{aligned}$$

Since $(\frac{1}{2},2)^* \wedge (\frac{2}{3},\frac{7}{3})^* \wedge (\frac{2}{3},\frac{5}{3})^* = (\frac{3}{4},-2) < (\frac{3}{4},-\frac{1}{4})$,

$$B = \{(\frac{2}{3},\frac{7}{3}),(\frac{2}{3},\frac{5}{3}) \mid (\frac{2}{3},\frac{5}{3})^* < (\frac{3}{4},-\frac{1}{4})\},$$

and

$$\mathbf{x}_1 = (\top,(\frac{1}{2},\frac{13}{6}),\top)$$

and

$$\mathbf{x}_2 = (\top,\top,(\frac{1}{2},\frac{3}{2}))$$

are maximal solutions of (II) because

$$(\frac{2}{3},\frac{7}{3}) \odot (\frac{3}{4},-\frac{1}{4}) = (\frac{1}{2},\frac{13}{6}),(\frac{2}{3},\frac{5}{3}) \odot (\frac{3}{4},-\frac{1}{4}) = (\frac{1}{2},\frac{3}{2}).$$

$$\begin{aligned} X = \{&\mathbf{x}_1 = ((x_1,y_1),(x_2,y_2),(\frac{1}{2},\frac{3}{2})),\\ &\mathbf{x}_2 = ((x_1,y_1),(\frac{1}{2},\frac{13}{6}),(x_3,y_3))\\ &\mid (x_1,y_1) \geq \bot,(x_2,y_2) \geq (\frac{1}{2},\frac{13}{6}),(x_3,y_3) \geq (\frac{1}{2},\frac{3}{2})\} \end{aligned}$$

is a solution set of (II).

(3) An equation is defined as

$$\begin{aligned} &((\tfrac{1}{2},2) \to (x_1,y_1)) \wedge ((\tfrac{2}{3},\tfrac{5}{3}) \to (x_2,y_2))\\ &\wedge((\tfrac{2}{3},\tfrac{5}{3}) \to (x_3,y_3)) = (\tfrac{3}{4},-\tfrac{1}{4}). \end{aligned}$$

Since $(\frac{1}{2},2)^0 \wedge (\frac{2}{3},\frac{5}{3})^0 = (\frac{3}{4},-\frac{1}{4})$,

$$\begin{aligned} X = \{&\mathbf{x} = ((x_1,y_1),(x_2,y_2),\bot)\\ &\text{or } \mathbf{x} = ((x_1,y_1),\bot,(x_3,y_3))\\ &\mid (x_1,y_1),(x_2,y_2),(x_3,y_3) \geq \bot\} \end{aligned}$$

is a solution set of (I).

Theorem 3.7. Let $\mathbf{a}_i = (a_{i1},a_{i2},...,a_{in}) \in L^n$ and $b_i \in L$. We define two equations with respect to an unknown $\mathbf{x} = (x_1,...,x_n) \in L^n$ as

$$\bigwedge_{j=1}^{n}(a_{ij} \to x_j) = b_i, i \in \{1,...,m\} \qquad \text{(III)}$$

$$\bigwedge_{j=1}^{n}(a_{ij} \Rightarrow x_j) = b_i,\ i \in \{1,...,m\} \qquad \text{(IV).}$$

Then, (1) (III) is solvable iff it has the least solution $\mathbf{x} = (x_1,...,x_n) \in L^n$ such that $x_j = \bigvee_{i=1}^{m}(b_i \odot a_{ij}), j = 1,...,n$.

(2) (IV) is solvable iff it has the least solution $\mathbf{x} = (x_1,...,x_n) \in L^n$ such that $x_j = \bigvee_{i=1}^{m}(a_{ij} \odot b_i), j = 1,...,n$.

(3) If (III) is solvable, then $b_i \geq \bigwedge_{j=1}^{n} a_{ij}^0$.

(4) If (IV) is solvable, then $b_i \geq \bigwedge_{j=1}^{n} a_{ij}^*$.

(5) If (III) (resp. (IV)) is solvable and $\mathbf{x}_1,...,\mathbf{x}_m$ is a solution of each ith equation, $i = 1,2,...,m$, then $\mathbf{x} = \bigwedge_{i=1}^{m}\mathbf{x}_i$ is a solution of (III) (resp. (IV)). Moreover, if each solution $\mathbf{x}_i$ of the ith equation is maximal, any maximal solution $\mathbf{x}$ of (III) (resp. (IV)) is $\mathbf{x} = \bigwedge_{i=1}^{m}\mathbf{x}_i$.

Proof. (1) (⇒) Let $\mathbf{y} = (y_1,...,y_n)$ denote a solution of (III). Since $b_i = \bigwedge_{j=1}^{n}(a_{ij} \to y_j) \leq a_{ij} \to y_j, b_i \odot a_{ij} \leq y_j$. Then, $\bigvee_{i=1}^{m}(b_i \odot a_{ij}) \leq y_j$.

Moreover,

$$\begin{aligned} b_i &= \bigwedge_{j=1}^{n}(a_{ij} \to b_i \odot a_{ij})\\ &\leq \bigwedge_{j=1}^{n}(a_{ij} \to \bigvee_{i=1}^{m}(b_i \odot a_{ij}))\\ &\leq \bigwedge_{j=1}^{n}(a_{ij} \to y_j) = b_i. \end{aligned}$$

Then, $\bigwedge_{j=1}^{n}(a_{ij} \to \bigvee_{i=1}^{m}(b_i \odot a_{ij})) = b_i,\ i \in \{1,...,m\}$. Substitute $x_j = \bigvee_{i=1}^{m}(b_i \odot a_{ij})$. Thus, $(x_1,...,x_n)$ is the least solution.

(⇐) It is trivial.

(3)

$$\begin{aligned} b_i &= \bigwedge_{j=1}^{n}(a_{ij} \to x_j)\\ &\geq \bigwedge_{j=1}^{n}(a_{ij} \to \bot)\\ &= \bigwedge_{j=1}^{n}(a_{ij})^0, i \in \{1,...,m\}. \end{aligned}$$

(2) and (4) are similarly proved as (1) and (3), respectively.

(5) Let $\mathbf{x}_i = (x_{i1},...,x_{in})$ denote a solution of the ith equation in (III) and $\mathbf{x} = \bigwedge_{i=1}^{m}\mathbf{x}_i = (x_1,...,x_n)$ with $x_j = \bigwedge_{i=1}^{m} x_{ij}$. Then,

$$\bigwedge_{j=1}^{n}(a_{ij} \to x_j) \leq \bigwedge_{j=1}^{n}(a_{ij} \to x_{ij}) = b_i.$$

Moreover,

$$\bigwedge_{j=1}^{n}(a_{ij} \to x_j) \geq \bigwedge_{j=1}^{n}(a_{ij} \to b_i \odot a_{ij}) = b_i.$$

Hence, $\bigwedge_{j=1}^{n}(a_{ij} \to x_j) = b_i$. Therefore, $\mathbf{x} = \bigwedge_{i=1}^{m}\mathbf{x}_i$ is a solution of (III).

Moreover, if $\mathbf{x}_i$ is a maximal solution of the ith equation in (III), then $\mathbf{x} = \bigwedge_{i=1}^{m}\mathbf{x}_i$ is a solution of (III). Let $\mathbf{y} =$

$(y_1, ..., y_n)$ denote a solution of (III). Then, $\mathbf{y} \leq \mathbf{x}_i$ for each $i = 1, ...m$. Then, $\mathbf{y} \leq \mathbf{x} = \bigwedge_{i=1}^m \mathbf{x}_i$. Hence, $\mathbf{x}$ is a maximal solution of (III).

Example 3.8. The structure $(L, \odot, \Rightarrow, \rightarrow, (\frac{1}{2}, 1), (1, 0))$ is defined as that in Example 3.3.

(1) An equation is defined as

$$((\tfrac{2}{3}, 2) \Rightarrow (x_1, y_1)) \wedge ((\tfrac{3}{4}, \tfrac{1}{3}) \Rightarrow (x_2, y_2)) \\ \wedge((\tfrac{2}{3}, \tfrac{5}{3}) \Rightarrow (x_3, y_3)) = (\tfrac{5}{6}, -1).$$

$\mathbf{X}_1 = \{((\frac{5}{9}, \frac{4}{3}), (x_2, y_2), (x_3, y_3)), ((x_1, y_1), (\frac{5}{8}, -\frac{5}{12}), (x_2, y_2)), ((x_1, y_1), (x_2, y_2), (\frac{5}{9}, 1)) \mid (x_1, y_1) \geq (\frac{5}{9}, \frac{4}{3}), (x_2, y_2) \geq (\frac{5}{8}, -\frac{5}{12}), (x_3, y_3) \geq (\frac{5}{9}, 1)\}$ is a solution set. $\mathbf{M}_1 = \{((\frac{5}{9}, \frac{4}{3}), \top, \top), (\top, (\frac{5}{8}, -\frac{5}{12}), \top), (\top, \top, (\frac{5}{9}, 1))\}$ is a maximal solution set.

(2) An equation is defined as

$$((\tfrac{5}{6}, 2) \Rightarrow (x_1, y_1)) \wedge ((\tfrac{2}{3}, \tfrac{1}{3}) \Rightarrow (x_2, y_2)) \\ \wedge((\tfrac{3}{4}, 0) \Rightarrow (x_3, y_3)) = (\tfrac{3}{4}, -1).$$

$$\mathbf{X}_2 = \{((\tfrac{5}{8}, \tfrac{7}{6}), (x_2, y_2), (x_3, y_3)), \\ ((x_1, y_1), (x_2, y_2), (\tfrac{9}{16}, -\tfrac{3}{4})) \\ \mid (x_1, y_1) \geq (\frac{5}{8}, \frac{7}{6}), (x_2, y_2) \geq \bot, (x_3, y_3) \geq (\tfrac{9}{16}, -\tfrac{3}{4})\}$$

is a solution set.

$\mathbf{M}_2 = \{((\frac{5}{8}, \frac{7}{6}), \top, \top)$ or $(\top, \top, (\frac{9}{16}, -\frac{3}{4}))\}$ is a maximal solution set.

$\mathbf{X} = \{((\frac{5}{8}, \frac{7}{6}), (\frac{5}{8}, -\frac{5}{12}), (x_3, y_3)), ((x_1, y_1), (\frac{5}{8}, -\frac{5}{12}), (\frac{9}{16}, -\frac{3}{4})) \mid (x_1, y_1) \geq (\frac{5}{8}, \frac{7}{6}), (x_3, y_3) \geq (\frac{9}{16}, -\frac{3}{4}))\}$ is a solution set of (1) and (2).

$$\mathbf{X} = \{((\tfrac{5}{8}, \tfrac{7}{6}), (\tfrac{5}{8}, -\tfrac{5}{12}), \top), (\top, (\tfrac{5}{8}, -\tfrac{5}{12}), (\tfrac{9}{16}, -\tfrac{3}{4})) \\ \mid (x_1, y_1) \geq (\tfrac{5}{8}, \tfrac{7}{6}), (x_3, y_3) \geq (\tfrac{9}{16}, -\tfrac{3}{4}))\}$$

is a maximal solution set of (1) and (2).

(3) An equation is defined as

$$((\tfrac{2}{3}, 2) \rightarrow (x_1, y_1)) \wedge ((\tfrac{3}{4}, \tfrac{1}{3}) \rightarrow (x_2, y_2)) \\ \wedge((\tfrac{2}{3}, \tfrac{5}{3}) \rightarrow (x_3, y_3)) = (\tfrac{5}{6}, -1).$$

$\mathbf{X}_3 = \{((\frac{5}{9}, \frac{2}{3}), (x_2, y_2), (x_3, y_3)), ((x_1, y_1), (\frac{5}{8}, -\frac{13}{18}), (x_2, y_2)), ((x_1, y_1), (x_2, y_2), (\frac{5}{9}, \frac{7}{18})) \mid (x_1, y_1) \geq (\frac{5}{9}, \frac{2}{3}), (x_2, y_2) \geq (\frac{5}{8}, -\frac{13}{18}), (x_3, y_3) \geq (\frac{5}{9}, \frac{7}{18})\}$ is a solution set. $\mathbf{M}_3 = \{((\frac{5}{9}, \frac{2}{3}), \top, \top), (\top, (\frac{5}{8}, -\frac{13}{18}), \top), (\top, \top, (\frac{5}{9}, \frac{7}{18}))\}$ is a maximal solution set.

(4) An equation is defined as

$$((\tfrac{5}{6}, 2) \rightarrow (x_1, y_1)) \wedge ((\tfrac{2}{3}, \tfrac{1}{3}) \rightarrow (x_2, y_2)) \\ \wedge((\tfrac{3}{4}, 0) \rightarrow (x_3, y_3)) = (\tfrac{3}{4}, -1).$$

$$\mathbf{X}_4 = \{((\tfrac{5}{8}, \tfrac{1}{2}), (x_2, y_2), (x_3, y_3)), \\ ((x_1, y_1), (x_2, y_2), (\tfrac{9}{16}, -1)) \\ \mid (x_1, y_1) \geq (\tfrac{5}{8}, \tfrac{1}{2}), (x_2, y_2) \geq \bot, (x_3, y_3) \geq (\tfrac{9}{16}, -1)\}$$

is a solution set.

$\mathbf{M}_4 = \{((\frac{5}{8}, \frac{1}{2}), \top, \top), (\top, \top, (\frac{9}{16}, -1))\}$ is a maximal solution set.

$\mathbf{X} = \{((\frac{5}{8}, \frac{1}{2}), (\frac{5}{8}, -\frac{13}{18}), (x_3, y_3)), ((x_1, y_1), (\frac{5}{8}, -\frac{13}{18}), (\frac{9}{16}, -1)) \mid (x_1, y_1) \geq (\frac{5}{8}, \frac{1}{2}), (x_3, y_3) \geq (\frac{9}{16}, -1))\}$ is a solution set of (3) and (4).

$\mathbf{X} = \{((\frac{5}{8}, \frac{1}{2}), (\frac{5}{8}, -\frac{13}{18}), \top)$ or $(\top, (\frac{5}{8}, -\frac{13}{18}), (\frac{9}{16}, -1))\}$ is a maximal solution set of (3) and (4).

4. Conclusion

Bandler and Kohout [6] investigated the solvability of fuzzy relation equations with inf-implication compositions in complete lattices. Perfilieva and Noskova investigated the solvability of fuzzy relation equations with inf-implication compositions in BL-algebras. In this paper, we investigated various solutions of fuzzy relation equations with inf-implication compositions in pseudo BL-algebras.

In the future, we will investigate various solutions of fuzzy relation equations with sup-compositions in pseudo BL-algebras and other algebraic structures.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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References

[1] E. Sanchez, "Resolution of composite fuzzy relation equations," *Information and Control*, vol. 30, no. 1, pp. 38-48, Jan. 1976.

[2] S. Gottwald, "On the existence of solutions of systems of fuzzy equations," *Fuzzy Sets and Systems*, vol. 12, no. 3, pp. 301-302, Apr. 1984.

[3] W. Pedrycz, "s-t fuzzy relational equations," *Fuzzy Sets and Systems*, vol. 59, no. 2, pp. 189-195, Oct. 1993.

[4] I. Perfilieva, "Fuzzy function as an approximate solution to a system of fuzzy relation equations," *Fuzzy Sets and Systems*, vol. 147, no. 3, pp. 363-383, Nov. 2004.

[5] I. Perfilieva and L. Noskova, "System of fuzzy relation equations with inf- composition: complete set of solutions," *Fuzzy Sets and Systems*, vol. 159, no. 17, pp. 2256-2271, Sep. 2008.

[6] W. Bandler and L. J. Kohout, "Semantics of implication operators and fuzzy relational products," *International Journal of Man-Machine Studies*, vol. 12, no. 1, pp. 89-116, Jan. 1980. http://dx.doi.org/10.1016/S0020-7373(80)80055-1

[7] G. Birkhoff, *Lattice Theory,* 3rd ed., Providence, RI: American Mathematical Society, 1967.

[8] A. Dvurecenskij, "Pseudo MV-algebras are intervals in l-groups," *Journal of the Australian Mathematical Society*, vol. 72, no. 3, pp. 427-445, Jun. 2002.

[9] A. Dvurecenskij, "On pseudo MV-algebras," *Soft Computing*, vol. 5, no. 5, pp. 347-354, Oct. 2001. http://dx.doi.org/10.1007/s005000100136

[10] N. Galatos and C. Tsinakis, "Generalized MV-algebras," *Journal of Algebra*, vol. 283, no. 1, pp. 254-291, Jan. 2005. http://dx.doi.org/10.1016/j.jalgebra.2004.07.002

[11] G. Georgescu and L. Leustean, "Some classes of pseudo-BL algebras," *Journal of the Australian Mathematical Society*, vol. 73, no. 1, pp. 127-154, Aug. 2002. http://dx.doi.org/10.1017/S144678870000851X

[12] G. Georgescu and A. lorgulescu, "Pseudo MV-algebras," *Multiple-Valued Logics*, vol. 6, pp. 193-215, 2001.

[13] G. Georgescu and A. Popescu, "Non-commutative fuzzy Galois connections," *Soft Computing*, vol. 7, no. 7, pp. 458-467, Jun. 2003. http://dx.doi.org/10.1007/s00500-003-0280-4

[14] G. Georgescu and A. Popescu, "Non-commutative fuzzy structures and pairs of weak negations," *Advances in Fuzzy Logic*, vol. 143, no. 1, pp. 129-155, Apr. 2004.

[15] U. Hoohle and E. P. Klement, *Non-Classical Logics and Their Applications to Fuzzy Subsets: A Handbook of the Mathematical Foundations of Fuzzy Set Theory*, Boston: Kluwer Academic Publishers, 1995.

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