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Special Fundamental Fields for Plane Cracks in Weight Function Theory

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Abstract

We construct the fundamental fields for 3-D arbitrarily shaped plane cracks in by differentiating with respect to a parameter of the crack. As an application of these fundamental fields, the total elastic energy release rate in combined mode cracking is computed.

Keywords: Stress Intensity Factors, Weight Function Theory, Plane Cracks, Reciprocity Theorem

1. Introduction

Bueckner (1970) introduced weight functions to determine the stress intensity factors in cracked bodies. These weight functions are displacements fields with a suitable singularity at the crack tip for the given crack configuration and body geometry. The displacement field produces stresses that are in equilibrium with the zero body forces and zero tractions on the surface of the body. The fields of the weight functions and corresponding strains and stresses are referred to as fundamental fields. Rice (1972) noted that when the dependence of the stress intensity factor and the displacements on the crack tip position are known under symmetric loading, the mode I weight function can be obtained by differentiating the displacements with respect to the crack position. Montenegro et al. (2006, 2008) used the weight function method to calculate the mode I stress intensity factor in embedded and surface irregular cracks and for a partially closed three-dimensional plane crack. In this paper, following Rice's method, we compute the fundamental field for a plane crack by partially differentiating with respect to a parameter of the plane crack. As an application of this fundamental field, the total energy release rate in combined mode cracking is obtained.

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2. Regular Fields

A rectangular Cartesian coordinate system with axes x, y, z and a system of cylindrical coordinates r, θ, z will be used. We assume that $x = r\cos(\theta - \theta_O)$ and $y = r\sin(\theta - \theta_O)$, with θ_O denoting some fixed angle. An elastic body V with a plane crack C in the plane $z = 0$ will be considered. C^+ and C^- refer to the crack faces in the upper and lower $z-$half spaces, respectively. C shall be chosen from a $1-$parametric family of cracks $C(p)$, defined by

$$r \le a(\theta) + pb(\theta), \quad -1 < p < 1 \tag{1}$$

where $a(\theta)$ and $b(\theta)$ are two smooth functions of θ with period 2π; $a + pb$ shall be positive for all θ and p and such that

$$r = a(\theta) + pb(\theta) \tag{2}$$

defines a simple contour $\Gamma(p)$, the edge of $C(p)$. The intersection of V with the plane $z=0$ is assumed to be large enough to accommodate the whole family (1). If $C=C(p)$, we let $S(p)$ denotes the boundary of V with the faces of C included. Let $V'\subset V$ and $S'\subset S(p)$ be portions of the elastic body and its boundary, respectively; these portions are to be the same for all p. In V', we assume a distribution of body forces F and on S' a distribution of tractions T. Both F and T shall be independent of p; the combined loading system F, T is to be self-equilibrated. We set

$$d=a(\theta)+pb(\theta)-r \tag{3}$$

This assigns to every point (r,θ) of C a distance d from $\Gamma(p)$. The generally accepted asymptotics of u, v, w can be written as

$$\begin{aligned} u&=u_o(\theta,p)+\alpha(\theta,p)d^{1/2}+u_1\\ v&=v_o(\theta,p)+\beta(\theta,p)d^{1/2}+v_1\\ w&=w_o(\theta,p)+\gamma(\theta,p)d^{1/2}+w_1 \end{aligned} \tag{4}$$

for the points of C^+. We postulate here that u_o, v_o, w_o; α, β, γ are smooth functions of θ,p, and that u_1, v_1, w_1 have the order $o(d^{1/2})$ as $d\to 0$.

3. Fundamental Fields

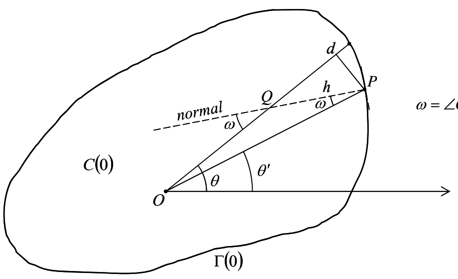

Fig. 1 Definitions of geometric variables in plane crack

A cracked body responds with a field of displacements, strains, and stresses. Let $\xi(p)$ denote this field; it depends on p through the choice $C=C(p)$ of the crack. The differentiation of all the field quantities with respect to p yields a field $\xi^*(p)$. Here, we assume a fixed load on the boundary and in the body with fixed functions. Differentiation with respect to parameter p leads to a field with vanishing boundary tractions and body forces. It has neither body forces nor boundary tractions. It is an ordinary fundamental with certain geometric intensity coefficients $m_i^*(\theta,p)$, which we shall link to the stress intensity factors $K_i(\theta,p)$ of $\xi(p)$. This can be done by comparing the Cartesian displacements u, v, w of $\xi(p)$ with the displacements $u^*=\partial u/\partial p$, $v^*=\partial v/\partial p$, and $w^*=\partial w/\partial p$ of $\xi^*(p)$ within C^+. If we differentiate equation (4) with respect to p, then

$$\begin{aligned} 2u^*&=b(\theta)\cdot\alpha(\theta,p)\cdot d^{-1/2}+o(d^{-1/2})\\ 2v^*&=b(\theta)\cdot\beta(\theta,p)\cdot d^{-1/2}+o(d^{-1/2})\\ 2w^*&=b(\theta)\cdot\gamma(\theta,p)\cdot d^{-1/2}+o(d^{-1/2}) \end{aligned} \tag{5}$$

There is no essential loss of generality to consider the approach to $\Gamma(p)$ from within C^+ for $p=0$. To this end, we pick a point p of angle θ' on $\Gamma(0)$ and let a point Q approach P along the inner normal to $\Gamma(0)$ through P (Fig. 1). We denote the distance from Q to P by h and observe that

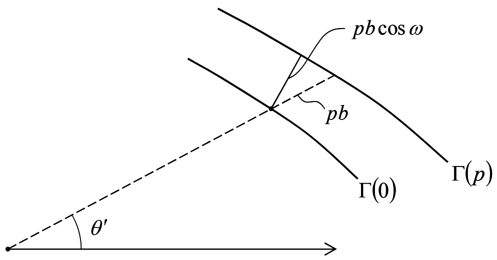

Fig. 2 Geometric meaning of $b\cos\omega$

$$\lim_{Q \to p} \frac{h}{d} = \cos\omega \tag{6}$$

where ω is the angle OPQ. We add here that

$$\cos\omega = a(\theta)\frac{d\theta}{ds} \tag{7}$$

at any generic point θ on $\Gamma(0)$; ds is the arc element on the edge. Turning now to K_i and m_i^*, we assume for simplicity that the tangent to $\Gamma(0)$ through P runs parallel to the $y-$axis. In this case,

$$\begin{aligned} K_I(\theta',0) &= \frac{\mu\sqrt{\pi}}{\sqrt{2}(1-\nu)}\lim(w-w_0)h^{-1/2} \\ K_{II}(\theta',0) &= \frac{\mu\sqrt{\pi}}{\sqrt{2}(1-\nu)}\lim(u-u_0)h^{-1/2} \\ K_{III}(\theta',0) &= -\frac{\mu\sqrt{\pi}}{\sqrt{2}}\lim(v-v_0)h^{-1/2} \end{aligned} \tag{8}$$

and furthermore

$$\begin{aligned} m_I^*(\theta',0) &= \lim w^* \cdot h^{1/2} \\ m_{II}^*(\theta',0) &= \lim u^* \cdot h^{1/2} \\ m_{III}^*(\theta',0) &= -\lim v^* \cdot h^{1/2} \end{aligned} \tag{9}$$

Now, the first relations (8) and (6) imply

$$K_I(\theta',0) = \frac{\mu\gamma(\theta',0)\sqrt{\pi}}{\sqrt{2}(1-\nu)}\cdot(\cos\omega)^{-1/2}$$

while the first relations (9) and (6) lead to

$$m_I^*(\theta',0) = 1/2b(\theta')\gamma(\theta',0)(\cos\omega)^{1/2}.$$

Hence,

$$m_I^*(\theta',0) = \frac{1-\nu}{\mu\sqrt{2\pi}}K_I(\theta',0)b(\theta')\cos\omega \tag{10}$$

In similar vein,

$$m_{II}^*(\theta',0) = \frac{1-\nu}{\mu\sqrt{2\pi}}K_{II}(\theta',0)b(\theta')\cos\omega \tag{11}$$

$$m_{III}^*(\theta',0) = \frac{1}{\mu\sqrt{2\pi}}K_{III}(\theta',0)b(\theta')\cos\omega \tag{12}$$

Relations (10)–(12) have been derived for the case where the tangent to $\Gamma(0)$ at p has the direction of the y-axis. However, this simplifying condition can always be established for any given p on $\Gamma(0)$ by a suitable choice for angle θ_0, which appears in the relations between the Cartesian coordinates x, y and the cylindrical ones r, θ. Therefore, relations (10)–(12) are valid for any p on $\Gamma(0)$. The product $b\cos\omega$ admits a simple geometric interpretation (Fig. 2). For small $|p|$, the quantity $|pb\cos\omega|$ represents the distance from p on $\Gamma(0)$ to the nearest point on $\Gamma(p)$.

Before we turn to an application, we consider a second family of cracks, $\bar{c}(\bar{p})$,

$$r \le a(\theta) + \bar{p}\cdot\bar{b}(\theta) \tag{13}$$

where $\bar{c}(0) = c(0)$. In analogy to the fields $\xi(p)$, we define a second family $\bar{\xi}(\bar{p})$ caused by applying a fixed load system $\bar{F}$, $\bar{T}$ to fixed regions $\bar{V}'$, $\bar{S}'$. Here "fixed" indicates the independence of $\bar{p}$. Differentiation with respect to $\bar{p}$ yields ordinary fundamental fields, and our results for the first family apply to the second family as well.

4. Application

Let us now take V with the crack $C(0)$ (the same for both families) and remove from V all those points whose distance from $\Gamma(0)$ is less than ε, where $\varepsilon > 0$ is given. For sufficiently small ε, the removed material is inside a torus surrounding $\Gamma(0)$; we apply the reciprocity theorem to the two fields $\xi^*(p)$ and $\overline{\xi}(p)$ in the reduced elastic body. Thereafter, we let $\varepsilon \rightarrow 0$. The result is an energy balance of the form (Bueckner (1973))

$$\sqrt{2\pi} \int_{\Gamma(0)} \left(\overline{K_I} m_I^* + \overline{K_{II}} m_{II}^* + \overline{K_{III}} m_{III}^* \right) ds = \int_S (w^*, \overline{T}) ds + \int_V (w^*, \overline{F}) dv \tag{14}$$

where the left hand side represents the contribution of the toroidal surface in the limit $\varepsilon = 0$. w^* is the displacement vector of $\xi^*(0)$, and the bar distinguishes the quantities of the second family. The results for (10)–(12) make it possible to rewrite the left hand side in terms of K_i, $\overline{K_i}$ so that

$$\frac{1}{\mu} \int_\Gamma \left[(1-\nu)\left(K_I \overline{K_I} + K_{II} \overline{K_{II}} \right) + K_{III} \overline{K_{III}} \right] \cdot b \cos\omega \cdot ds = \int_S (w^*, \overline{T}) ds + \int_V (w^*, \overline{F}) dv \tag{15}$$

In the same vein, the reciprocity theorem can be applied to $\overline{\xi^*}(0)$ and $\xi(0)$. The result is

$$\frac{1}{\mu} \int_\Gamma \left[(1-\nu)\left(K_I \overline{K_I} + K_{II} \overline{K_{II}} \right) + K_{III} \overline{K_{III}} \right] \cdot \overline{b} \cos\omega \cdot ds = \int_S (\overline{w^*}, T) ds + \int_V (\overline{w^*}, F) dv \tag{15`}$$

Proceeding with (15), we choose a special system $\overline{F}$, $\overline{T}$ for illustration. To this end, we take two points Q_1, Q_2 of V and denote the unit vector pointing from Q_1 in the direction of Q_2 by n. At Q_1, we apply a unit force in direction n; and apply a unit force in direction n at Q_2. No other forces or tractions shall act. Hence,

$$\frac{1}{\mu} \int_\Gamma \left[(1-\nu)\left(K_I \overline{K_I} + K_{II} \overline{K_{II}} \right) + K_{III} \overline{K_{III}} \right] \cdot b \cos\omega \cdot ds = \left(w^*(Q_1) - w^*(Q_2), n \right) \tag{16}$$

This is a particular case of (16): Q_1, Q_2 are opposite points on the crack faces, and n is normal to C.

One can multiply both sides of (15) by p and rewrite the balance with the aid of the denotations

$$pb\cos\omega = \delta a, \quad pw^* = \delta w \tag{17}$$

For positive p and b, we can interpret $C(p)$ to be an extension of $C(0)$ and $\delta_a ds$ as a local gain in area; δ_w is the change in $w(0)$ caused by the extension. The new form of (15), namely

$$\frac{1}{\mu} \int_\Gamma \left[(1-\nu)\left(K_I \overline{K_I} + K_{II} \overline{K_{II}} \right) + K_{III} \overline{K_{III}} \right] \delta a ds = \int_S (\delta w, \overline{T}) ds + \int_V (\delta w, \overline{F}) dv \tag{18}$$

can be interpreted as an exchange of energies that accompanies the extension. If, in particular, the two families are identical, so that $K_i = \overline{K_i}$, $T = \overline{T}$, and $F = \overline{F}$, then the right hand side of (18) represents the work of the externally impressed forces F, T in the process of crack extension under a constant load. The left hand side of (18) is equal to twice the energy released (Broek (1982)).

Formulas (10)–(12) are valid beyond the assumptions on S' and T. It suffices to have T defined on the union of all $S(p)$ and to use the restriction of T on $S(p)$ as traction in the case of $C = C(p)$.

5. Conclusion

In this paper, by differentiating with respect to crack geometric parameter p, we obtained the weight function for a plane crack in a 3-D linear elastic body. Differentiation gave a field with vanishing boundary tractions and body forces. As an application of this fundamental field, the total energy release rate was obtained in combined mode cracking for a plane 3-D crack.

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