

## NOTES ON THE SPACE OF DIRICHLET TYPE AND WEIGHTED BESOV SPACE

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**ABSTRACT.** For  $0 < p < \infty$ ,  $\alpha > -1$  and  $0 < r < 1$ , we show that if  $f$  is in the space of Dirichlet type  $\mathfrak{D}_{p-1}^p$ , then  $\int_0^1 M_p^p(r, f')(1-r)^{p-1}rdr < \infty$  and  $\int_0^1 M_{(2+\alpha)p}^{(2+\alpha)p}(r, f')(1-r)^{(2+\alpha)p+\alpha}rdr < \infty$  where  $M_p(r, f) = \left[ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right]^{1/p}$ . For  $1 < p < q < \infty$  and  $\alpha + 1 < p$ , we show that if there exists some positive constant  $c$  such that  $\|f\|_{L^q(d\mu)} \leq c \|f\|_{\mathfrak{D}_\alpha^p}$  for all  $f \in \mathfrak{D}_\alpha^p$ , then  $\|f\|_{L^q(d\mu)} \leq c \|f\|_{\mathcal{B}_p(g)}$  where  $\mathcal{B}_p(g)$  is the weighted Besov space. We also find the condition of measure  $\mu$  such that  $\sup_{a \in D} \int_D (k_a(z)(1-|a|^2)^{(p-\alpha-1)})^{q/p} d\mu(z) < \infty$ .

### 1. Introduction

Let  $C$  be the complex plane and  $D = \{z \in C : |z| < 1\}$  be the open unit disk in  $C$ . Let  $dA(z)$  be the area measure on  $D$  normalized so that the area of  $D$  is 1. In rectangular and polar coordinates,

$$dA(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr d\theta.$$

Let  $\mathcal{H}(D)$  be the space of analytic functions in  $D$ . For  $1 \leq p < +\infty$ , the Bergman space  $L_a^p(D, dA)$  consists of those functions  $f \in \mathcal{H}(D)$  such that

$$\|f\|_{L^p(dA)} = \left[ \int_D |f(z)|^p dA(z) \right]^{1/p} < +\infty.$$

For any  $\alpha > -1$ , let  $dA_\alpha$  be the measure on  $D$  defined by

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z).$$

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Then  $dA_\alpha(z)$  is a probability measure on  $D$ . For any  $1 \leq p < +\infty$ , the weighted Bergman space  $L_{a,\alpha}^p$  is the set of all  $f \in \mathcal{H}(D)$  for which  $\int_D |f(z)|^p dA_\alpha(z) < \infty$  (See [13]).

If  $0 < p$  and  $\alpha > -1$ , the space of Dirichlet type  $\mathfrak{D}_\alpha^p$  consists of those functions  $f \in \mathcal{H}(D)$  such that

$$\|f\|_{\mathfrak{D}_\alpha^p} = \left( |f(0)|^p + \int_D |f'(z)|^p dA_\alpha(z) \right)^{1/p} < \infty.$$

See [9, 12] as general references for the space of Dirichlet type  $\mathfrak{D}_\alpha^p$ .

For  $0 < p < \infty$  and  $0 < r < 1$ , we set

$$M_p(r, f) = \left[ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right]^{1/p}.$$

As usual,  $M_\infty(r, f) = \sup\{|f(rt)| : t \in \partial D\}$ .

The classical result of Hardy and Littlewood is that  $M_p(f, r) = O((1-r)^{-\alpha})$  if and only if  $M_p(f', r) = O((1-r)^{-(\alpha+1)})$ . In [10], the important various properties of  $M_p(r, f)$  were investigated.

In section 2, we will show that if  $f \in \mathfrak{D}_{p-1}^p$ , then

$$\int_0^1 M_p^p(r, f')(1-r)^{p-1} r dr < \infty$$

and

$$\int_0^1 M_{(2+\alpha)p}^{(2+\alpha)p}(r, f')(1-r)^{(2+\alpha)p+\alpha} r dr < \infty$$

where  $\alpha > -1$ .

Riesz representation theorem implies that there exists a unique function  $K_z$  in  $L_a^2(D, dA)$  such that

$$f(z) = \int_D f(w) \overline{K_z(w)} dA(w)$$

for all  $f$  in  $L_a^2(D, dA)$ . Let  $K(z, w)$  be the function on  $D \times D$  defined by  $K(z, w) = \overline{K_z(w)}$ .  $K(z, w)$  is called the Bergman kernel of  $D$ .

For  $1 < p < +\infty$ , the Besov space  $\mathcal{B}_p$  of  $D$  is defined to be the space of  $f \in \mathcal{H}(D)$  such that

$$\int_D (1 - |z|^2)^p |f'(z)|^p d\lambda(z) < +\infty$$

where  $d\lambda(z) = K(z, z) dA(z) = \frac{dA(z)}{(1-|z|^2)^2}$  is the Möbius invariant measure on  $D$ . Note that  $\mathcal{B}_2$  is the classical Dirichlet space.

If  $g$  is a positive Borel measurable function in  $D$ , the weighted Besov space  $\mathcal{B}_p(g)$  is defined to be the space of  $f \in \mathcal{H}(D)$  such that

$$\|f\|_{\mathcal{B}_p(g)}^p = |f(0)|^p + \int_D (1 - |z|^2)^p |f'(z)|^p g(z) d\lambda(z) < \infty.$$

In section 3, we will show that if  $-1 < \alpha < p - 1$ , then  $\mathfrak{D}_\alpha^p = \mathcal{B}_p(g)$ .

For a large class of spaces  $X$  of analytic functions in  $D$ , a characterization of those positive Borel measures  $\mu$  in  $D$  such that  $X \subset L^p(d\mu)$  is known and such a characterization is useful to study the boundedness of operators acting on  $X$ .

In section 3, we will show that if there exists some positive constant  $c$  such that  $\|f\|_{L^q(d\mu)} \leq c \|f\|_{\mathfrak{D}_\alpha^p}$  for all  $f \in \mathfrak{D}_\alpha^p$ , then  $\|f\|_{L^q(d\mu)} \leq c \|f\|_{\mathcal{B}_p(g)}$ .

In this paper,  $c$  will stand for positive constants whose value may change from line to line but not depend on the functions.

For any  $a \in D$  and  $z \in D$ , let

$$k_a(z) = \frac{K(z, a)}{\sqrt{K(a, a)}} = \frac{1 - |a|^2}{(1 - z\bar{a})^2}.$$

Then  $k_a(z)$  are unit vectors in  $L_a^2(D, dA)$ . Let  $0 < p, q < \infty$  and  $-1 < \alpha < p$ . Let  $\mu$  be a positive Borel measure on  $D$ . In section 3, we will also show that if there exists some positive constant  $c$  such that  $\|f\|_{L^q(d\mu)} \leq c \|f\|_{\mathfrak{D}_\alpha^p}$  for all  $f \in \mathcal{H}(D)$ , then

$$\sup_{a \in D} \int_D (k_a(z)(1 - |a|^2)^{(p-\alpha-1)})^{q/p} d\mu(z) < \infty.$$

## 2. Properties of $M_p(r, f)$

LEMMA 2.1. If  $1 \leq p < \alpha + 1$ , then

$$f \in L_{a,\alpha}^p \iff f' \in L_{a,\alpha+p}^p.$$

*Proof.* By Theorem 6 in [8],

$$\mathfrak{D}_\alpha^p = L_{a,\alpha-p}^p$$

if  $1 \leq p < \alpha + 1$ . By the definition of  $\mathfrak{D}_\alpha^p$  and  $L_{a,\alpha}^p$ ,  $f \in \mathfrak{D}_\alpha^p$  if and only if  $f' \in L_{a,\alpha}^p$ . This implies that

$$\begin{aligned} f \in L_{a,\alpha}^p (= L_{a,\alpha+p-p}^p) &\iff f \in \mathfrak{D}_{\alpha+p}^p \\ &\iff f' \in L_{a,\alpha+p}^p. \end{aligned}$$

□

**THEOREM 2.2.** *If  $1 \leq p < \alpha + 1$  and  $\int_0^1 M_p^p(r, f') dr < \infty$ , then  $f \in L_{a,\alpha-p}^p$ .*

*Proof.* We have

$$\begin{aligned} & \int_D |f'(z)|^p (1 - |z|)^\alpha dA(z) \\ &= \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |f'(re^{it})|^p (1 - r)^\alpha r dr dt \\ &= \int_0^1 \left[ \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{it})|^p dt \right] (1 - r)^\alpha r dr \\ &\leq \int_0^1 M_p^p(r, f') dr \\ &< \infty. \end{aligned}$$

This implies that  $f' \in L_{a,\alpha}^p$ . By Lemma 2.1,  $f \in L_{a,\alpha-p}^p$ .  $\square$

**LEMMA 2.3.** *If  $f \in \mathfrak{D}_{p-1}^p$ , then*

$$\int_0^1 M_p^p(r, f') (1 - r)^{p-1} r dr < \infty.$$

*Proof.* We have

$$\begin{aligned} & \int_0^1 M_p^p(r, f') (1 - r)^{p-1} r dr \\ &= \int_0^1 \left[ \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{it})|^p dt \right] (1 - r)^{p-1} r dr \\ &= \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |f'(re^{it})|^p (1 - r)^{p-1} r dt dr \\ &= \int_D |f'(z)|^p (1 - |z|)^{p-1} dA(z) \\ &< c \int_D |f'(z)|^p dA_{p-1}(z) \\ &< \infty. \end{aligned}$$

$\square$

**LEMMA 2.4.** *For  $0 < s < t \leq \infty$ , there exists a positive constant  $c_{s,t}$  depending only on  $s$  and  $t$  such that, for each  $f \in \mathcal{H}(D)$  and each*

$r \in (0, 1)$ , then

$$M_t(r, f) \leq c_{s,t} M_s \left( \frac{1+r}{2}, f \right) (1-r)^{1/t-1/s}.$$

*Proof.* See Lemma 3.4 in [10].  $\square$

**THEOREM 2.5.** If  $f \in \mathfrak{D}_{p-1}^p$  and  $\alpha > -1$ , then

$$\int_0^1 M_{(2+\alpha)p}^{(2+\alpha)p}(r, f')(1-r)^{(2+\alpha)p+\alpha} r dr < \infty.$$

*Proof.* Since

$$M_{(2+\alpha)p}(r, f') \leq c M_p \left( \frac{1+r}{2}, f' \right) (1-r)^{\frac{1}{(2+\alpha)p}-\frac{1}{p}}$$

by Lemma 2.4,

$$M_{(2+\alpha)p}^{(2+\alpha)p}(r, f') \leq c M_p^{(2+\alpha)p} \left( \frac{1+r}{2}, f' \right) (1-r)^{-1-\alpha}.$$

This implies that

$$\begin{aligned} & \int_0^1 M_{(2+\alpha)p}^{(2+\alpha)p}(r, f')(1-r)^{(2+\alpha)p+\alpha} r dr \\ & \leq c \int_0^1 M_p^{(2+\alpha)p} \left( \frac{1+r}{2}, f' \right) (1-r)^{(2+\alpha)p-1} r dr \\ & \leq c \int_{1/2}^1 M_p^{(2+\alpha)p}(s, f')(2(1-s))^{(2+\alpha)p-1} (2s-1) 2 ds \\ & \leq c 2^{(2+\alpha)p+1} \int_{1/2}^1 M_p^{(2+\alpha)p}(s, f')(1-s)^{(2+\alpha)p-1} s ds \\ & \leq c 2^{(2+\alpha)p+1} \int_0^1 M_p^{(2+\alpha)p}(r, f')(1-r)^{(2+\alpha)p-1} r dr. \end{aligned}$$

By Lemma 2.3,

$$\begin{aligned} & \int_0^1 (M_p(r, f')(1-r))^p \frac{r}{1-r} dr \\ & = \int_0^1 M_p^p(r, f')(1-r)^{p-1} r dr \\ & < \infty. \end{aligned}$$

This implies that  $M_p(r, f')(1 - r) \rightarrow 0$  as  $r \rightarrow 1^-$ . By Lemma 2.3,

$$\begin{aligned} & \int_0^1 M_{(2+\alpha)p}^{(2+\alpha)p}(r, f')(1 - r)^{(2+\alpha)p+\alpha} r dr \\ & \leq c 2^{(2+\alpha)p+1} \int_0^1 M_p^{(2+\alpha)p}(r, f')(1 - r)^{(2+\alpha)p-1} r dr \\ & \leq c 2^{(2+\alpha)p+1} \int_0^1 M_p^p(r, f')(1 - r)^{p-1} M_p^{(1+\alpha)p}(r, f')(1 - r)^{(1+\alpha)p} r dr \\ & \leq c \int_0^1 M_p^p(r, f')(1 - r)^{p-1} r dr \\ & < \infty. \end{aligned}$$

□

### 3. Space of Dirichlet type and weighted Besov space

Here and throughout the paper

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$$

where  $p, q > 1$  and  $g(z) = (1 - |z|)^{\alpha-p+2}$  where  $-1 < \alpha < p - 1$ .

**THEOREM 3.1.** *If  $-1 < \alpha < p - 1$ , then*

$$\mathfrak{D}_\alpha^p = \mathcal{B}_p(g).$$

*Proof.* If  $f \in \mathfrak{D}_\alpha^p$ , then  $\int_D (1 - |z|^2)^\alpha |f'(z)|^p dA(z) < \infty$ .

$$\begin{aligned} & \int_D (1 - |z|^2)^p g(z) |f'(z)|^p d\lambda(z) \\ & = \int_D (1 - |z|^2)^p (1 - |z|)^{\alpha-p+2} |f'(z)|^p d\lambda(z) \\ & = \int_D (1 - |z|)^\alpha (1 + |z|)^{p-2} |f'(z)|^p dA(z) \\ & < 2^{p-2} \int_D |f'(z)|^p (1 - |z|)^\alpha dA(z) \\ & < \infty. \end{aligned}$$

This implies that  $\mathfrak{D}_\alpha^p \subset \mathcal{B}_p(g)$ .

If  $p - 2 \geq 0$ , then

$$\begin{aligned} & \int_D (1 - |z|)^\alpha |f'(z)|^p dA(z) \\ & \leq \int_D (1 - |z|)^\alpha (1 + |z|)^{p-2} |f'(z)|^p dA(z). \end{aligned}$$

If  $-1 < p - 2 < 0$ , then

$$\begin{aligned} & \int_D (1 - |z|)^\alpha |f'(z)|^p dA(z) \\ & = 2^{2-p} 2^{p-2} \int_D (1 - |z|)^\alpha |f'(z)|^p dA(z) \\ & \leq 2^{2-p} \int_D (1 - |z|)^\alpha (1 + |z|)^{p-2} |f'(z)|^p dA(z). \end{aligned}$$

If  $f \in \mathcal{B}_p(g)$  for  $p > 1$ , then

$$\begin{aligned} & \int_D (1 - |z|)^\alpha |f'(z)|^p dA(z) \\ & < c \int_D (1 - |z|^2)^\alpha (1 + |z|)^{p-2} |f'(z)|^p dA(z) \\ & = c \int_D (1 - |z|^2)^p |f'(z)|^p (1 - |z|)^{\alpha-p+2} d\lambda(z) \\ & < \infty. \end{aligned}$$

This implies that  $\mathcal{B}_p(g) \subset \mathfrak{D}_\alpha^p$ . □

**THEOREM 3.2.** Suppose that  $1 < p < q < \infty$  and  $\alpha + 1 < p$ . Let  $\mu$  be a positive Borel measure in  $D$ . If there exists some positive constant  $c$  such that  $\|f\|_{L^q(d\mu)} \leq c \|f\|_{\mathfrak{D}_\alpha^p}$  for all  $f \in \mathfrak{D}_\alpha^p$ , then  $\|f\|_{L^q(d\mu)} \leq c \|f\|_{\mathcal{B}_p(g)}$ .

*Proof.* If  $f \in \mathfrak{D}_\alpha^p$ , then  $f \in \mathcal{B}_p(g)$  by Theorem 3.1. Since  $\|f\|_{L^q(d\mu)} \leq c \|f\|_{\mathfrak{D}_\alpha^p}$ ,

$$\begin{aligned}
& \int |f(z)|^q d\mu(z) \\
& \leq c \left( |f(0)|^p + \int_D |f'(z)|^p dA_\alpha(z) \right)^{q/p} \\
& = c \left( |f(0)|^p + (\alpha+1) \int_D (1-|z|^2)^p (1-|z|^2)^{\alpha-p+2} |f'(z)|^p \frac{dA(z)}{(1-|z|^2)^2} \right)^{q/p} \\
& \leq c \left( |f(0)|^p + \int_D (1-|z|^2)^p g(z) |f'(z)|^p d\lambda(z) \right)^{q/p}.
\end{aligned}$$

This implies that  $\|f\|_{L^q(d\mu)} \leq c \|f\|_{\mathcal{B}_p(g)}$ .

□

LEMMA 3.3. Suppose that  $z \in D$  and  $c$  is real,  $t > -1$ , and

$$I_{c,t}(z) = \int_D \frac{(1-|w|^2)^t}{|1-z\bar{w}|^{2+t+c}} dA(w);$$

then we have

- (1) if  $c < 0$ , then  $I_{c,t}(z)$  is bounded in  $z$ ;
- (2) If  $c > 0$ , then  $I_{c,t}(z) \sim \frac{1}{(1-|z|^2)^c}$  ( $|z| \rightarrow 1^{-1}$ );
- (3) If  $c = 0$ , then  $I_{c,t}(z) \sim \log \frac{1}{1-|z|^2}$  ( $|z| \rightarrow 1^{-1}$ )

*Proof.* See Lemma 4.2.2 in [13]. □

THEOREM 3.4. Let  $0 < p, q < \infty$  and  $-1 < \alpha < p$ . Let  $\mu$  be a positive Borel measure in  $D$ . If there exists some positive constant  $c$  such that

$$\|f\|_{L^q(d\mu)} \leq c \|f\|_{\mathfrak{D}_\alpha^p}$$

for all  $f \in \mathcal{H}(D)$ , then

$$\sup_{a \in D} \int_D (k_a(z)(1-|a|^2)^{(p-\alpha-1)})^{q/p} d\mu(z) < \infty.$$

*Proof.* We have

$$\begin{aligned}
& \int_D k_a(z)^{q/p} (1-|a|^2)^{(p-\alpha-1)q/p} d\mu(z) \\
& = \int_D \left( \frac{(1-|a|^2)^{p-\alpha}}{|1-\bar{a}z|^2} \right)^{q/p} d\mu(z) \\
& = ((1-|a|^2)^{p-\alpha})^{q/p} \int_D \left( \frac{1}{|1-\bar{a}z|^2} \right)^{q/p} d\mu(z).
\end{aligned}$$

Since  $\|f\|_{L^q(d\mu)} \leq c \|f\|_{\mathfrak{D}_\alpha^p}$  for  $f(z) = (1 - \bar{a}z)^{-2/p}$ ,

$$\begin{aligned} & \int_D k_a(z)^{q/p} (1 - |a|^2)^{(p-\alpha-1)q/p} d\mu(z) \\ & \leq c((1 - |a|^2)^{p-\alpha})^{q/p} \left(1 + \left(\frac{2|a|}{p}\right)^p \int_D \frac{(1 - |z|^2)^\alpha}{|1 - \bar{a}z|^{p+2}} dA(z)\right)^{q/p} \\ & \leq c((1 - |a|^2)^{p-\alpha})^{q/p} \left(1 + \frac{|a|^p}{(1 - |a|^2)^{p-\alpha}}\right)^{q/p} \\ & \leq c((1 - |a|^2)^{p-\alpha} + |a|^p)^{q/p} \\ & < \infty \end{aligned}$$

where the second inequality follows from Lemma 3.3.  $\square$

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