# FUNCTIONAL INEQUALITIES IN PARANORMED SPACES 

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#### Abstract

In this paper, we investigate additive functional inequalities in paranormed spaces. Furthermore, we prove the HyersUlam stability of additive functional inequalities in paranormed spaces.


## 1. Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [3] and Steinhaus [26] independently and since then several generalizations and applications of this notion have been investigated by various authors (see $[5,14,16,17,25]$ ). This notion was defined in normed spaces by Kolk [15].

We recall some basic facts concerning Fréchet spaces.
Definition 1.1. [28] Let $X$ be a vector space. A paranorm $P: X \rightarrow$ $[0, \infty)$ is a function on $X$ such that
(1) $P(0)=0$;
(2) $P(-x)=P(x)$;
(3) $P(x+y) \leq P(x)+P(y)$ (triangle inequality)
(4) If $\left\{t_{n}\right\}$ is a sequence of scalars with $t_{n} \rightarrow t$ and $\left\{x_{n}\right\} \subset X$ with $P\left(x_{n}-x\right) \rightarrow 0$, then $P\left(t_{n} x_{n}-t x\right) \rightarrow 0$ (continuity of multiplication).

The pair $(X, P)$ is called a paranormed space if $P$ is a paranorm on $X$.

The paranorm is called total if, in addition, we have
(5) $P(x)=0$ implies $x=0$.

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A Fréchet space is a total and complete paranormed space.
The stability problem of functional equations originated from a question of Ulam [27] concerning the stability of group homomorphisms. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [21] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

In 1990, Rassias [22] during the $27^{\text {th }}$ International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [6] following the same approach as in Rassias [21], gave an affirmative solution to this question for $p>1$. It was shown by Gajda [6], as well as by Rassias and Šemrl [23] that one cannot prove a Rassias' type theorem when $p=1$ (cf. the books of Czerwik [2], Hyers, Isac and Rassias [11]).

In 1982, J.M. Rassias [20] followed the innovative approach of the Th.M. Rassias' theorem [21] in which he replaced the factor $\|x\|^{p}+\|y\|^{p}$ by $\|x\|^{p} \cdot\|y\|^{q}$ for $p, q \in \mathbb{R}$ with $p+q \neq 1$. Găvruta [7] provided a further generalization of Th.M. Rassias' Theorem. During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings (see $[12,13,18]$ ).

Throughout this paper, assume that $(X, P)$ is a Fréchet space and that $(Y,\|\cdot\|)$ is a Banach space.

In [8], Gilányi showed that if $f$ satisfies the functional inequality

$$
\begin{equation*}
\left\|2 f(x)+2 f(y)-f\left(x y^{-1}\right)\right\| \leq\|f(x y)\| \tag{1.1}
\end{equation*}
$$

then $f$ satisfies the Jordan-von Neumann functional equation

$$
2 f(x)+2 f(y)=f(x y)+f\left(x y^{-1}\right)
$$

See also [24]. Fechner [4] and Gilányi [9] proved the Hyers-Ulam stability of the functional inequality (1.1).

Park, Cho and Han [19] proved the Hyers-Ulam stability of the following functional inequalities

$$
\begin{align*}
\|f(x)+f(y)+f(z)\| & \leq\left\|2 f\left(\frac{x+y+z}{2}\right)\right\|  \tag{1.2}\\
\|f(x)+f(y)+f(z)\| & \leq\|f(x+y+z)\|  \tag{1.3}\\
\|f(x)+f(y)+2 f(z)\| & \leq\left\|2 f\left(\frac{x+y}{2}+z\right)\right\| \tag{1.4}
\end{align*}
$$

In Section 2, we prove the Hyers-Ulam stability of the functional inequality (1.2) in paranormed spaces. In Section 3, we prove the HyersUlam stability of the functional inequality (1.3) in paranormed spaces. In Section 4, we prove the Hyers-Ulam stability of the functional inequality (1.4) in paranormed spaces.
2. Stability of a functional inequality associated with a 3variable Jensen additive functional equation

We prove the Hyers-Ulam stability of a functional inequality associated with a Jordan-von Neumann type 3 -variable Jensen additive functional equation.

Note that $P(2 x) \leq 2 P(x)$ for all $x \in Y$.
Proposition 2.1. ([19, Proposition 2.1]) Let $f: X \rightarrow Y$ be a mapping such that

$$
\|f(x)+f(y)+f(z)\| \leq\left\|2 f\left(\frac{x+y+z}{2}\right)\right\|
$$

for all $x, y, z \in X$. Then $f$ is Cauchy additive.
Theorem 2.2. Let $r$ be a positive real number with $r<1$, and let $f: X \rightarrow Y$ be a mapping such that

$$
\begin{align*}
& \|f(x)+f(y)+f(z)\|  \tag{2.1}\\
& \leq\left\|2 f\left(\frac{x+y+z}{2}\right)\right\|+P(x)^{r}+P(y)^{r}+P(z)^{r}
\end{align*}
$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h: X \rightarrow Y$ such that

$$
\begin{equation*}
\|f(x)-h(x)\| \leq \frac{4+2^{r+2}+2^{2 r}}{4-4^{r}} P(x)^{r}=\frac{2+2^{r}}{2-2^{r}} P(x)^{r} \tag{2.2}
\end{equation*}
$$

for all $x \in X$.
Proof. Letting $x=y=z=0$ in (2.1), we get $\|3 f(0)\| \leq\|2 f(0)\|$. So $f(0)=0$.

Letting $y=x$ and $z=-2 x$ in (2.1), we get

$$
\|2 f(x)+f(-2 x)\| \leq\left(2+2^{r}\right) P(x)^{r}
$$

and so

$$
\|2 f(-2 x)+f(4 x)\| \leq\left(2+2^{r}\right) 2^{r} P(x)^{r}
$$

for all $x \in X$. Thus

$$
\|4 f(x)-f(4 x)\| \leq\left(4+2^{r+2}+2^{2 r}\right) P(x)^{r}
$$

and so

$$
\left\|f(x)-\frac{1}{4} f(4 x)\right\| \leq \frac{4+2^{r+2}+2^{2 r}}{4} P(x)^{r}
$$

for all $x \in X$. Hence

$$
\begin{align*}
\left\|\frac{1}{4^{l}} f\left(4^{l} x\right)-\frac{1}{4^{m}} f\left(4^{m} x\right)\right\| & \leq \sum_{j=l}^{m-1}\left\|\frac{1}{4^{j}} f\left(4^{j} x\right)-\frac{1}{4^{j+1}} f\left(4^{j+1} x\right)\right\| \\
& \leq \frac{4+2^{r+2}+2^{2 r}}{4} \sum_{j=l}^{m-1} \frac{4^{r j}}{4^{j}} P(x)^{r} \tag{2.3}
\end{align*}
$$

for all nonnegative integers $m$ and $l$ with $m>l$ and all $x \in X$. It follows from (2.3) that the sequence $\left\{\frac{1}{4^{n}} f\left(4^{n} x\right)\right\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\left\{\frac{1}{4^{n}} f\left(4^{n} x\right)\right\}$ converges. So one can define the mapping $h: X \rightarrow Y$ by

$$
h(x):=\lim _{n \rightarrow \infty} \frac{1}{4^{n}} f\left(4^{n} x\right)
$$

for all $x \in X$. Moreover, letting $l=0$ and passing the limit $m \rightarrow \infty$ in (2.3), we get (2.2).

It follows from (2.1) that

$$
\begin{aligned}
& \|h(x)+h(y)+h(z)\|=\lim _{n \rightarrow \infty} \frac{1}{4^{n}}\left\|f\left(4^{n} x\right)+f\left(4^{n} y\right)+f\left(4^{n} z\right)\right\| \\
& \leq \lim _{n \rightarrow \infty} \frac{1}{4^{n}}\left\|2 f\left(4^{n} \frac{x+y+z}{2}\right)\right\|+\lim _{n \rightarrow \infty} \frac{4^{n r}}{4^{n}}\left(P(x)^{r}+P(y)^{r}+P(z)^{r}\right) \\
& =\left\|2 h\left(\frac{x+y+z}{2}\right)\right\|
\end{aligned}
$$

for all $x, y, z \in X$. So

$$
\|h(x)+h(y)+h(z)\| \leq\left\|2 h\left(\frac{x+y+z}{2}\right)\right\|
$$

for all $x, y, z \in X$. By Proposition 2.1, the mapping $h: X \rightarrow Y$ is Cauchy additive.

Now, let $T: X \rightarrow Y$ be another Cauchy additive mapping satisfying (2.2). Then we have

$$
\begin{aligned}
\|h(x)-T(x)\| & =\frac{1}{4^{n}}\left\|h\left(4^{n} x\right)-T\left(4^{n} x\right)\right\| \\
& \leq \frac{1}{4^{n}}\left(\left\|h\left(4^{n} x\right)-f\left(4^{n} x\right)\right\|+\left\|T\left(4^{n} x\right)-f\left(4^{n} x\right)\right\|\right) \\
& \leq \frac{2\left(4+2^{r+2}+2^{2 r}\right) 4^{n r}}{\left(4-4^{r}\right) 4^{n}} P(x)^{r}
\end{aligned}
$$

which tends to zero as $n \rightarrow \infty$ for all $x \in X$. So we can conclude that $h(x)=T(x)$ for all $x \in X$. This proves the uniqueness of $h$. Thus the mapping $h: X \rightarrow Y$ is a unique Cauchy additive mapping satisfying (2.2).

Theorem 2.3. Let $r$ be a positive real number with $r<\frac{1}{3}$, and let $f: X \rightarrow Y$ be a mapping such that

$$
\begin{align*}
& \|f(x)+f(y)+f(z)\|  \tag{2.4}\\
& \leq\left\|2 f\left(\frac{x+y+z}{2}\right)\right\|+P(x)^{r} \cdot P(y)^{r} \cdot P(z)^{r}
\end{align*}
$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h: X \rightarrow Y$ such that

$$
\begin{equation*}
\|f(x)-h(x)\| \leq \frac{2^{r+1}+2^{4 r}}{4-4^{3 r}} P(x)^{3 r} \tag{2.5}
\end{equation*}
$$

for all $x \in X$.
Proof. Letting $x=y=z=0$ in (2.4), we get $\|3 f(0)\| \leq\|2 f(0)\|$. So $f(0)=0$.

Letting $y=x$ and $z=-2 x$ in (2.4), we get

$$
\|2 f(x)+f(-2 x)\| \leq 2^{r} P(x)^{3 r}
$$

and so

$$
\|2 f(-2 x)+f(4 x)\| \leq 2^{4 r} P(x)^{3 r}
$$

for all $x \in X$. Thus

$$
\|4 f(x)-f(4 x)\| \leq\left(2^{r+1}+2^{4 r}\right) P(x)^{3 r}
$$

and so

$$
\left\|f(x)-\frac{1}{4} f(4 x)\right\| \leq \frac{2^{r+2}+2^{4 r}}{4} P(x)^{3 r}
$$

for all $x \in X$. Hence

$$
\begin{align*}
\left\|\frac{1}{4^{l}} f\left(4^{l} x\right)-\frac{1}{4^{m}} f\left(4^{m} x\right)\right\| & \leq \sum_{j=l}^{m-1}\left\|\frac{1}{4^{j}} f\left(4^{j} x\right)-\frac{1}{4^{j+1}} f\left(4^{j+1} x\right)\right\| \\
& \leq \frac{2^{r+1}+2^{4 r}}{4} \sum_{j=l}^{m-1} \frac{4^{3 r j}}{4^{j}} P(x)^{3 r} \tag{2.6}
\end{align*}
$$

for all nonnegative integers $m$ and $l$ with $m>l$ and all $x \in X$. It follows from (2.6) that the sequence $\left\{\frac{1}{4^{n}} f\left(4^{n} x\right)\right\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\left\{\frac{1}{4^{n}} f\left(4^{n} x\right)\right\}$ converges. So one can define the mapping $h: X \rightarrow Y$ by

$$
h(x):=\lim _{n \rightarrow \infty} \frac{1}{4^{n}} f\left(4^{n} x\right)
$$

for all $x \in X$. Moreover, letting $l=0$ and passing the limit $m \rightarrow \infty$ in (2.6), we get (2.5).

The rest of the proof is similar to the proof of Theorem 2.2.
3. Stability of a functional inequality associated with a 3variable Cauchy additive functional equation

We prove the Hyers-Ulam stability of a functional inequality associated with a Jordan-von Neumann type 3 -variable Cauchy additive functional equation.

Proposition 3.1. ([19, Proposition 2.2]) Let $f: X \rightarrow Y$ be a mapping such that

$$
\|f(x)+f(y)+f(z)\| \leq\|f(x+y+z)\|
$$

for all $x, y, z \in X$. Then $f$ is Cauchy additive.
Theorem 3.2. Let $r$ be a positive real number with $r<1$, and let $f: X \rightarrow Y$ be a mapping such that

$$
\begin{align*}
& \|f(x)+f(y)+f(z)\|  \tag{3.1}\\
& \leq\|f(x+y+z)\|+P(x)^{r}+P(y)^{r}+P(z)^{r}
\end{align*}
$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h: X \rightarrow Y$ such that

$$
\|f(x)-h(x)\| \leq \frac{2+2^{r}}{2-2^{r}} P(x)^{r}
$$

for all $x \in X$.

Proof. Letting $x=y=z=0$ in (3.1), we get $\|3 f(0)\| \leq\|f(0)\|$. So $f(0)=0$.

Letting $y=x$ and $z=-2 x$ in (3.1), we get

$$
\|2 f(x)+f(-2 x)\| \leq\left(2+2^{r}\right) P(x)^{r}
$$

and so

$$
\|2 f(-2 x)+f(4 x)\| \leq\left(2+2^{r}\right) 2^{r} P(x)^{r}
$$

for all $x \in X$. Thus

$$
\|4 f(x)-f(4 x)\| \leq\left(4+2^{r+2}+2^{2 r}\right) P(x)^{r}
$$

and so

$$
\left\|f(x)-\frac{1}{4} f(4 x)\right\| \leq \frac{4+2^{r+2}+2^{2 r}}{4} P(x)^{r}
$$

for all $x \in X$.
The rest of the proof is the same as in the proof of Theorem 2.2.
Theorem 3.3. Let $r$ be a positive real number with $r<\frac{1}{3}$, and let $f: X \rightarrow Y$ be a mapping such that
$\|f(x)+f(y)+f(z)\| \leq\|f(x+y+z)\|+P(x)^{r} \cdot P(y)^{r} \cdot P(z)^{r}$
for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h: X \rightarrow Y$ such that

$$
\|f(x)-h(x)\| \leq \frac{2^{r+1}+2^{4 r}}{4-4^{3 r}} P(x)^{3 r}
$$

for all $x \in X$.
Proof. The proof is similar to the proofs of Theorems 2.2 and 2.3.

## 4. Stability of a functional inequality associated with the Cauchy-Jensen functional equation

We prove the Hyers-Ulam stability of a functional inequality associated with a Jordan-von Neumann type Cauchy-Jensen functional equation.

Proposition 4.1. ([19, Proposition 2.3]) Let $f: X \rightarrow Y$ be a mapping such that

$$
\|f(x)+f(y)+2 f(z)\| \leq\left\|2 f\left(\frac{x+y}{2}+z\right)\right\|
$$

for all $x, y, z \in X$. Then $f$ is Cauchy additive.

Theorem 4.2. Let $r$ be a positive real number with $r<1$, and let $f: X \rightarrow Y$ be a mapping such that

$$
\begin{align*}
& \|f(x)+f(y)+2 f(z)\|  \tag{4.1}\\
& \leq\left\|2 f\left(\frac{x+y}{2}+z\right)\right\|+P(x)^{r}+P(y)^{r}+P(z)^{r}
\end{align*}
$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h: X \rightarrow Y$ such that

$$
\|f(x)-h(x)\| \leq \frac{2+3 \cdot 2^{r}+2^{2 r}}{4-4^{r}} P(x)^{r}=\frac{1+2^{r}}{2-2^{r}} P(x)^{r}
$$

for all $x \in X$.
Proof. Letting $x=y=z=0$ in (4.1), we get $\|4 f(0)\| \leq\|2 f(0)\|$. So $f(0)=0$.

Replacing $x$ by $-2 x$ and letting $y=0$ and $z=x$ in (4.1), we get

$$
\|f(-2 x)+2 f(x)\| \leq\left(1+2^{r}\right) P(x)^{r}
$$

and so

$$
\|2 f(-2 x)+f(4 x)\| \leq\left(1+2^{r}\right) 2^{r} P(x)^{r}
$$

for all $x \in X$. Thus

$$
\|4 f(x)-f(4 x)\| \leq\left(2+3 \cdot 2^{r}+2^{2 r}\right) P(x)^{r}
$$

and so

$$
\left\|f(x)-\frac{1}{4} f(4 x)\right\| \leq \frac{2+3 \cdot 2^{r}+2^{2 r}}{4} P(x)^{r}
$$

for all $x \in X$.
The rest of the proof is the same as in the proof of Theorem 2.2.

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