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# STRONG CONVERGENCE THEOREMS FOR EQUILIBRIUM PROBLEMS, FIXED POINT PROBLEMS OF QUASI-NONEXPANSIVE MAPPINGS AND VARIATIONAL INEQUALITY PROBLEMS[†]

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Abstract. In this paper, a new iterative algorithm involving quasi-nonexpansive mapping in Hilbert space is proposed and proved to be strongly convergent to a point which is simultaneously a fixed point of a quasi-nonexpansive mapping, a solution of an equilibrium problem and the set of solutions of a variational inequality problem. The results of the paper extend previous results, see, for instance, Takahashi and Takahashi (J Math Anal Appl 331:506-515, 2007), P.E.Maing é (Computers and Mathematics with Applications, 59: 74–79,2010) and other results in this field.

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## 1. Introduction

Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $\phi$ be a bifunction of $C \times C$ into $R$, where $R$ is the set of real numbers. The equilibrium problem for $\phi : C \times C \longrightarrow R$ is to find $x \in C$ such that

$$\phi(x, y) \geq 0, \ \forall \ y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(\phi)$. Given a mapping $T : C \to H$, let $\phi(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in EP(\phi)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., $z$ is a solution of the variational inequality. Numerous problems in physics, optimization and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem; see, for instance, [1-13].

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A mapping $T$ of $C$ into $H$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall \ x, y \in C.$$

We denote by $F(T)$ the set of fixed points of $T$. If $C \subset H$ is bounded, closed and convex and $T$ is a nonexpansive mapping of $C$ into itself, then $F(T)$ is nonempty; for instance, see [14].There are some methods for approximation of fixed points of a nonexpansive mapping. In 2000, Moudafi [15] proved the following strong convergence theorem.

**Theorem 1.1** ([15]). *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $T$ be a nonexpansive mapping of $C$ into itself such that $F(T)$ is nonempty. Let $f$ be a contraction of $C$ into itself and let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \frac{1}{1+\varepsilon_n} T(x_n) + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n), \tag{1.2}$$

*for all $n \in N$, where $\{\varepsilon_n\} \subset (0,1)$ satisfies*

$$\lim_{n\to\infty} \varepsilon_n = 0, \ \sum_{n=1}^{\infty} \varepsilon_n = \infty \text{ and } \lim_{n\to\infty} |\frac{1}{\varepsilon_n + 1} - \frac{1}{\varepsilon_n}| = 0.$$

*Then $\{x_n\}$ converges strongly to $z \in F(T)$, where $z = P_{F(T)} f(z)$ and $P_{F(T)}$ is the metric projection of $H$ onto $F(T)$.*

Such a method for approximation of fixed points is called the viscosity approximation method. In 2007, Takahashi and Takahashi [8] proved the following fixed point theorem.

**Theorem 1.2.** *Let $C$ be a nonempty closed convex subset of $H$. Let $\phi$ be a bifunction from $C \times C$ to $R$ satisfying $(A1) - (A4)$ and let $T$ be a nonexpansive mapping of $C$ into $H$ such that $F(T) \cap EP(\phi) \neq \emptyset$. Let $f$ be a contraction of $H$ into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall \ y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T u_n. \end{cases} \tag{1.3}$$

*for all $n \in N$, where $\alpha_n \subset [0,1]$ and $r_n \subset (0,\infty)$ satisfy*

(1) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty;$

(2) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$

*Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(T) \cap EP(\phi)$, where $z = P_{F(T)\bigcap EP(\phi)} f(z)$.*

A mapping $T$ of $C$ into $H$ is called quasi-nonexpansive if

$$\|Tx - v\| \leq \|x - v\|, \ \forall \ (x, v) \in C \times F(T).$$

If $T : C \longrightarrow H$ is nonexpansive and the set $F(T)$ of fixed points of $T$ is nonempty, then $T$ is quasi-nonexpansive.

In 2010, P.E.Maing é [16] proved the following convergence result of fixed point for the quasi-nonexpansive mappings in Hilbert spaces.

**Theorem 1.3.** *Let $C$ be a nonempty closed convex subset of $H$, and let $\{x_n\}$ be a sequence defined as follows,*

$$x_1 \in H \text{ and } x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_\omega x_n, \tag{1.4}$$

*where $\{\alpha_n\}$ is a slow vanishing sequence, i.e.*

$$\lim_{n\longrightarrow\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty,$$

*$\omega \in (0,1)$, $f : C \longrightarrow C$ a contration of modulus $\rho \in [0,1), T_\omega := (1-\omega)I + \omega T$ (I being the identity mapping on $C$), with two main conditions on $T$:*
*(i1) $T : C \longrightarrow C$ is quasi-nonexpansive;*
*(i2) $T$ is demiclosed on $C$, that is $\{y_k\} \subset C, y_k \rightharpoonup y$ weakly, $(I-T)(y_k) \to 0$ strongly $\Rightarrow y \in F(T)$.*

*Then $\{x_n\}$ converges strongly to the unique element $z \in F(T)$,where $z = P_{F(T)\bigcap EP(\phi)} f(z)$, which equivalently solves the following variational inequality problem:*

$$z \in F(T) \text{ and } (\forall v \in F(T)),\ \langle (I-f)z, v-z\rangle \geq 0. \tag{1.5}$$

In this paper, motivated and inspired by the above results, we introduce a new iterative algorithm in Hilbert space $H$. Let $C$ be a nonempty closed convex subset of $H$. Let $\phi$ be a bifunction from $C \times C$ to $R$ satisfying (A1)–(A4) and let $T_\omega : (1-\omega)I + \omega T$ ($I$ being the identity mapping on $C$) be a mapping with $T :\ C \longrightarrow H$ being quasi-nonexpansive and demi-closed on $C, \omega \in (0,1)$, such that $F(T) \cap EP(\phi) \neq \emptyset$. Let $f : H \longrightarrow H$ be a contraction of modulus $\rho \in [0,1)$, and let $\{x_n\}$and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0,\ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_w u_n, \end{cases} \tag{1.6}$$

for all $n \in N$, where $\{\alpha_n\} \subset (0,1)$, and $\{r_n\} \subset (0,\infty)$ satisfy
(1) $\lim\limits_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$; (2) $\liminf\limits_{n\to\infty} r_n > 0,\ \Sigma_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$,
for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of an equilibrium problem in Hilbert space. Furthermore, we also proved that $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(T) \cap EP(\phi)$, where $z = P_{F(T)\cap EP(\phi)} f(z)$, which equivalently solves the following variational inequality problem:

$$z \in F(T) \cap EP(\phi), and (\forall v \in F(T) \cap EP(\phi)), \langle (I-f), v-z\rangle \geq 0.$$

The results of this paper extend some previously published results, see for instance [5,6].

## 2. Preliminaries

Throughout this paper, we assume that $H$ is a real Hilbert space endowed with an inner product and its induced norm denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively. $C$ is a closed convex subset of $H$. When $\{x_n\}$ is a sequence in $H$, $x_n \rightharpoonup x$ implies that $x_n$ converges weakly to $x$, and $x_n \longrightarrow x$ means the strong convergence. In a real Hilbert space $H$, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,$$

for all $x, y \in H$, and $\lambda \in R$. Let $C$ be a nonempty closed convex subset of $H$. Then, for any $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C(x)$, such that

$$\|x - P_C(x)| \leq \|x - y|, \ \forall\, y \in C.$$

Such a $P_C$ is called the metric projection of $H$ onto $C$. We know that $P_C$ is nonexpansive.

For solving the equilibrium problem for a bifunction $\phi : C \times C \longrightarrow R$, let us assume that $\phi$ satisfies the following conditions:
(A1)$\phi(x, x) = 0$ for all $x \in C$;
(A2)$\phi$ is monotone, i.e. $\phi(x, y) + \phi(y, x) \leq 0$ for all $x, y \in C$;
(A3)for each $x, y, z \in C, \lim\limits_{t \to 0} \phi(tz + (1-t)x, y) \leq \phi(x, y)$;
(A4)for each $x \in C$, $y \longrightarrow \phi(x, y)$ is convex and lower semicontinous.

**Lemma 2.1** ([1]). *Let $T$ be a quasi-nonexpansive mapping on $C$ with $F(T) \neq \emptyset$, and set $T_\omega := (1-\omega)I + \omega T$ for $\omega \in (0, 1]$. Then the following statements are reached:*
*(i)* $\langle x - T_\omega x, x - v\rangle \geq \omega\|x - Tx\|^2, \ \forall (x, v) \in C \times F(T)$;
*(ii)* $\|T_\omega x - v\|^2 \leq \|x - q\|^2 - \omega(1-\omega)\|Tx - x\|^2, \ \forall (x, v) \in C \times F(T)$;
*(iii) $T_\omega$ is quasi-nonexpansive mappings;*
*(iv)* $F(T) = F(T_\omega)$.

**Lemma 2.2** ([1]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j\geq 1}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 1$. Also consider the sequence of integers $\{\tau(n)\}_{n\geq n_1}$ defined by*

$$\tau(n) = max\{k \leq n \mid \Gamma_k < \Gamma_{k+1}\}.$$

*Then $\{\tau(n)\}_{n\geq n_1}$ is a nondecreasing sequence verifying $\lim_{n\to\infty} \tau(n) = \infty$, and for all $n \geq n_1$, it holds that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and we have*

$$\Gamma_n \leq \Gamma_{\tau(n)+1}. \tag{2.1}$$

**Lemma 2.3** ([1]). *Let $C$ be a nonempty closed convex subset of $H$ and let $\phi$ be a bifunction of $C \times C \longrightarrow R$ satisfying (A1)–(A4). Let $r > 0$, and $x \in H$, then there exists $z \in C$ such that*

$$\phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C.$$

**Lemma 2.4** ([1]). *Assume that $\phi : C \times C \longrightarrow R$ satisfies $(A1)-(A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r(x) = \{z \in C : \phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C$$

*for all $z \in H$. Then, the following hold:*
*(1)$T_r$ is single-valued;*
*(2)$T_r$ is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

*(3)$F(T_r) = EP(\phi), \ \forall\ r > 0$;*
*(4)$EP(\phi)$ is closed and convex.*

**Lemma 2.5** ([17]). *Let $\{\alpha_n\}$ be a sequence of non-negative real numbers satisfying $\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n$, where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\} \subset (-\infty, +\infty)$ satisfying the condition:*
(1)$\Sigma_{n=1}^{\infty}\gamma_n = \infty$;
(2) $\limsup\limits_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$, *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
*Then* $\lim\limits_{n\to\infty} \alpha_n = 0$.

**Lemma 2.6** ([17]). *If $z$ is solution of* (1.5) *with $T : C \longrightarrow C$ demi-closed and $\{y_n\} \subset C$ is a bounded sequence such that $\|Ty_n - y_n\| \longrightarrow 0$,then*

$$\liminf_{n\to\infty}\langle (I - f)z, y_n - z\rangle \geq 0.$$

## 3. Main results

**Theorem 3.1.** *Let $C$ be a nonempty closed convex subset of $H$. Let $\phi$ be a bifunction from $C \times C$ to $R$ satisfying (A1)–(A4), and let $T_\omega : (1 - \omega)I + \omega T$ ($I$ being the identity mapping on $C$) be a mapping with $T : C \longrightarrow H$ being quasi-nonexpansive and demi-closed on $C$, $\omega \in (0, 1)$ , such that $F(T) \cap EP(\phi) \neq \emptyset$. Let $f : H \longrightarrow H$ be a contraction of modulus $\rho \in [0, 1)$, and let $\{x_n\}$and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall\ y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T_w u_n, \end{cases} \tag{3.1}$$

*for all $n \in N$, where $\{\alpha_n\} \subset [0, 1]$, and $\{r_n\} \subset (0, \infty)$ satisfy*
(1) $\lim\limits_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$; (2) $\liminf\limits_{n\to\infty} r_n > 0, \ \Sigma_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$, *then* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to $z \in F(T) \cap EP(\phi)$, where $z = P_{F(T)\bigcap EP(\phi)}f(z)$, which equivalently solves the following variational inequality problem:*

$$z \in F(T) \cap EP(\phi), \ (\forall v \in F(T) \cap EP(\phi)), \ \langle (I - f), v - z\rangle \geq 0.$$

*Proof.* Let $Q = P_{F(T)\bigcap EP(\phi)}$. Then $Qf$ is a contraction of $H$ into itself. In fact, there exists $\rho \in [0, 1)$, such that $\|f(x) - f(y)\| \leq \rho\|x - y\|$ for all $x, y \in H$. So we have that

$$\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq \rho\|x - y\|, \tag{3.2}$$

for all $x, y \in H$. So $Qf$ is a contraction of $H$ into itself. Since $H$ is complete, there exists a unique element $z \in H$ such that $z = Qf(z)$. Such a $z \in H$ is an element of $C$.

Let $v \in F(T) \cap EP(\phi)$, then from $u_n = T_{r_n}x_n$, we have

$$\|u_n - v\| \leq \|T_{r_n}x_n - T_{r_n}v\| \leq \|x_n - v\|, \ \forall\ x, y \in C, \tag{3.3}$$

$$\|T_w u_n - v\|^2 \leq \|u_n - v\|^2 - w(1-w)\|Tu_n - u\|^2 \leq \|u_n - v\|^2 \leq \|x_n - v\|^2, \tag{3.4}$$

for all $n \in N$. Put $M = max\{\|x_1 - v\|,\ \frac{1}{1-\rho}\|f(v) - v\|\}$. It is obvious that $\|x_1 - v\| \leq M$. Suppose $\|x_n - v\| \leq M$. Then, we have

$$\begin{aligned}
\|x_{n+1} - v\| &= \|\alpha_n f(x_n) + (1-\alpha_n)T_w u_n - v\| \\
&\leq \alpha_n\|f(x_n) - f(v)\| + \alpha_n\|f(v) - v\| + (1-\alpha_n)\|T_w x_n - v\| \\
&\leq [\alpha_n\rho + (1-\alpha_n)]\|x_n - v\| + \alpha_n\|f(v) - v\| \\
&= [1 - \alpha_n(1-\rho)]\|x_n - v\| + \alpha_n(1-\rho)\frac{\|f(v) - v\|}{1-\rho} \\
&\leq [1 - \alpha_n(1-\rho)]M + \alpha_n(1-\rho)M = M.
\end{aligned} \tag{3.5}$$

So we have that $\|x_n - v\| \leq M$ for any $n \in N$ and hence $\{x_n\}$ is bounded. We also obtain that $\{u_n\}, \{T_w u_n\}, \{T_w x_n\}, \{f(x_n)\}$ and $\{f(u_n)\}$ are bounded. Then we have

$$\begin{aligned}
\|x_{n+1} - v\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)T_\omega u_n - v\|^2 \\
&\leq \|\alpha_n f(x_n) + (1-\alpha_n)T_\omega u_n - v\|^2 \\
&\leq \alpha_n\|f(x_n) - v\|^2 + (1-\alpha_n)\|T_\omega u_n - v\|^2.
\end{aligned} \tag{3.6}$$

By Lemma 2.1, $v \in F(T_\omega)$, so

$$\|T_\omega u_n - v\|^2 \leq \|x_n - v\|^2 - \omega(1-\omega)\|Tu_n - u_n\|^2,$$

and (3.6) equivalently

$$\begin{aligned}
&\|x_{n+1} - v\|^2 \\
&\leq \alpha_n\|f(x_n) - v\|^2 + (1-\alpha_n)(\|x_n - v\|^2 - \omega(1-\omega)\|Tu_n - u_n\|^2) \\
&\leq \alpha_n\|f(x_n) - v\|^2 + (1-\alpha_n)(\|x_n - v\|^2 - \omega(1-\omega)\|Tu_n - u_n\|^2) \\
&\leq \alpha_n\|f(x_n) - v\|^2 + (1-\alpha_n)\|x_n - v\|^2 - (1-\alpha_n)\omega(1-\omega)\|Tu_n - u_n\|^2 \\
&\leq (1-\alpha_n)\|x_n - v\|^2 + \alpha_n\|f(x_n - v)\|^2 - (1-\alpha_n)\omega(1-\omega)\|Tu_n - u_n\|^2,
\end{aligned} \tag{3.7}$$

(3.7) can be equivalently rewritten as

$$\begin{aligned}
&\|x_{n+1} - v\|^2 - \|x_n - v\|^2 + (1-\alpha_n)\omega(1-\omega)\|Tu_n - u_n\|^2 \\
&\leq -\alpha_n\|x_n - v\|^2 + \alpha_n\|f(x_n) - v\|^2.
\end{aligned} \tag{3.8}$$

Setting $\Gamma_n = \|x_n - v\|^2$, we have

$$\begin{aligned}
&\Gamma_{n+1} - \Gamma_n + (1-\alpha_n)w(1-w)\|Tu_n - u_n\|^2 \\
&\le \alpha_n(\|f(x_n) - x_n\|^2 + 2\langle (f-I)x_n, x_n - v\rangle) \\
&\le \alpha_n\|f(x_n) - x_n\|^2 + 2\alpha_n\langle (f-I)x_n, x_n - v\rangle.
\end{aligned} \tag{3.9}$$

The rest of the proof will be divided into two parts:

**Case 1.** Suppose that there exists $n_1$ such that $\Gamma_n := \|x_n - v\|^2$, $n \ge n_1$ is nonincreasing, i.e. $\|x_n - v\|^2 \ge \|x_{n+1} - v\|^2$. In this situation, $\{\Gamma_n\}$ is then convergent because it is also nonnegative(hence it is bounded from below), so that $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$; together with (3.9), and $\alpha_n \longrightarrow 0$, and the boundness of $\{x_n\}$, we obtain

$$\lim_{n\to\infty} \|Tu_n - u_n\|^2 = 0.$$

Next, we show that $\|x_n - u_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Let $v \in F(T) \cap EP(\phi)$, we have

$$\begin{aligned}
\|u_n - v\|^2 &= \|T_{r_n}x_n - T_{r_n}v\|^2 \le \langle T_{r_n}x_n - T_{r_n}v, x_n - v\rangle \\
&= \langle u_n - v, x_n - v\rangle = \frac{1}{2}(\|u_n - v\|^2 + \|x_n - v\|^2 - \|x_n - u_n\|^2),
\end{aligned}$$

and hence

$$\|u_n - v\|^2 \le \|x_n - v\|^2 - \|x_n - u_n\|^2.$$

Therefore, from the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\|x_{n+1} - v\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)T_\omega u_n - v\|^2 \\
&\le \alpha_n\|f(x_n) - v\|^2 + (1-\alpha_n)\|u_n - v\|^2 \\
&\le (1-\alpha_n)\|x_n - v\|^2 + \alpha_n\|f(x_n) - v\|^2 - (1-\alpha_n)\|x_n - u_n\|^2
\end{aligned}$$

and hence,

$$\begin{aligned}
(1-\alpha_n)\|x_n - u_n\|^2 &\le (1-\alpha_n)\|x_n - v\|^2 + \alpha_n\|f(x_n) - v\|^2 - \|x_{n+1} - v\|^2 \\
&\le \alpha_n\|f(x_n) - v\|^2 - \|x_{n+1} - v\|^2 + \|x_n - v\|^2 + \alpha_n\|x_n - v\|^2,
\end{aligned} \tag{3.10}$$

because that

$$\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0,$$

we obtain that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0.$$

Next, we show that

$$\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle \le 0,$$

where $z = P_{F(T)\cap EP(\phi)}f(z)$. To show this inequality, we choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\lim_{i\to\infty}\langle f(z) - z, x_{n_i} - z\rangle = \limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle.$$

Since $\{u_{n_i}\}$ is bounded ,there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$, which converges weakly to $\varepsilon$ without loss of generality, we can assume that $\{u_{n_i}\} \rightharpoonup \varepsilon$. Since $\|Tu_n - u_n\| \to 0$, $T$ is demi-closed, we known that any weak cluster-point of $\{u_n\}$ belongs to $F(T)$. So, we get $\varepsilon \in F(T)$. Let us show $\varepsilon \in EP(\phi)$. By $u_n = T_{r_n}$, we have

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall\ y \in C$$

From $(A2)$, we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \phi(y, u_n)$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq \phi(y, u_{n_i})$$

since$\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \longrightarrow 0$ and $u_{n_i} \rightharpoonup \varepsilon$, from $(A4)$ we have $\phi(y, \varepsilon) \leq 0, \ \forall\ y \in C$.

For $t$ with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)\varepsilon$. Since $y \in C$ and $\varepsilon \in C$, we have $y_t \in C$, and hence $\phi(y_t, \varepsilon) \leq 0$, so from $(A_4)$ we have

$$0 = \phi(y_t, y_t) \leq t\phi(y_t, y) + (1-t)\phi(y_t, \varepsilon) \leq t\phi(y_t, y),$$

and hence $0 \leq \phi(y_t, y)$. From $(A_3)$, we have $0 \leq \phi(\varepsilon, y)$ for all $y \in C$, and hence $\varepsilon \in EP(\phi)$. Therefore $\varepsilon \in F(T) \cap EP(\phi)$. Since $z = P_{F(T)\cap EP(\phi)} f(z)$, we have

$$\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle = \lim_{i\to\infty}\langle f(z) - z, x_{n_i} - z\rangle = \langle f(z) - z, \varepsilon - z\rangle \leq 0. \quad (3.11)$$

So we have

$$\begin{aligned}
&\|x_{n+1} - z\|^2 = \|\alpha_n f(x_n) + (1-\alpha_n)T_\omega u_n - z\|^2\\
&\leq \alpha_n^2\|f(x_n) - z\|^2 + 2\alpha_n(1-\alpha_n)\langle f(x_n) - z, T_\omega u_n - z\rangle + (1-\alpha_n)^2\|T_\omega u_n - z\|^2\\
&\leq \alpha_n^2\|f(x_n) - z\|^2 + (1-\alpha_n)^2(\|x_n - z\|^2 - \omega(1-\omega)\|Tu_n - u_n\|)\\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - f(z), T_\omega u_n - z\rangle + 2\alpha_n(1-\alpha_n)\langle f(z) - z, T_\omega u_n - z\rangle \qquad (3.12)\\
&\leq (1 - 2\alpha_n + \alpha_n^2)\|x_n - z\|^2 + \alpha_n^2\|f(x_n) - z\|^2 - (1-\alpha_n)^2\omega(1-\omega)\|Tu_n - u_n\|\\
&\quad + 2\alpha_n(1-\alpha_n)\rho\|x_n - z\|^2 + 2\alpha_n(1-\alpha_n)\langle f(z) - z, T_\omega u_n - z\rangle\\
&= (1-\gamma_n)\|x_n - z\|^2 + \delta_n
\end{aligned}$$

where

$$\begin{aligned}
&\gamma_n = \alpha_n[2 - \alpha_n - 2\rho(1-\alpha_n)],\\
&\delta_n = \alpha_n^2\|f(x_n) - z\|^2 + 2\alpha_n(1-\alpha_n)\langle f(z) - z, T_\omega u_n - z\rangle
\end{aligned}$$

because of $\sum_{n=1}^{\infty}\gamma_n = \infty$, and $\limsup_{n\to\infty}\frac{\gamma_n}{\delta_n} \leq 0$, by Lemma 2.5, we have

$$\lim_{n\to\infty}\|x_n - z\|^2 = 0.$$

**Case 2.** Suppose there exists subsequence $\{\Gamma_{n_k}\}_{k\geq 0}$ of $\{\Gamma_n\}_{n\geq 0}$, such that $\{\Gamma_{n_k}\} \leq \{\Gamma_{n_{k+1}}\}$, $\forall k \geq 0$. In this situation, we consider the sequence of indices $\{\tau(n)\}$ as defined in *Lemma* 2.2. It follows that $\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} > 0$, which from (3.9) amounts to

$$(1 - \alpha_{\tau(n)})\omega(1-\omega)\|Tu_{\tau(n)} - u_{\tau(n)}\|^2$$

$< \alpha_{\tau(n)}\|f(x_{\tau(n)}) - x_{\tau(n)}\|^2 + 2\alpha_{\tau(n)}\langle (f-I)x_{\tau(n)}, x_{\tau(n)} - v\rangle,$
hence, by the boundedness of $\{x_n\}$ and $\alpha_n \longrightarrow 0$, we immediately obtain

$$\lim_{n\to\infty} \|Tu_{\tau(n)} - u_{\tau(n)}\| = 0.$$

As $\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} > 0$, which from (3.10), amounts to

$$\lim_{n\to\infty} \|x_{\tau(n)} - u_{\tau(n)}\| = 0,$$

which from (3.11), amounts to

$$\limsup_{n\to\infty} \langle f(z) - z, x_{\tau(n)} - z\rangle \leq 0,$$

which from (3.12), amounts to

$$\lim_{n\to\infty} \|x_{\tau(n)} - z\|^2 = 0,$$

Then, recalling that $\Gamma_n \leq \Gamma_{\tau(n)+1}$, by Lemma 2.2, we conclude that $\lim_{n\to\infty} \|x_n - z\|^2 = 0$. Following the proof of case 1 and case 2 we obtain that:
$\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(T) \cap EP(\phi)$, where $z = P_{F(T)\bigcap EP(\phi)} f(z)$. Which equivalently solves the following variational inequality problem:

$$z \in F(T) \cap EP(\phi), \ \textit{and} \ (\forall v \in F(T) \cap EP(\phi)), \ \langle (I-f), v - z\rangle \geq 0.$$

□

As direct consequences of Theorem 3.1, we obtain two corollaries.

**Corollary 3.2.** *Let $C$ be a nonempty closed convex subset of $H$ and let $T_\omega : (1-\omega)I + \omega T$ be a mapping with $T : C \longrightarrow H$ being quasi-nonexpansive and demi-closed on $C$, $\omega \in (0,1)$ , such that $F(T) \neq \emptyset$. Let $f : H \longrightarrow H$ be a contraction of modulus $\rho \in [0,1)$, and let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T_w P_C x_n,$$

*for all $n \in N$, where $\{\alpha_n\} \subset [0,1]$, satisfy $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$; then $\{x_n\}$ converges strongly to $z \in F(T)$, where $z = P_{F(T)} f(z)$.*

*Proof.* Put $\phi(x,y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in N$ in Theorem 3.1. Then, we have $u_n = P_C x_n$. So, from Theorem 3.1, the sequence $\{x_n\}$ generated by $x_1 \in H$ and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T_w P_C x_n,$$

for all $n \in N$, converges strongly to $z \in F(T)$, where $z = P_{F(T)} f(z)$. □

**Corollary 3.3.** *Let $C$ be a nonempty closed convex subset of $H$. Let $\phi$ be a bifunction from $C \times C$ to $R$ satisfying (A1)–(A4) such that $EP(\phi) \neq \emptyset$. Let $f : H \longrightarrow H$ be a contraction of modulus $\rho \in [0,1)$, and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall \ y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) u_n, \end{cases}$$

*for all* $n \in N$*, where* $\{\alpha_n\} \subset [0,1]$*, and* $\{r_n\} \subset (0,\infty)$ *satisfy*

(1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$; (2) $\liminf_{n\to\infty} r_n > 0,\ \Sigma_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$, *then* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z \in EP(\phi)$*, where* $z = P_{EP(\phi)}f(z)$.

*Proof.* Put $T_\omega x = x$ for all $x \in C$ in Theorem 3.1. Then, from Theorem 3.1, the sequence $\{x_n\}$ and $\{u_n\}$ generated in Corollaty 3.3 converge strongly to $z \in EP(\phi)$, where $z = P_{EP(\phi)}f(z)$. □

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