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STRONG CONVERGENCE OF A MODIFIED ISHIKAWA ITERATIVE ALGORITHM FOR LIPSCHITZ PSEUDOCONTRACTIVE MAPPINGS

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Abstract. Let H be a real Hilbert space and let $T : H \to H$ be a Lipschitz pseudocontractive mapping. We introduce a modified Ishikawa iterative algorithm and prove that if $F(T) = \{x \in H : Tx = x\} \neq \emptyset$, then our proposed iterative algorithm converges strongly to a fixed point of T. No compactness assumption is imposed on T and no further requirement is imposed on $F(T)$.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and induced norm $||.||$. Let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be *L-Lipschitzian* if there exists $L \geq 0$ such that

$$||Tx - Ty|| \leq L||x - y||, \ \forall x, y \in C. \tag{1.1}$$

T is said to be a *contraction* if $L \in [0, 1)$ and T is said to be *nonexpansive* if $L = 1$. T is said to be *pseudocontractive* in the terminology of Browder and Petryshyn [1] if

$$||Tx - Ty||^2 \leq ||x - y||^2 + ||x - Tx - (y - Ty)||^2, \ \forall x, y \in C. \tag{1.2}$$

It is easy to verify that (1.1) is equivalent to the monotonicity condition

$$\langle (I - T)x - (I - T)y, x - y \rangle \geq 0, \ \forall x, y \in C, \tag{1.3}$$

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where I is the identity operator. Inequalities (1.2) and (1.3) are also equivalent to the inequality

$$\langle Tx - Ty, x - y\rangle \leq ||x - y||^2,\ \forall x, y \in C. \tag{1.4}$$

An important proper subclass of the class of pseudocontractive mappings is the class of *k-strictly pseudocontractive mappings.* T is said to be *k-strictly pseudocontractive* (see for example [1]) if there exists $k \in [0, 1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||x - Tx - (y - Ty)||^2,\ \forall x, y \in C \tag{1.5}$$

It is well known that if T is k-strictly pseudocontractive, then T is Lipschitz with Lipschitz constant $L = \frac{1+\sqrt{k}}{1-\sqrt{k}}$.

If $F(T) = \{x \in C : Tx = x\} \neq \emptyset$ and inequality (1.2) (or (1.3) or (1.4)) is satisfied for all $x \in C$ and for all $y \in F(T)$, then T is said to be *hemicontractive* (see for example [2]). T is said to be *demicontractive* if $F(T) \neq \emptyset$ and inequality (1.5) is satisfied for some $k \in [0, 1)$ and for all $x \in C$, and $y \in F(T)$. T is said to be demiclosed at p if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in C which converges weakly to $x^* \in C$ and $\{Tx_n\}_{n=1}^{\infty}$ converges strongly to p, then $Tx^* = p$.

The Mann iteration scheme $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n,\ n \geq 1, \tag{1.6}$$

where the *control sequence* $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $[0, 1]$ satisfying some appropriate conditions has been used by several authors for the approximation of fixed points of nonexpansive maps, strictly pseudocontractive maps and demicontractive maps (see for example [1-8]). It is now well known (see for example [9]) that Mann iteration scheme may not in general be applicable for the iterative construction of fixed points of a Lipschitz pseudocontractive map in Hilbert spaces. For Lipschitz pseudocontractive maps, the *Ishikawa iteration sequence* $\{x_n\}_{n=1}^{\infty}$ generated from arbitrary $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T[(1 - \beta_n)x_n + \beta_n Tx_n],\ n \geq 1, \tag{1.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are control sequences in $[0, 1]$ is usually applicable.

In real Hilbert spaces, one of the most general well known convergence theorem using the Mann iteration algorithm for the class of k-strictly pseudocontractive mappings is the following.

Theorem 1.1 ([10]). *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a $k-$strictly pseudocontractive map with a nonempty fixed point set $F(T)$ and let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in $(0, 1-k)$ satisfying the conditions (i) $\lim_{n\to\infty} \alpha_n = 0$ (ii) $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n - k) = \infty$. Then the Mann iteration algorithm $\{x_n\}_{n=1}^{\infty}$ converges weakly to a fixed point of T.*

If $k = 0$ in Theorem 1.1, we obtain weak convergence theorem for nonexpansive maps.

To obtain strong convergence of Mann to a fixed of a k-strictly pseudocontractive maps or even a nonexpansive map in the setting of Theorem 1.1, additional conditions are usually required on T or the subset C (see for example [1-8]). In [11], Genel and Lindenstraus provided an example of a nonexpansive mapping defined on a bounded closed convex subset of a Hilbert space for which the Mann iteration does not converge to a fixed point of T.

Recently Yao, Zhou and Liou [12] (see also [13]) studied a modified Mann iteration algorithm and proved strong convergence of the modified algorithm to a fixed point of a nonexpansive mapping in real Hilbert spaces. They proved the following.

Theorem 1.2 *Let H be a real Hilbert space. Let $T : H \to H$ be a non-expansive mapping with $F(T) \neq \emptyset$. Let $\{t_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ be two real sequences in $[0,1]$. Assume the following conditions are satisfied:*
(c1) $\lim\limits_{n\to\infty} t_n = 0$;
(c2) $\sum_{n=1}^{\infty} t_n = \infty$;
(c3) $\alpha_n \in [a,b] \subset (0,1)$.
Then the modified Mann iteration sequence $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in H$ by

$$x_{n+1} = (1-\alpha_n)(1-t_n)x_n + \alpha_n T[(1-t_n)x_n], \ n \geq 1 \tag{1.8}$$

converges strongly to a fixed point of T.

Observe that $\{x_n\}$ can be expressed in the form

$$\begin{cases} \nu_n = (1-t_n)x_n \\ x_{n+1} = (1-\alpha_n)\nu_n + \alpha_n T\nu_n. \end{cases} \tag{1.9}$$

Clearly, the modified Mann iteration algorithm reduces to the normal Mann iteration algorithm when $t_n \equiv 0$.

More recently, Paul-Emile Maingé and Ştefan Măruşter [14] employed the modified Mann algorithm (1.8) and proved that it converges strongly to a fixed point of a demicontractive map in real Hilbert spaces. They proved the following:

Theorem 1.3 *Let H be a real Hilbert space and let $T : H \to H$ be a demicontractive map with constant $k \in [0,1)$ and let $(I-T)$ be demiclosed at 0. Let $\{t_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ be two real numbers in $[0,1]$. Assume the following conditions are satisfied:*
(c1) $\lim\limits_{n\to\infty} t_n = 0$;
(c2) $\sum_{n=1}^{\infty} t_n = \infty$;
(c3) $\{\alpha_n\} \subset (0,b]$, *with* $0 < b < 1-k$;
(c4) $\lim\limits_{n\to\infty} \frac{t_n}{\alpha_n} = 0$.
Then the modified Mann iteration sequence $\{x_n\}_{n=1}^{\infty}$ generated from an $x_1 \in H$ by (1.8) converges strongly to $P_{F(T)}(0)$ the least norm element of $F(T)$.

For L-Lipschitzian pseudocontractive maps for which the Ishikawa algorithm rather than the Mann algorithm has been applicable, Ishikawa [15] first proved the following:

Theorem 1.4 *Let C be a nonempty* compact *subset of a real Hilbert space H and $T : C \to C$ an L-Lipschitzian pseudocontractive mapping. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be real sequences satisfying the conditions:*
(i) $0 \leq \alpha_n \leq \beta_n < 1$; (ii) $\lim_{n\to\infty} \beta_n = 0$; (iii) $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$. *Then the Ishikawa iteration sequence $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in C$ by (1.7) converges strongly to a fixed point of T.*

Since the appearance of Theorem 1.4, many authors have extended it in various forms (see for example [16-23]). However, strong convergence have not been achieved without compactness assumption on T or C; or other requirements on the set of fixed point $F(T)$; or complete modification of the scheme to a hybrid algorithm (see for examples [16-23]). Recently, Zegeye, Shahzad and Alghamdi [17] assumed that the interior of $F(T)$ is nonempty ($\text{int}F(T) \neq \emptyset$) to achieve strong convergence when T is a self mapping of a nonempty closed convex subset of a real Hilbert space. This appears very restrictive since even in $\Re$ with the usual norm, Lipschitz pseudocontractive maps with finite number of fixed points do not enjoy this condition that $\text{int}F(T) \neq \emptyset$.

It is our purpose in this paper to complement Yao, Xu and Liou [12]; and Maingé and Măruşter [14] by introducing a modified Ishikawa iteration algorithm analogous to the modified Mann iteration algorithm studied in [12] and [14]. We further prove that our modified Ishikawa algorithm converges strongly to a fixed point of a Lipschitz pseodocontive map in real Hilbert spaces.

2. Preliminaries

In what follows, we shall need the following results.

Lemma 2.1 ([Kolmogorov Criterium]). *Let C be a closed convex subset of a real Hilbert space H and let x be a point in H. Let $P_C(x)$ denote the projection of x onto C. Then $z = P_C(x)$ if and only if $\langle x - z, z - y\rangle \geq 0$, $\forall y \in C$.*

Lemma 2.1 ([24]). *Let H be a real Hilbert space, C a closed convex subset of H and $T : C \to C$ a continuous pseudocontractive mapping, then*
(i) $F(T)$ is a closed convex subset of C.
(ii) $(I - T)$ is demiclosed at zero.

Lemma 2.2 ([14]). *Let $\{a_n\}$ be a sequence of nonnegative numbers such that $a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n r_n$, where $\{r_n\}$ is a bounded sequence of real numbers and $\{\lambda_n\} \subset [0, 1]$ satisfies $\sum_{n=1}^{\infty} \lambda_n = \infty$. Then $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} r_n$.*

3. Main results

We now introduce the following modified form of the Ishikawa algorithm:

Modified Ishikawa Algorithm. Let H be a real Hilbert space and let $T : H \to H$ be a giving mapping. For arbitrary $x_1 \in H$ our modified Ishikawa iteration sequence $\{x_n\}$ is given by

$$x_{n+1} = (1-\alpha_n)(1-t_n)x_n + \alpha_n T\Big[(1-\beta_n)(1-t_n)x_n + \beta_n T[(1-t_n)x_n]\Big],\ n \geq 1 \tag{3.1}$$

where $\{t_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$ satisfying some appropriate conditions that will be made precise in our strong convergence theorem. Observe that (3.1) can be written in the form

$$\begin{cases} \nu_n = (1-t_n)x_n \\ y_n = (1-\beta_n)\nu_n + \beta_n T\nu_n, \\ x_{n+1} = (1-\alpha_n)\nu_n + \alpha_n T y_n. \end{cases} \tag{3.2}$$

Observe that as in the case of the modified Mann iteration algorithm of [12] and [14], our modified Ishikawa iteration scheme reduces to the normal Ishikawa iteration when $t_n \equiv 0$.

We now prove the following strong convergence theorems which applies for L-Lipschitzian pseudocontractive maps in real Hilbert spaces.

Theorem 3.1 *Let H be a real Hilbert space and let $T : H \to H$ be an L-Lipschitzian hemicontractive map such that $(I-T)$ is demiclosed at 0. Let $\{t_n\}_{n=1}^\infty$, $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}$ be sequences in $(0,1)$ satisfying the conditions*
(c1) $\lim_{n\to\infty} t_n = 0$;
(c2) $\sum_{n=1}^\infty t_n = \infty$;
(c3) $\alpha_n \leq \beta_n,\ n \geq 1$; *and* $0 < \epsilon \leq \beta_n \leq b < 1$ *for some* $\epsilon > 0$ *and for some* $b \in \left(0, \frac{1}{\sqrt{1+L^2}+1}\right)$;
(c4) $\lim_{n\to\infty} \frac{t_n}{\alpha_n} = 0$.
Then the modified Ishikawa iteration sequence $\{x_n\}_{n=1}^\infty$ generated from an $x_1 \in H$ by (3.1) converges strongly to $P_{F(T)}(0)$ the least norm element of $F(T)$.

Proof. Using the form (3.2) we set $G_n\nu_n := T[(1-\beta_n)\nu_n + \beta_n T\nu_n],\ n \geq 1$. Then using the Lipschitz property of T and the well known identity

$$||(1-t)x + ty||^2 = (1-t)||x||^2 + t||y||^2 - t(1-t)||x-y||^2 \tag{3.3}$$

which holds for all x, y in H and for all t in $[0,1]$ we obtain for arbitrary $p \in F(T)$ that

$$\begin{aligned} ||G_n\nu_n - p||^2 &= ||T[(1-\beta_n)\nu_n + \beta_n T\nu_n] - p||^2 \\ &\leq ||(1-\beta_n)\nu_n + \beta_n T\nu_n - p||^2 + ||(1-\beta_n)\nu_n + \beta_n T\nu_n - G_n\nu_n||^2 \\ &= ||(1-\beta_n)(\nu_n - p) + \beta_n(T\nu_n - p)||^2 \\ &\quad + ||(1-\beta_n)(\nu_n - G_n\nu_n) + \beta_n(T\nu_n - T[(1-\beta_n)\nu_n + \beta_n T\nu_n])||^2 \\ &= (1-\beta_n)||\nu_n - p||^2 + \beta_n||T\nu_n - p||^2 - \beta_n(1-\beta_n)||\nu_n - T\nu_n||^2 \\ &\quad + (1-\beta_n)||\nu_n - G_n\nu_n||^2 \end{aligned}$$

$$
\begin{aligned}
& +\beta_n||T\nu_n - T[(1-\beta_n)\nu_n + \beta_n T\nu_n]||^2 \\
& -\beta_n(1-\beta_n)||\nu_n - T\nu_n||^2 \\
\leq \; & ||\nu_n - p||^2 + \beta_n||\nu_n - T\nu_n||^2 - \beta_n(1-\beta_n)||\nu_n - T\nu_n||^2 \\
& +(1-\beta_n)||\nu_n - G_n\nu_n||^2 + L^2\beta_n^3||\nu_n - T\nu_n||^2 \\
& -\beta_n(1-\beta_n)||\nu_n - T\nu_n||^2 \\
= \; & ||\nu_n - p||^2 + (1-\beta_n)||\nu_n - G_n\nu_n||^2 \\
& -\beta_n[1 - 2\beta_n - \beta_n^2 L^2]||\nu_n - T\nu_n||^2. \qquad (3.4)
\end{aligned}
$$

Using (3.4) we obtain for arbitrary $p \in F(T)$ that

$$
\begin{aligned}
||x_{n+1} - p||^2 &= ||(1-\alpha_n)(\nu_n - p) + \alpha_n(G_n\nu_n - p)||^2 \\
&= (1-\alpha_n)||\nu_n - p||^2 + \alpha_n||G_n\nu_n - p||^2 \\
&\quad -\alpha_n(1-\alpha_n)||\nu_n - G_n\nu_n||^2 \\
&\leq (1-\alpha_n)||\nu_n - p||^2 + \alpha_n\Big[||\nu_n - p||^2 \\
&\quad +(1-\beta_n)||\nu_n - G_n\nu_n||^2 - \beta_n(1 - 2\beta_n - \beta_n^2L^2)||\nu_n - T\nu_n||^2\Big] \\
&\quad -\alpha_n(1-\alpha_n)||\nu_n - G_n\nu_n||^2 \\
&= ||\nu_n - p||^2 - \alpha_n(\beta_n - \alpha_n)||\nu_n - G_n\nu_n||^2 \\
&\quad -\alpha_n\beta_n[1 - 2\beta_n - \beta_n^2L^2]||\nu_n - T\nu_n||^2. \qquad (3.6)
\end{aligned}
$$

Since $\alpha_n \leq \beta_n$, $\forall n \geq 1$, we obtain

$$||x_{n+1} - p||^2 \leq ||\nu_n - p||^2 - \alpha_n\beta_n[1 - 2\beta_n - \beta_n^2L^2]||\nu_n - T\nu_n||^2. \qquad (3.7)$$

From (3.2) we obtain $\frac{1}{\alpha_n}(\nu_n - x_{n+1}) = \nu_n - G_n\nu_n$. Hence

$$
\begin{aligned}
\frac{1}{\alpha_n}||\nu_n - x_{n+1}|| &= ||\nu_n - G_n\nu_n|| \\
&\leq ||\nu_n - T\nu_n|| + ||T\nu_n - G_n\nu_n|| \\
&\leq ||\nu_n - T\nu_n|| + L\beta_n||\nu_n - T\nu_n|| \\
&= [1 + L\beta_n]||\nu_n - T\nu_n||.
\end{aligned}
$$

Thus

$$\frac{1}{\alpha_n(1 + L\beta_n)}||\nu_n - x_{n+1}|| \leq ||\nu_n - T\nu_n||. \qquad (3.8)$$

Using (3.8) in (3.7) we obtain

$$||x_{n+1} - p||^2 \leq ||\nu_n - p||^2 - \lambda_n||\nu_n - x_{n+1}||^2, \qquad (3.9)$$

where

$$\lambda_n := \frac{\beta_n[1 - 2\beta_n - L^2\beta_n^2]}{\alpha_n(1 + L\beta_n)^2} \geq \frac{\epsilon[1 - 2b - L^2b^2]}{b(1 + Lb)^2} > 0.$$

Furthermore, it follows from (3.2) and (3.9) that

$$||x_{n+1} - p|| \leq ||\nu_n - p|| \leq (1 - t_n)||x_n - p|| + t_n||p||,$$

and this yields

$$||x_{n+1} - p|| \leq \max\{||x_1 - p||, ||p||\}.$$

Hence $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence.

It also follows from (3.2) that

$$\begin{aligned} ||\nu_n - p||^2 &= ||(1 - t_n)(x_n - p) - t_n p||^2 \\ &= (1 - t_n)^2 ||x_n - p||^2 + t_n^2 ||p||^2 - 2t_n(1 - t_n)\langle x_n - p, p\rangle \\ &\leq (1 - t_n)||x_n - p||^2 + t_n^2 ||p||^2 - 2t_n(1 - t_n)\langle x_n - p, p\rangle. \end{aligned} \tag{3.10}$$

Using (3.10) in (3.9) we obtain

$$\begin{aligned} ||x_{n+1} - p||^2 &\leq (1 - t_n)||x_n - p||^2 + t_n^2||p||^2 - 2t_n(1 - t_n)\langle x_n - p, p\rangle \\ &\quad -\lambda_n ||\nu_n - x_{n+1}||^2 \\ &= (1 - t_n)||x_n - p||^2 - t_n\Big[-t_n||p||^2 + 2(1 - t_n)\langle x_n - p, p\rangle \\ &\quad +\frac{\lambda_n}{t_n}||\nu_n - x_{n+1}||^2\Big] \\ &= (1 - t_n)||x_n - p||^2 - t_n Z_n, \end{aligned} \tag{3.11}$$

where

$$Z_n := -t_n||p||^2 + 2(1 - t_n)\langle x_n - p, p\rangle + \frac{\lambda_n}{t_n}||\nu_n - x_{n+1}||^2. \tag{3.12}$$

Thus,

$$||x_{n+1} - p||^2 \leq (1 - t_n)||x_n - p||^2 - t_n Z_n. \tag{3.13}$$

Observe that

$$\begin{aligned} -Z_n &= t_n||p||^2 - 2(1 - t_n)\langle x_n - p, p\rangle - \frac{\lambda_n}{t_n}||\nu_n - x_{n+1}||^2 \\ &\leq ||p||^2 + 2||x_n - p||||p|| \leq D_1, \text{ for some } D_1, \text{ since } \{x_n\} \text{ is bounded.} \end{aligned}$$

It thus follows that $\{Z_n\}$ is bounded below since $\{x_n\}$ is bounded. From (3.13), Lemma 2.3 and condition (c2) of our theorem we obtain

$$\limsup_{n\to\infty} ||x_n - p||^2 \leq \limsup_{n\to\infty}(-Z_n) = -\liminf_{n\to\infty} Z_n. \tag{3.14}$$

Thus $\liminf_{n\to\infty} Z_n$ is a finite real number. Since $\lim_{n\to\infty} t_n = 0$, it follows from (3.12) that

$$\liminf_{n\to\infty} Z_n = \liminf_{n\to\infty}\Big[2\langle x_n - p, p\rangle + \frac{\lambda_n}{t_n}||\nu_n - x_{n+1}||^2\Big].$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}$ such that

$$\liminf_{n\to\infty} Z_n = \lim_{k\to\infty}\Big[2\langle x_{n_k} - p, p\rangle + \frac{\lambda_{n_k}}{t_{n_k}}||\nu_{n_k} - x_{n_k+1}||^2\Big]. \tag{3.15}$$

Since $\{x_n\}$ is bounded and $\liminf_{n\to\infty} Z_n$ is finite, it follows that $\frac{\lambda_{n_k}}{t_{n_k}}||\nu_{n_k} - x_{n_k+1}||^2$ is bounded. Observe that since

$$\frac{\lambda_n}{t_n} = \frac{\beta_n[1-2\beta_n - L^2\beta_n^2]}{\alpha_n t_n(1+L\beta_n)^2} \geq \frac{\epsilon[1-2b-L^2b^2]}{\alpha_n t_n(1+Lb)^2},$$

then

$$\frac{1}{\alpha_n t_n} \leq \frac{(1+Lb)^2\lambda_n}{\epsilon[1-2b-L^2L^2]t_n}.$$

Thus

$$\frac{1}{\alpha_{n_k}t_{n_k}}||\nu_{n_k} - x_{n_k+1}||^2 \leq \frac{(1+Lb)^2}{\epsilon[1-2b-L^2b^2]}\frac{\lambda_{n_k}}{t_{n_k}}||v_{n_k} - x_{n_k+1}||^2.$$

Hence $\frac{1}{\alpha_{n_k}t_{n_k}}||\nu_{n_k} - x_{n_k+1}||^2$ is bounded. Observe that from (3.2) we have $\frac{1}{\alpha_n}(\nu_n - x_{n+1}) = \nu_n - G_n\nu_n$. Hence

$$\begin{aligned}
\frac{1}{\alpha_n}||\nu_n - x_{n+1}|| &= ||\nu_n - G_n\nu_n|| \\
&\geq \Big|||\nu_n - T\nu_n|| - ||T\nu_n - G_n\nu_n||\Big| \\
&\geq ||\nu_n - T\nu_n|| - L\beta_n||\nu_n - T\nu_n|| \\
&= [1-L\beta_n]||\nu_n - T\nu_n||.
\end{aligned}$$

Thus

$$||\nu_n - T\nu_n|| \leq \frac{1}{\alpha_n(1-L\beta_n)}||\nu_n - x_{n+1}|| \leq \frac{1}{\alpha_n(1-Lb)}||\nu_n - x_{n+1}||. \quad (3.16)$$

It now follows that

$$||\nu_n - T\nu_n||^2 \leq \frac{t_n}{\alpha_n}\Big(\frac{1}{(1-Lb)^2}\frac{||\nu_n - x_{n+1}||^2}{\alpha_n t_n}\Big). \quad (3.17)$$

Furthermore

$$\begin{aligned}
||x_n - x_{n+1}|| &\leq ||x_n - \nu_n|| + ||\nu_n - x_{n+1}|| \\
&\leq t_n||x_n|| + \alpha_n||\nu_n - Ty_n|| \\
&\leq t_n||x_n|| + ||\nu_n - T\nu_n|| + L||\nu_n - T\nu_n|| \\
&= t_n||x_n|| + (1+L)||\nu_n - T\nu_n||. \quad (3.18)
\end{aligned}$$

Since $\lim_{n\to\infty} t_n = \lim_{n\to\infty}\frac{t_n}{\alpha_n} = 0$, and since $\frac{||\nu_{n_k} - x_{n_k+1}||^2}{\alpha_{n_k}t_{n_k}}$ is bounded, it now follows from (3.17) and (3.18) that

$$\lim_{k\to\infty}||T\nu_{n_k} - \nu_{n_k}||^2 = 0, \text{ and } \lim_{k\to\infty}||x_{n_k} - x_{n_k+1}||^2 = 0.$$

Since $(I-T)$ is demiclosed at 0, it follows that any weak cluster point of $\{\nu_{n_k}\}_{k=1}^\infty$ is in $F(T)$. Furthermore, $\{\nu_{n_k}\}_{k=1}^\infty$ and $\{x_{n_k}\}_{k=1}^\infty$ have the same set of weak cluster points. Since $F(T)$ is closed and convex, then setting $p = P_{F(T)}(0)$ in Lemma 2.1 and using (3.15) we obtain

$$\liminf_{n\to\infty} Z_n \geq 2\liminf_{k\to\infty}\langle x_{n_k} - p, p\rangle = 2\liminf_{k\to\infty}\langle x_{n_k} - P_{F(T)}(0), P_{F(T)}(0)\rangle \geq 0. \tag{3.19}$$

From (3.19) and (3.14) we now obtain

$$\limsup_{n\to\infty} ||x_n - p||^2 = \limsup_{n\to\infty} ||x_n - P_{F(T)}(0)||^2 = 0. \qquad \square$$

Corollary 3.1 *Let H be a real Hilbert space and let $T : H \to H$ be an L-Lipschitzian pseudocontractive map with a nonempty fixed point set $F(T)$. Let $\{t_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}$ be sequences in $(0,1)$ satisfying the conditions*
(c1) $\lim\limits_{n\to\infty} t_n = 0$;
(c2) $\sum_{n=1}^{\infty} t_n = \infty$;
(c3) $\alpha_n \leq \beta_n,\ n \geq 1$; *and* $0 < \epsilon \leq \beta_n \leq b < 1$ *for some* $\epsilon > 0$ *and for some* $b \in \left(0, \frac{1}{\sqrt{1+L^2}+1}\right)$;
(c4) $\lim\limits_{n\to\infty} \frac{t_n}{\alpha_n} = 0$.
Then the modified Ishikawa iteration sequence $\{x_n\}_{n=1}^{\infty}$ generated from an $x_1 \in H$ by (3.1) converges strongly to $P_{F(T)}(0)$ the least norm element of $F(T)$.

Corollary 3.2 *Let H be a real Hilbert space and let $T : H \to H$ be a k-strictly pseudocontractive map with a nonempty fixed point set $F(T)$. Let $\{t_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}$ be sequences in $(0,1)$ satisfying the conditions*
(c1) $\lim\limits_{n\to\infty} t_n = 0$;
(c2) $\sum_{n=1}^{\infty} t_n = \infty$;
(c3) $\alpha_n \leq \beta_n,\ n \geq 1$; *and* $0 < \epsilon \leq \beta_n \leq b < 1$ *for some* $\epsilon > 0$ *and for some* $b \in \left(0, \frac{1}{\sqrt{1+L^2}+1}\right)$, *where* $L := \frac{1+\sqrt{k}}{1-\sqrt{k}}$;
(c4) $\lim\limits_{n\to\infty} \frac{t_n}{\alpha_n} = 0$.
Then the modified Ishikawa iteration sequence $\{x_n\}_{n=1}^{\infty}$ generated from an $x_1 \in H$ by (3.1) converges strongly to $P_{F(T)}(0)$ the least norm element of $F(T)$.

Corollary 3.3 *Let H be a real Hilbert space and let $T : H \to H$ be an L-Lipschitzian k-demicontractive map such that $(I - T)$ is demiclosed at 0. Let $\{t_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}$ be sequences in $(0,1)$ satisfying the conditions*
(c1) $\lim\limits_{n\to\infty} t_n = 0$;
(c2) $\sum_{n=1}^{\infty} t_n = \infty$;
(c3) $\alpha_n \leq \beta_n,\ n \geq 1$; *and* $0 < \epsilon \leq \beta_n \leq b < 1$ *for some* $\epsilon > 0$ *and for some* $b \in \left(0, \frac{1}{\sqrt{1+L^2}+1}\right)$;
(c4) $\lim\limits_{n\to\infty} \frac{t_n}{\alpha_n} = 0$.
Then the modified Ishikawa iteration sequence $\{x_n\}_{n=1}^{\infty}$ generated from an $x_1 \in$

H by (3.1) converges strongly to $P_{F(T)}(0)$ *the least norm element of* $F(T)$.

Remark 3.1 Our Theorem and Corollaries remain true if H is replaced with a nonempty closed convex subset K of H with $0 \in K$. Furthermore, for arbitrary nonempty closed convex subset K of H, our iteration scheme could be appropriately modified with the projection $P_K : H \to K$ to achieve our results.

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