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# MIRROR $d$-ALGEBRAS†

KEUM SOOK SO AND YOUNG HEE KIM*

Abstract. In this paper we investigate necessary conditions for the mirror algebra $(M(X), \oplus, (0,0))$ to be a $d$-algebra (having the condition $(D5)$, resp.) when $(X, *, 0)$ is a $d$-algebra (having the condition $(D5)$, resp.). Moreover, we obtain the necessary conditions for $M(X)$ of a $d^*$-algebra $X$ to be a $d^*$-algebra.

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## 1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras [7, 8]. It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. We refer useful textbooks for $BCK/BCI$-algebra to [6, 9, 15]. J. Neggers et al. [10] introduced the notion of $Q$-algebras which is a generalization of $BCK/BCI/BCH$-algebras, and obtained several properties and discussed quadratic $Q$-algebras. S. S. Ahn and H. S. Kim [1] introduced the notion of $QS$-algebras, and S. S. Ahn et al. [2] studied positive implicativity in $Q$-algebras and discussed some relations between $R-(L-)$ maps and positive implicativity. J. Neggers and H. S. Kim introduced the notion of $d$-algebras which is another useful generalization of $BCK$-algebras, and then investigated several relations between $d$-algebras and $BCK$-algebras as well as several other relations between $d$-algebras and oriented digraphs [13]. After that some further aspects were studied [3, 4, 11, 12]. P. J. Allen et al. [5] introduced the notion of mirror image of a given algebras, and obtained some interesting properties: a mirror algebra of a $d$-algebra is also a $d$-algebra, and a mirror algebra of an implicative $BCK$-algebra is a left $L$-up algebra. Recently, K. S. So [14] investigated how to construct mirror $Q$-algebras of a $Q$-algebra, and she obtained the necessary conditions for $M(X)$ to be a $Q$-algebra.

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*Corresponding author. †This work is supported by Chungbuk National University fund 2012.

In this paper we investigate necessary conditions for the mirror algebra $(M(X), \oplus, (0,0))$ to be a $d$-algebra (having the condition $(D5)$, resp.) when $(X, *, 0)$ is a $d$-algebra (having the condition $(D5)$, resp.). Moreover, we obtain the necessary conditions for $M(X)$ of a $d^*$-algebra $X$ to be a $d^*$-algebra.

## 2. Preliminaries

An (*ordinary*) *d-algebra* [13] is a non-empty set $X$ with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(D1) $x * x = 0$,
(D2) $0 * x = 0$,
(D3) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

A $BCK$-algebra is a $d$-algebra $X$ satisfying the following additional axioms:

(D4) $(x * y) * (x * z)) * (z * y) = 0$,
(D5) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

**Example 2.1** ([3])**.** Consider the real numbers $\mathbf{R}$, and suppose that $(\mathbf{R}; *, \mathbf{e})$ has the multiplication

$$x * y = (x - y)(x - e) + e$$

Then $x*x = e; e*x = e; x*y = y*x = e$ yields $(x-y)(x-e) = 0$, $(y-x)(y-e) = e$ and $x = y$ or $x = e = y$, i.e., $x = y$, i.e., $(\mathbf{R}; *, e)$ is a $d$-algebra.

A $d$-algebra $X$ is said to be a $d^*$-algebra [12] if it satisfies the following axiom: for all $x, y \in X$,

(D6) $(x * y) * x = 0$.

P. J. Allen et al. [5] introduced the notion of mirror algebras of a given algebra as follows:

Let $(X, *, 0)$ be an algebra. Let $M(X) := X \times \{0, 1\}$ and define a binary operation "$*$" on $M(X)$ as follows:

$$(x, 0) * (y, 0) := (x * y, 0),$$
$$(x, 1) * (y, 1) := (y * x, 0),$$
$$(x, 0) * (y, 1) := (x * (x * y), 0),$$
$$(x, 1) * (y, 0) := \begin{cases} \text{(y, 1)} & \text{when } x * y = 0, \\ \text{(x, 1)} & \text{when } x * y \neq 0. \end{cases}$$

Then we say that $M(X) := (M(X), *, (0, 0))$ is a *left mirror algebra* of the algebra $(X, *, 0)$. Similarly, if we define

$$(x, *) * (y, 1) := (y * (y * x), 0)$$

then $M(X) := (M(X), *, (0, 0))$ is a *right mirror algebra* of the algebra $(X, *, 0)$.

It was shown in [5] that the mirror algebra of a $d$(resp., $d - BH$)-algebra is also a $d$(resp., $d - BH$)-algebra, but the mirror algebra of a $BCK$-algebra need not be a $BCK$-algebra.

In [5] Allen et al. defined (left, right) mirror algebras of an algebra, but it is not known how to construct mirror algebras of any given algebra. K. S. So [14] investigated a construction of a mirror algebra in $Q$-algebras.

A *$Q$-algebra* [10] is a non-empty set $X$ with a constant 0 and a binary operation "$*$" satisfying the axioms $(D1), (D2)$ and

(D7) $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$.

Let $(X, *, 0)$ be a $Q$-algebra. Define a binary operation "$\oplus$" on $M(X)$ by

(M1) $(x, 0) \oplus (y, 0) = (x * y, 0)$,
(M2) $(x, 1) \oplus (y, 1) = (y * x, 0)$,
(M3) $(x, 0) \oplus (y, 1) = (\alpha(x, y), 0)$,
(M4) $(x, 1) \oplus (y, 0) = (\beta(x, y), 1)$

where $\alpha, \beta : X \times X \to X$ are mappings. K. S. So obtained the necessary conditions for $(M(X), \oplus, (0, 0))$ to be a $Q$-algebra. K. S. So's definition for mirror algebras is more generalized case of P. J. Allen et al.'s method. In this paper we apply this idea to $d$-algebras, and obtain the necessary conditions for mirror $d$-algebras and mirror $d^*$-algebras.

## 3. Constructions of mirror $d(d^*)$-algebras

Let $(X, *, 0)$ be a $d$-algebra and let $M(X) := X \times \{0, 1\}$. Define a binary operation "$\oplus$" on $M(X)$ by $(M1) \sim (M4)$ as in $Q$-algebras.

**Theorem 3.1.** *Let $(X, *, 0)$ be a $d$-algebra. If $\alpha(0, y) = 0$ for all $y \in X$, then the mirror algebra $(M(X), \oplus, (0, 0))$ is also a $d$-algebra.*

*Proof.* By $(M1)$ and $(M2)$, the axiom $(D1)$ holds trivially. For any $(y, 0) \in M(X)$, we have $(0, 0) \oplus (y, 0) = (0 * y, 0) = (0, 0)$ by $(D2)$. For any $(y, 1) \in M(X)$, $(0, 0) \oplus (y, 1) = (\alpha(0, y), 0)$. If $\alpha(0, y) = 0$ for all $y \in X$, then $(D2)$ holds. Assume $(x, i) \oplus (y, j) = (0, 0) = (y, j) \oplus (x, i)$ where $x, y \in X$ and $i, j \in \{0, 1\}$. We claim that $i = j$. In fact, if $i = 0, j = 1$, then $(0, 0) = (y, 1) \oplus (x, 0) = (\beta(y, x), 1)$ and hence we obtain $\beta(y, x) = 0$ and $0 = 1$, a contradiction. If $i = 1, j = 0$, then $(0, 0) = (x, 1) \oplus (y, 0) = (\beta(x, y), 1)$, a contradiction also. It follows that $(x, i) \oplus (y, i) = (0, 0) = (y, i) \oplus (x, i)$ and hence $(x * y, i) = (0, 0) = (y * x, i)$. Since $(X, *, 0)$ is a $d$-algebra, we obtain $x = y$, proving the theorem. □

**Example 3.2.** Consider a set $X := \{0, 1, 2, \cdots\}$ with a binary operation "$*$" on $X$ defined by

$$x * y := \begin{cases} 0 & x \leq y, \\ 1 & \text{otherwise} \end{cases}$$

Then $(X, *, 0)$ is a $d$-algebra [12]. In order to construct for $M(X)$ to be a $d$-algebra, if we define $\alpha(x, y) = xy^2$ and $\beta(x, y)$ is an arbitrary function on $X \times X \to X$, then $M(X)$ is a $d$-algebra.

In Example 3.2, if we change the functions $\alpha, \beta$, then we can obtain very many $d$-algebras.

A $d$-algebra $(X, *, 0)$ is said to be *bounded* if there exists $m \in X$ such that $x * m = 0$ for all $x \in X$. We call such an element $m$ the *maximal element* of $X$.

**Proposition 3.3.** *Let $(X, *, 0)$ be a d-algebra. If $\alpha(x, 0) = 0$ for all $x \in X$, then the mirror algebra $(M(X), \oplus, (0, 0))$ is bounded.*

*Proof.* Consider $(0, 1)$. Given $x \in X$, we have $(x, 0) \oplus (0, 1) = (\alpha(x, 0), 0)$ and $(x, 1) \oplus (0, 1) = (0 * x, 0) = (0, 0)$. It follows that $(0, 1)$ is the maximal element of $M(X)$ if $\alpha(x, 0) = 0$ for all $x \in X$, proving the proposition. □

The mirror $d$-algebra $M(X)$ in Example 3.2 is bounded, since $\alpha(x, y) = xy^2$ and $\alpha(0, y) = 0$. If we define $\alpha(x, y) = y^3$, then $M(X)$ is a non-bounded mirror $d$-algebra.

Give a $d$-algebra $X$, we consider a mapping $\varphi : M(X) \to M(X)$ defined by $\varphi(x, 0) = (x, 0), \varphi(x, 1) = (x, 0)$ for all $x \in X$. Such a map $\varphi$ is called an *exchange function* on $M(X)$. Note that the exchange function is self-inverse, i.e., $\varphi(\varphi(x, i)) = (x, i)$ for all $(x, i) \in M(X)$.

Let $(X, *, 0)$ be a $d$-algebra. A map $f : X \to X$ is said to be *order-reversing* if $x * y = 0$, $x, y \in X$, then $f(y) * f(x) = 0$.

**Theorem 3.4.** *Let $(M(X), \oplus, (0, 0))$ be a mirror d-algebra of a d-algebra $(X, *, 0)$. Then the exchange function $\varphi : M(X) \to M(X)$ is order-reversing if $\alpha(x, y) = 0$ implies $\alpha(y, x) = 0$ for all $x, y \in X$.*

*Proof.* Given $x, y \in X$, we consider 4 cases. If $(x, 0) \oplus (y, 0) = (0, 0)$, then $(x * y, 0) = (0, 0)$ and hence $x * y = 0$. It follows that $\varphi(y, 0) \oplus \varphi(x, 0) = (y, 1) \oplus (x, 1) = (x * y, 0) = (0, 0)$. If $(x, 1) \oplus (y, 1) = (0, 0)$, then $(y * x, 0) = (0, 0)$ and hence $y * x = 0$. It follows that $\varphi(y, 1) \oplus \varphi(x, 1) = (y, 0) \oplus (x, 0) = (y * x, 0) = (0, 0)$. If $(x, 0) \oplus (y, 1) = (0, 0)$, then $(\alpha(x, y), 0) = (0, 0)$ and hence $\alpha(x, y) = 0$. By assumption, we have $\alpha(y, x) = 0$. It follows that $\varphi(y, 1) \oplus \varphi(x, 0) = (y, 0) \oplus (x, 1) = (\alpha(y, x), 0) = (0, 0)$. The case $(x, 1) \oplus (y, 0) = (0, 0)$ does not happen, since $(x, 1) \oplus (y, 0) = (\beta(x, y), 1) \neq (0, 0)$. This proves the theorem. □

**Remark.** There are no restrictions on the function $\beta$ on $M(X)$ for the exchange function $\varphi$ of $M(X)$ to be order-reversing.

In the above Theorem 3.4, if we define $\alpha(x, y) \equiv (0, 0)$, then the exchange function $\varphi$ is order-reversing. In this case, notice that $(x, 0) \oplus (y, 1) = (0, 0)$ is our version of $X \times \{0, 1\}$ "*lies below*" $X \times \{1\}$. Thus we have an "ordinal sum" defined in this way, with $\beta : X \times X \to X$ arbitrary.

**Theorem 3.5.** *Let $(X, *, 0)$ be a d-algebra with $(D5)$. Then the necessary conditions for the mirror d-algebra $(M(X), \oplus, (0, 0)$ to have the condition $(D5)$ are*

(i) $\alpha(0, y) = 0$,
(ii) $\alpha(x * \alpha(x, y), y) = 0$,
(iii) $(\beta(x, y) * x) * y = 0$,
(iv) $y * \beta(x, y * x) = 0$

*for all $x, y \in X$.*

*Proof.* Given $x, y \in X$, we consider 4 cases. Case 1. $(x, 0)$ and $(y, 0)$: Since $X$ has the condition $(D5)$, we have $[(x, 0) \oplus ((x, 0) \oplus (y, 0))] \oplus (y, 0) = ((x * (x * y)) * y, 0) = (0, 0)$. Case 2. $(x, 0)$ and $(y, 1)$: $[(x, 0) \oplus ((x, 0) \oplus (y, 1))] \oplus (y, 1) = [(x, 0) \oplus (\alpha(x, y), 0)] \oplus (y, 1) = (x * \alpha(x, y), 0) \oplus (y, 1) = (\alpha(x * \alpha(x, y), y), 0)$. Hence the requirement is $\alpha(x * \alpha(x, y), y) = 0$. Case 3. $(x, 1)$ and $(y, 0)$: $[(x, 1) \oplus ((x, 1) \oplus (y, 0))] \oplus (y, 0) = [(x, 1) \oplus (\beta(x, y), 1)] \oplus (y, 0) = ((\beta(x, y) * x) * y, 0)$. Hence the requirement is $(\beta(x, y) * x) * y = 0$. Case 4. $(x, 1)$ and $(y, 1)$: $[(x, 1) \oplus ((x, 1) \oplus (y, 1))] \oplus (y, 1) = [(x, 1) \oplus (y * x, 0)] \oplus (y, 1) = (\beta(x, y * x), 1) \oplus (y, 1) = (y * \beta(x, y * x), 0)$. Hence the requirement is $y * \beta(x, y * x) = 0$. This proves the theorem. □

Note that finding suitable examples of $\alpha(x, y)$ and $\beta(x, y)$ satisfying the above conditions (i) $\sim$ (iv) may enrich the chance of analytic investigation of algebraic structures.

**Theorem 3.6.** *Let $(X, *, 0)$ be a $d^*$-algebra. Then the necessary conditions for the mirror $d$-algebra $(M(X), \oplus, (0, 0)$ to be a $d^*$-algebra are*

(i) $\alpha(0, y) = 0$, (ii) $\alpha(x, y) * x = 0$,
(iii) $x * \beta(x, y) = 0$, (iv) $\alpha(y * x, x) = 0$

*for all $x, y \in X$.*

*Proof.* Given $x, y \in X$, we consider 4 cases. Case 1. $(x, 0)$ and $(y, 0)$: Since $X$ is a $d^*$-algebra, we have $((x, 0) \oplus (y, 0)) \oplus (x, 0) = ((x * y) * x, 0) = (0, 0)$. Case 2. $(x, 0)$ and $(y, 1)$: $((x, 0) \oplus (y, 1)) \oplus (x, 0) = (\alpha(x, y), 0) \oplus (x, 0) = (\alpha(x, y) * x, 0)$. Hence the requirement is $\alpha(x, y) * x = 0$. Case 3. $(x, 1)$ and $(y, 0)$: $((x, 1) \oplus (y, 0)) \oplus (x, 1) = (\beta(x, y), 1) \oplus (x, 1) = (x * \beta(x, y), 0) = (0, 0)$. Hence the requirement is $x * \beta(x, y) = 0$. Case 4. $(x, 1)$ and $(y, 1)$: $[(x, 1) \oplus (y, 1)] \oplus (x, 1) = (y * x, 0) \oplus (x, 1) = (\alpha(y * x, x), 0) = (0, 0)$. It follows that $\alpha(y * x, x) = 0$. This proves the theorem. □

**Example 3.7.** Let $(X, *, 0)$ be a $d^*$-algebra. If we define a binary operation "$\oplus$" on $M(X)$ by

(i) $(x, 0) \oplus (y, 0) = (x * y, 0)$, (ii) $(x, 1) \oplus (y, 1) = (y * x, 0)$,
(iii) $(x, 0) \oplus (y, 1) = (0, 0)$, (iv) $(x, 1) \oplus (y, 0) = (x, 1)$

for all $x, y \in X$. Then it is easy to see that $(M(X), \oplus, (0, 0))$ is a $d^*$-algebra.

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**Young Hee Kim** Department of Mathematics, Chungbuk National University, Chongju 361-763, Korea.
e-mail: yhkim@chungbuk.ac.kr

**Keum Sook So** Department of Mathematics, Hallym University, Chuncheon 200-702, Korea.
e-mail: ksso@hallym.ac.kr