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A STUDY ON SINGULAR INTEGRO-DIFFERENTIAL EQUATION OF ABEL'S TYPE BY ITERATIVE METHODS

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Abstract. In this article, Adomian decomposition method (ADM) ,variation iteration method(VIM) and homotopy analysis method (HAM)for solving integro-differential equation with singular kernel have been investigated. Also,we study the existence and uniqueness of solutions and the convergence of present methods. The accuracy of the proposed method are illustrated with solving some numerical examples.

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1. Introduction

The mathematical formulation of solid state physics,plasma physics, fluid mechanics, chemical kinetics and mathematical biology often involve singular integral and integro-differential equations.

In the three decades , many powerful and simple methods have been proposed and applied successfully to approximate various type of singular integral and integro-differential equations with a wide range of applications [1-10].In this work,we discuss the three different methods such as Adomian decomposition method (ADM) that introduced in 1986 [11],variation iteration method (VIM) and homotopy analysis method (HAM) that proposed by Chinese mathematician Ji-Huan He [12-15] and apply these to solve singular integro-differential equation whit Abel's kernel as follows:

$$\sum_{j=0}^{k} p_j(x)u^{(j)}(x) = f(x) + \int_a^x \frac{G(t, u^{(l)}(t))}{\sqrt{g(x-t)}}\, dt,\ x \in J^{'} = [a, b]. \qquad (1)$$

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With initial conditions

$$u^{(r)}(a) = b_r, \quad r = 0, 1, ..., k-1$$

where a is a real constant and the functions $f(x)$, $G(x, u^{(l)}(x)), l \geq 0$ and $p_j(x), j = 0, 1, ..., k$ that $p_k(x) \neq 0$ are given ,and $u(x)$ is the solution to be determined.
To solve Eq.(1),we consider (1) as follows:

$$u(x) = L^{-1}(\frac{f(x)}{p_k(x)}) + \sum_{r=0}^{k-1} \frac{1}{(r!)}(x-a)^r b_r + L^{-1}(\int_a^x \frac{G(t, u^{(l)}(t))}{p_k(t)\sqrt{g(x-t)}} \, dt) - L^{-1}(\sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)} \, u^{(j)}(x)). \tag{2}$$

Where L^{-1} is the multiple integration operator as follows:

$$L^{-1}(.) = \int_a^x \int_a^x ... \int_a^x \int_a^x (.) \underbrace{dx \; dx \; ... \; dx \; dx}_{k \; times}.$$

We can obtain the term $\sum_{r=0}^{k-1} \frac{1}{(r!)} \, (x-a)^r \, b_r$ from the initial conditions. From [a10], we have

$$L^{-1}(\int_a^x \frac{G(t, u^{(l)}(t))}{p_k(t)\sqrt{g(x-t)}} \, dt) = \int_a^x \frac{(x-t)^k}{(k!)} \frac{G(t, u^{(l)}(t))}{p_k(t)\sqrt{g(x-t)}} \, dt, \tag{3}$$

$$\sum_{j=0}^{k-1} L^{-1}(\frac{p_j(x)}{p_k(x)}) \, u^{(j)}(x)) = \sum_{j=0}^{k-1} \int_a^x \frac{(x-t)^{k-1}}{(k-1)!} \frac{p_j(x)}{p_k(x)} \, u^{(j)}(x) \, dt, \tag{4}$$

By substituting (3)and (4) into (2), we obtain

$$\begin{aligned} u(x) = L^{-1}(\tfrac{f(x)}{p_k(x)}) + \textstyle\sum_{r=0}^{k-1} \tfrac{1}{(r!)}(x-a)^r b_r + \int_a^x \tfrac{(x-t)^k}{(k!)} \tfrac{G(t,u^{(l)}(t))}{p_k(t)\sqrt{g(x-t)}} \, dt \\ - \textstyle\sum_{j=0}^{k-1} \int_a^x \tfrac{(x-t)^{k-1}}{(k-1)!} \tfrac{p_j(x}{p_k(x)} \, u^{(j)}(x) \, dt \end{aligned} \tag{5}$$

For convenient, we set

$$\begin{aligned} L^{-1}(\tfrac{f(x)}{p_k(x)}) + \textstyle\sum_{r=0}^{k-1} \tfrac{1}{(r!)}(x-a)^r b_r = F(x) \\ \tfrac{(x-t)^k}{(k!) \; p_k(t)\sqrt{g(x-t)}} = k_1(x,t) \\ \tfrac{(x-t)^{k-1}}{(k-1)!} \tfrac{p_j(x}{p_k(x)} = k_2(x,t) \end{aligned} \tag{6}$$

So,we have

$$u(x) = F(x) + \int_a^x k_1(x,t) \; G(t, u^{(l)}(t)) \; dt - \sum_{j=0}^{k-1} \int_a^x k_2(x,t) \; u^{(j)}(t) \; dt. \tag{7}$$

The structure of this paper is organized as follows:
In section 2, we apply three iterative method to solving integro-differential equation with singular kernel. The existance and uniqueness of the solution and convergence of the mentioned proposed methods are brought in section 3.We solve several examples in section 4,and a brief conclusion is given in section 5.

2. Methodology

Here we represent briefly the main point of each the methods,for more detail can be refer to[]:

2.1. Adomian decomposition method. consider the functional equation

$$Au = g(x), \tag{8}$$

where A represent a general nonlinear differential operator involving both linear and nonlinear terms,the linear term is decomposed into $L + R$, where L is invertible and R is remainder of linear operator and N is nonlinear term. The operator L can be taken as the highest order derivate .Thus the Eq.(8) can be written as

$$Lu + Ru + Nu = g,$$

because L is invertible , the equivalent expression is

$$u = f(x) - L^{-1}Ru - L^{-1}Nu, \tag{9}$$

where the function $f(x)$ represent the term arising from integrate the function $g(x)$.
Adomian decomposition method [16,17] defines the unknown function $u(x)$ by an infinite series

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{10}$$

where the components $u_n(x)$ are usually determined recurrently. The nonlinear term Nu can be decomposed into an infinite series of polynomials given by

$$Nu = \sum_{n=0}^{\infty} A_n, \tag{11}$$

where $A_n, n \geq 0$ are the Adomian polynomials defined by

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty} \lambda^i u_i)] \mid_{\lambda=0} . \tag{12}$$

Now, substituting (10) , (11) into (9) ,we have

$$\sum_{n=0}^{\infty} u_n(x) = f(x) - L^{-1}R(\sum_{n=0}^{\infty} u_n(x)) - L^{-1}N(\sum_{n=0}^{\infty} u_n(x)), \tag{13}$$

Consequently, we can write recursively by

$$\begin{aligned} &u_0 = f(x), \\ &u_{n+1} = -L^{-1}R(u_n) - L^{-1}(A_n), \quad n \geq 0. \end{aligned} \tag{14}$$

2.1.1. Using ADM. in this part,the Adomian decomposition method is applied to solve singular integro-differential equation of Abel's type,according to the ADM ,we can write the iterative formula (14) as follows:

$$\begin{aligned} &u_0(x) = f(x), \\ &u_{n+1}(x) = \int_a^x k_1(x,t)\ A_n\ dt - \sum_{j=0}^{k-1} \int_a^x k_2(x,t)\ L_{n_j}\ dt, \quad n \geq 0. \end{aligned} \tag{15}$$

the nonlinear terms $G(t, u^l(t))$ and $D^j(u(x))$ $(D^j = \frac{\partial^j}{\partial x^j})$, are usually represented by an infinite series of the so called Adomian polynomials as follows:

$$G(t, u^{(l)}(t)) = \sum_{i=0}^{\infty} A_i, \quad D^j(u(x)) = \sum_{i=0}^{\infty} L_{i_j}.$$

where A_i and $L_{i_j}(i \geq 0,\ j = 0, 1, ..., k-1)$ are the Adomian polynomials were introduced in [22].

2.2. He's variational iteration method. consider the functional equation

$$Lu(x) + Nu(x) = g(x),$$

where L and N are linear and nonlinear operators, respectively and $g(x)$ is a given continuous function. Ji-Huan He has modified the general Lagrange multiplier method [18] into an iteration method, which is called correction functional, in the following way [19,20,21],

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\{Lu_n(\tau) + Nu_n(\tau) - g(\tau)\}d\tau, \quad n \geq 0, \tag{16}$$

It is obvious that the successive approximations $u_j, j \geq o$ can be established by determining λ, a general Lagrange multiplier, which can be identified optimally via the variational theory. the function $\widetilde{u}_n$ is restricted variation where is , and is considered as a restricted variation i.e.$\delta\widetilde{u}_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x), n \geq 0$ of the solution $u(x)$ will be readily obtained upon using the obtained zeroth approximation u_0 may be selected by any function that justifies at least two of the prescribed boundary conditions. Whit λ determined , then several approximations $u_j(x), j \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{17}$$

2.2.1. Using VIM. In this part,the variation iteration method is applied to solve singular integro-differential equation of Abel type,according to the VIM, we can write the iterative formula (16) as follows:

$$\begin{aligned} u_{n+1}(x) = u_n(x) + \textstyle\int_0^x \lambda(\tau)[u_n(\tau) - f(\tau) - \int_a^\tau k_1(\tau,t)G(t,u^{(l)})(t)dt \\ + \textstyle\sum_{j=0}^{k-1} \int_a^\tau k_2(\tau,t)u^{(j)}(t)dt \ ] \ d\tau. \end{aligned} \tag{18}$$

To find the optimal λ, we proceed as follows:

$$\begin{aligned} \delta u_{n+1}(x) = \delta u_n(x) + \delta(\textstyle\int_0^x \lambda(\tau)[u_n(\tau) - f(\tau) - \int_a^\tau k_1(\tau,t)G(t,u^{(l)})(t)dt \\ + \textstyle\sum_{j=0}^{k-1} \int_a^\tau k_2(\tau,t)u^{(j)}(t)dt \ ] \ d\tau) = \delta u_n + \lambda\delta u_n \mid_{\tau=x} - \int_0^x \lambda'(\tau)\delta u_n(\tau)d\tau = 0 \end{aligned} \tag{19}$$

this yields the stationary conditions
$\lambda' = 0$
$1 + \lambda = 0$
this in turn gives $\lambda = 1$
Substituting this value of the Lagrange multiplier into the functional (18) gives the iteration formula

$$\begin{aligned} & u_0 = f(x), \\ & u_{n+1}(x) = u_n(x) - \textstyle\int_0^x [u_n(\tau) - f(\tau) - \int_a^\tau k_1(\tau,t)G(t,u^{(l)})(t)dt \\ & + \textstyle\sum_{j=0}^{k-1} \int_a^\tau k_2(\tau,t)u^{(j)}(t)dt \ ] \ d\tau. \end{aligned} \tag{20}$$

2.3. Homotopy analysis method. Consider

$$N[u] = 0,$$

where N is a nonlinear operator, $u(x)$ is unknown function. Let $u_0(x)$ denote an initial guess of the exact solution $u(x)$, $h \neq 0$ an auxiliary parameter, $H(x) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[r(x)] = 0$ when $r(x) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$(1-q)L[\phi(x;q) - u_0(x)] - qhH(x)N[\phi(x;q)] = \hat{H}[\phi(x;q); u_0(x), H(x), h, q]. \tag{21}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x)$, the auxiliary nonlinear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H(x)$. Enforcing the homotopy (21) to be zero, i.e.,

$$\hat{H}[\phi(x;q); u_0(x), H(x), h, q] = 0, \tag{22}$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x;q) - u_0(x)] = qhH(x)N[\phi(x;q)]. \tag{23}$$

when $q = 0$, the zero-order deformation Eq.(23) becomes

$$\phi(x;0) = u_0(x), \tag{24}$$

and when $q = 1$, since $h \neq 0$ and $H(x) \neq 0$, the zero-order deformation Eq.(23) is equivalent to

$$\phi(x;1) = u(x). \tag{25}$$

Thus, according to (24) and (25), as the embedding parameter q increases from 0 to 1, $\phi(x;q)$ varies continuously from the initial approximation $u_0(x)$ to the exact solution $u(x)$. Such a kind of continuous variation is called deformation in homotopy [24,25].

Due to Taylor's theorem, $\phi(x;q)$ can be expanded in a power series of q as follows

$$\phi(x;q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x)q^m, \tag{26}$$

where,

$$u_m(x) = \frac{1}{m!}\frac{\partial^m \phi(x;q)}{\partial q^m} \mid_{q=0} .$$

Let the initial guess $u_0(x)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H(x)$ be properly chosen so that the power series (26) of $\phi(x;q)$ converges at $q = 1$, then, we have under these assumptions the solution series

$$u(x) = \phi(x;1) = u_0(x) + \sum_{m=1}^{\infty} u_m(x). \tag{27}$$

From Eq.(26), we can write Eq.(21) as follows

$$\begin{array}{l}(1-q)L[\phi(x,q) - u_0(x)] = (1-q)L[\sum_{m=1}^{\infty} u_m(x)\ q^m] = q\ h\ H(x)N[\phi(x,q)] \Rightarrow \\ L[\sum_{m=1}^{\infty} u_m(x)\ q^m] - q\ L[\sum_{m=1}^{\infty} u_m(x)q^m] = q\ h\ H(x)N[\phi(x,q)]\end{array} \tag{28}$$

By differentiating (26) m times with respect to q, we obtain

$$\begin{array}{l}\{L[\sum_{m=1}^{\infty} u_m(x)\ q^m] - q\ L[\sum_{m=1}^{\infty} u_m(x)q^m]\}^{(m)} = \{q\ h\ H(x)N[\phi(x,q)]\}^{(m)} = \\ m!\ L[u_m(x) - u_{m-1}(x)] = h\ H(x)\ m\ \frac{\partial^{m-1}N[\phi(x;q)]}{\partial q^{m-1}} \mid_{q=0} .\end{array}$$

therefore,

$$L[u_m(x) - \chi_m u_{m-1}(x)] = hH(x)\Re_m(y_{m-1}(x)), \tag{29}$$

where,

$$\Re_m(u_{m-1}(x)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x;q)]}{\partial q^{m-1}} \mid_{q=0}, \tag{30}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}$$

Note that the high-order deformation Eq.(29) is governing the linear operator L, and the term $\Re_m(y_{m-1}(x))$ can be expressed simply by (30) for any nonlinear operator N.

2.3.1. Using HAM. in this part,the homotopy analysis method is applied to solve singular integro-differential equation of Abel's type,according to the HAM, we have

$$N[u(x)] = u(x) - f(x) - \int_a^x k_1(x,t)\ G(t, u^{(l)}(t))\ dt + \sum_{j=0}^{k-1} \int_a^x k_2(x,t)\ D^j(u(t))\ dt,$$

so,

$$\begin{aligned} \Re_m(u_{m-1}(x)) = u_{m-1}(x) - \int_a^x k_1(x,t)\ G(t, u_{m-1}^{(l)}(t))\ dt \\ + \sum_{j=0}^{k-1} \int_a^n k_2(x,t)\ D^j(u_{m-1}(t))\ dt. \end{aligned} \tag{31}$$

Substituting (31) into (29)

$$\begin{aligned} L[u_m(x) - \chi_m u_{m-1}(x)] = hH(x)[u_{m-1}(x) - \int_a^x k_1(x,t)\ G(t, u_{m-1}^{(l)}(t))\ dt \\ + \sum_{j=0}^{k-1} \int_a^x k_2(x,t)\ D^j(u_{m-1}(t))\ dt]. \end{aligned} \tag{32}$$

we take an initial guess $u_0(x) = f(x)$, an auxiliary linear operator $Lu = u$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H(x) = 1$. This is substituted into (32) to give the recurrence relation

$$\begin{aligned} u_0(x) &= f(x), \\ u_n(x) &= \int_a^x k_1(x,t)\ G(t, u_{n-1}^{(l)}(t))\ dt \\ &\quad - \sum_{j=0}^{k-1} \int_a^x k_2(x,t)\ D^j(u_{n-1}(t))\ dt \quad n \geq 1. \end{aligned} \tag{33}$$

3. Existence and convergency of iterative methods

In this section we study the existence and uniqness of the solutions and convergence of the methods.Consider the Eq.(7), we assume $F(x)$ is bounded for all x in $J^{'}$ and

$$\begin{aligned} &\mid k_1(x,y) \mid \leq N_1, \\ &\mid k_2(x,y) \mid \leq N_{1_j}, \quad j = 0, 1, ..., k-1,\ \forall x \in J^{'}. \end{aligned}$$

Also, we suppose the nonlinear terms $G(x, u^{(l)}(x))$ and $D^j(u(x))$ are Lipschitz continuous with $\mid G(u^{(l)}(x)) - G(u^{(l)^*}(x)) \mid \leq d \mid u(x) - u^*(x) \mid$, $\mid D^j(u(x)) - D^j(u^*(x)) \mid \leq Z_j \mid u(x) - u^*(x) \mid$, $\quad j = 0, 1, ..., k-1$. If we set

$$\begin{aligned} &\gamma = (b-a)\ (d\ N_1 + k\ Z\ N), \\ &\quad Z = max \mid Z_j \mid,\ N = max \mid N_{1_j} \mid, \quad j = 0, 1, ..., k-1. \end{aligned}$$

In what follow, we will prove theorems by considering the above assumptions.

Theorem 3.1. *Assuming that* $0<\gamma<1$,*then singular integro-differential equation of Abel type in Eq.(1), has a unique solution.*

Proof. Let u and u^* be two different solutions of (6) then

$$\begin{aligned}&| u(x)-u^*(x) |=| \int_a^x k_1(x,t)\ [G(t,u^{(l)}(t))-G(t,u*^{(l)}(t))]\ dt\\ &-\sum_{j=0}^{k-1}\int_a^x k_2(x,t)\ [D^j(u(t))-D^j(u^*(t))]\ dt\ |\\ &\leq \int_a^x |\ k_1(x,t)\ |\ |\ G(t,u^{(l)}(t))-G(t,u*^{(l)}(t))\ |\ \ dt\\ &+\sum_{j=0}^{k-1}\int_a^x |\ k_2(x,t)\ |\ |\ D^j(u(t))-D^j(u^*(t))\ |\ \ dt\\ &\leq (b-a)\ (d\ N_1+k\ Z\ N_2)\ |\ u(x)-u^*(x)\ |=\gamma\ |\ u(x)-u^*(x)\ |\ .\end{aligned}$$

from which we get $(1-\gamma)|u-u^*|\leq 0$. Since $0<\gamma<1$, so $|u-u^*|=0$. therefore, $u=u^*$ and this completes the proof. □

Theorem 3.2. *The series solution* $u(x,y)=\sum_{i=0}^{\infty}u_i(x,y)$ *of problem(1) using ADM convergence when* $0<\gamma<1$ *and* $\parallel u_1(x,y)\parallel<\infty$.

Proof. Denote as $(C[J'],\parallel . \parallel)$ the Banach space of all continuous functions on J' with the norm $\parallel f(x)\parallel = max\mid f(x)\mid$ for all x in J'. Define the sequence of partial sums s_n, let s_n and s_m be arbitrary partial sums with $n\geq m$. We are going to prove that $s_n=\sum_{i=0}^{n}u_i(x)$ is a Cauchy sequence in this Banach space:

$$\begin{aligned}&\parallel s_n-s_m\parallel = max_{\forall x\in J'}\mid s_n-s_m\mid = max_{\forall x\in J'}\mid \sum_{i=m+1}^{n}u_i(x)\mid\\ &= max_{\forall x\in J'}\mid \sum_{i=m+1}^{n}[\int_a^y k_1(x,t)\ A_i\ dt\\ &-\sum_{j=0}^{k-1}\int_a^x k_2(x,t)\ L_{i_j}]\mid=\\ &max_{\forall x\in J'}\mid \int_a^x k_1(x,t)\ (\sum_{i=m}^{n-1}A_i)\ dt+\sum_{j=0}^{k-1}\int_a^x\ k_2(x,t)\ (\sum_{i=m}^{n-1}L_{i_j})\ dt\mid .\end{aligned}$$

From [22], we have

$$\sum_{i=m}^{n-1}A_i=G(s_{n-1})-G(s_{m-1}),$$

$$\sum_{i=m}^{n-1}L_{i_j}=D^j(s_{n-1})-D^j(s_{m-1}).$$

So,

$$\begin{aligned}&\parallel s_n-s_m\parallel\\ &= max_{\forall x\in J'}\mid \int_a^x k_1(x,t)\ [G(s_{n-1})-G(s_{m-1})]\ dt-\sum_{j=0}^{k-1}\int_a^x k_2(x,t)\ D^j(s_{n-1})\\ &-D^j(s_{m-1})\ dt)\mid\\ &\leq max_{\forall x}(\int_a^x\mid k_1(x,t)\mid\mid G(s_{n-1})-G(s_{m-1})\mid\ \ dt\\ &+\int_a^x\mid k_2(x,t)\mid\mid D^j(s_{n-1})-D^j(s_{m-1})\mid\ \ dt)\\ &\leq\gamma\parallel s_{n-1}-s_{m-1}\parallel .\end{aligned}$$

Let $n=m+1$, then

$$\parallel s_n-s_m\parallel\leq\gamma\parallel s_m-s_{m-1}\parallel\leq\gamma^2\parallel s_{m-1}-s_{m-2}\parallel\leq ...\leq\gamma^m\parallel s_1-s_0\parallel .$$

$$\begin{aligned}&\parallel s_n-s_m\parallel\leq\parallel s_{m+1}-s_m\parallel+\parallel s_{m+2}-s_{m+1}\parallel+...+\parallel s_n-s_{n-1}\parallel\\ &\leq[\gamma^m+\gamma^{m1}+...+\gamma^{n-m-1}]\parallel s_1-s_0\parallel\\ &\leq\ \gamma^m[1+\gamma+\gamma^2+...+\gamma^{n-m-1}]\parallel s_1-s_0\parallel\leq[\tfrac{1-\gamma^{n-m}}{1-\gamma}]\parallel u_1(x)\parallel .\end{aligned}$$

Since $0 < \gamma < 1$, we have $(1 - \gamma^{n-m}) < 1$, then

$$\| s_n - s_m \| \leq \frac{\gamma^m}{1-\gamma} \| u_1(x) \| .$$

But $| u_1(x) | < \infty$ (since $f(x)$ is bounded), so, as $m \to \infty$, then $\| s_n - s_m \| \to 0$. We conclude that s_n is a Cauchy sequence in $C[J^{'}]$, therefore the series is convergence and the proof is complete. □

Theorem 3.3. *When solving singular integro-differential equation of Abel's type Eq.(1) with VIM,* $\lim_{n\to\infty} u_n(x)$ *is convergence to exact solution whenever* $0 < \gamma < 1$ *and* $p_k(x, y) = 1$

Proof.

$$\begin{aligned} u_{n+1}(x) = u_n(x) - \textstyle\int_0^x [u_n(\tau) - f(\tau) - \int_a^\tau \frac{(\tau-t)^k}{(k!)} \frac{G(t,u_n^{(l)}(t))}{p_k(t)\sqrt{g(\tau-t)}}\, dt \\ + \textstyle\sum_{j=0}^{k-1} \int_a^\tau \frac{(\tau-t)^{k-1}}{(k-1)!} \frac{p_j(t)}{p_k(t)}\, u_n^{(j)}(t)\, dt\ ]\, d\tau, \end{aligned} \tag{34}$$

$$\begin{aligned} u(x) = u(x) - \textstyle\int_0^x [u(\tau) - f(\tau) - \int_a^\tau \frac{(\tau-t)^k}{(k!)} \frac{G(t,u^{(l)}(t))}{p_k(t)\sqrt{g(\tau-t)}}\, dt \\ + \textstyle\sum_{j=0}^{k-1} \int_a^\tau \frac{(\tau-t)^{k-1}}{(k-1)!} \frac{p_j(t)}{p_k(t)u^{(j)}(t)} dt\ ] d\tau. \end{aligned} \tag{35}$$

By subtracting relation (34) from (35),

$$\begin{aligned} &u_{n+1}(x) - u(x) = \\ &u_n(x) - u(x) - \textstyle\int_0^x [u_n(\tau) - u(\tau) - \int_a^\tau \frac{(\tau-t)^k}{(k!)p_k(t)\sqrt{g(\tau-t)}} [G(t, u_n^{(l)}(t)) - G(t, u^{(l)}(t))]\, dt \\ &- \textstyle\sum_{j=0}^{k-1} \int_a^\tau \frac{g(\tau-t)^{k-1}}{(k-1)!} \frac{p_j(t)}{p_k(t)} [D^j(u_n(t)) - D^j(u(t))]\ dt] d\tau \end{aligned} \tag{36}$$

If we set, $e_{n+1}(x, y) = u_{n+1}(x, y) - u(x, y), \quad e_n(x) = u_n(x) - u(x)$ then

$$\begin{aligned} e_{n+1}(x) = e_n(x) - \int_0^x [u_n(\tau) - u(\tau) \\ - \int_a^\tau \frac{(\tau-t)^k}{(k!)p_k(t)\sqrt{g(\tau-t)}} [G(t, u_n^{(l)}(t)) - G(t, u^{(l)}(t))]\ dt \\ - \sum_{j=0}^{k-1} \int_a^\tau \frac{(\tau-t)^{k-1}}{(k-1)!} \frac{p_j(t)}{p_k(t)} [D^j(u_n(t)) - D^j(u(t))]\ dt] d\tau \\ - (e_n(x) - e_n(x_0)) \\ \leq e_n(x)(1 - (b-a)\ (d\ N_1 + k\ Z\ N_2)) = (1-\gamma)e_n(x). \end{aligned}$$

therefore,

$$\| e_{n+1} \| = max_{\forall x \epsilon J'} \mid e_{n+1} \mid \leq (1-\gamma) max_{\forall x \epsilon J'} \mid e_n \mid = \| e_n \| . \tag{37}$$

since $0 < \gamma < 1$, then $\| e_n \| \to 0$. So, the series converges and the proof is complete. □

Theorem 3.4. *Singular integro-differential equation of Abel's type is convergent to the exact solution when using HAM.*

Proof. We assume:

$$\hat{G}(u(x)) = \sum_{m=0}^{\infty} G(u_m(x)),$$
$$\hat{D^j}(u(x)) = \sum_{m=0}^{\infty} D^j(u_m(x)),$$
$$u(x) = \sum_{m=0}^{\infty} u_m(x),$$

where,

$$\lim_{m\to\infty} u_m(x) = 0.$$

We can write,

$$\sum_{m=1}^{n}[u_m(x) - \chi_m u_{m-1}(x)] = u_1 + (u_2 - u_1) + ... + (u_n - u_{n-1}) = u_n(x). \quad (38)$$

Hence, from (38)

$$\lim_{n\to\infty} u_n(x) = 0. \quad (39)$$

So, using (39) and the definition of the linear operator L, we have

$$\sum_{m=1}^{\infty} L[u_m(x) - \chi_m u_{m-1}(x)] = L[\sum_{m=1}^{\infty}[u_m(x) - \chi_m u_{m-1}(x)]] = 0.$$

Therefore from (29), we can obtain that,

$$\sum_{m=1}^{\infty} L[u_m(x) - \chi_m u_{m-1}(x)] = hH(x)\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x,y)) = 0.$$

Since $h \neq 0$ and $H(x,y) \neq 0$, we have

$$\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x)) = 0. \quad (40)$$

By substituting $\Re_{m-1}(u_{m-1}(x))$ into the relation (40) and simplifying it , we have

$$\begin{aligned}&\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x)) = \\ &\sum_{m=1}^{\infty}[u_{m-1}(x) - \int_a^x k_1(x,t)\ G(t,u_{m-1}^{(l)}(t))\ dt - \sum_{j=0}^{k-1}\int_a^x k_2(x,t)\ D^j(u_{m-1}(t))\ dt \\ &-(1-\chi_m)f(x)] = u(x) - f(x) - \int_a^x k_1(x,t)\ [\sum_{m=1}^{\infty} G(t,u_{m-1}^{(l)}(t))]\ dt \\ &-\sum_{j=0}^{k-1}\int_a^x k_2(x,t)\ [\sum_{m=1}^{\infty} D^j(u_{m-1}(t))]\ dt.\end{aligned} \quad (41)$$

From (40) and (41), we have

$$u(x) = f(x) + \int_a^x k_1(x,t)\ \hat{G}(t,u(t))\ dt - \sum_{j=0}^{k-1}\int_a^x k_2(x,t)\ \hat{D^j}(u(t))\ dt,$$

therefore, $u(x)$ must be the exact solution of Eq.(1). □

TABLE 1. Numerical results of Example 4.1

x	Errors(ADM,n=15)	Errors(VIM,n=10)	Errors(HAM,n=6)
(0.15)	0.082479	0.062402	0.042336
(0.25)	0.083648	0.063263	0.042564
(0.4)	0.085534	0.064713	0.044859
(0.55)	0.086385	0.066538	0.046126
(0.75)	0.087605	0.067349	0.047089

TABLE 2. Numerical results of Example 4.2

x	Errors (ADM,n=24)	Errors (VIM,n=17)	Errors (HAM, n=11)
(0.1)	0.0073202	0.0052437	0.0032336
(0.3)	0.0073644	0.0053289	0.0033428
(0.5)	0.0074358	0.0054437	0.0035347
(0.7)	0.0075188	0.0055601	0.0036785
(0.8)	0.0075653	0.0056379	0.0037024

4. Numerical example

In this section, we compute a numerical example which is solved by the ADM, VIM and HAM. The program has been provided with Mathematica 6.

Example 4.1. Consider the singular integro-differential equation as follow

$$u'(x) + xu(x) = \frac{1}{2\sqrt{x}} + x\sqrt{x} - \frac{\pi}{2} + \int_0^x \frac{u(t)}{\sqrt{x-t}}dt, \tag{42}$$

whit initial condition

$$u(0) = 0.$$

The exact solution is $u(x) = \sqrt{x}$, $\epsilon = 10^{-2}$.

Example 4.2. Consider the following equation given by

$$u^{'''}(x) + e^x u'(x) = e^x - 2\sqrt{x} + \int_0^x \frac{u'(t)}{\sqrt{x-t}}dt$$

whit initial condition

$$u''(0) = u(0) = 0 \ and \ u'(0) = 1.$$

The exact solution is $u(x) = x$, $\epsilon = 10^{-3}$.

Tables 1 and 2 show that the error of the HAM is less than the error of the ADM and VIM.

5. Conclusion

The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which are convergent are rapidly to the exact solutions. In this work, the HAM has been successfully employed to obtain the approximate solution of the singular integro-differential equation . For this purpose in examples, we showed that the HAM is more rapid convergence than the ADM and VIM.

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