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UNIFORM L^p-CONTINUITY OF THE SOLUTION OF STOCHASTIC DIFFERENTIAL EQUATIONS†

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Abstract. This note is concerned with the uniform L^p-continuity of solution for the stochastic differential equations under Lipschitz condition and linear growth condition. Furthermore, uniform L^p-continuity of the solution for the stochastic functional differential equation is given.

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1. Introduction

In the past few decades, stochastic models have received a great deal of research attention since they have been successfully used in variety of application fields, including biology, epidemiology, mechanics, economics and finance (see [1, 3, 4, 5, 11, 14, 15] and references therein for details). Stochastic differential equation(SDEs) is the most fundamental concept in modern stochastic models. Consequently, there is an increasing interest in stochastic differential equations. For instance, in 2006, Henderson et al. [4] published the Stochastic Differential Equations in Science and Engineering, in 2007, Mao [9] published the stochastic differential equations and applications, in 2007, Li and Fu [8] considered the stability analysis of stochastic functional differential equations with infinite delay and its application to recurrent neural networks.

On the one hand, Mao [9] introduced the following d-dimensional stochastic differential equations of Itô type:

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \tag{1}$$

on $t_0 \le t \le T$, and he showed that there exists a unique solution $x(t)$ to equation (1) and the solution belongs to $\mathcal{M}^2([t_0, T]; R^d)$ under the Lipschitz condition (2)

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and linear growth condition hold (3); For any $x, y \in R^d$ and $t \in [t_0, T]$, it follows that

$$|f(x,t) - f(y,t)|^2 \vee |g(x,t) - g(y,t)|^2 \leq \overline{K}|x-y|^2. \tag{2}$$

For any $t \in [t_0, T]$ and $x \in R^d$ it follows that

$$|f(x,t)|^2 \vee |g(x,t)|^2 \leq K(1 + |x|^2). \tag{3}$$

Since then, the theory of existence and uniqueness of the solution for SDEs has been developed by researchers (see [1, 3, 5, 8, 13, 15]). Moreover, the classical and powerful techniques applied in the study of the existence and uniqueness of the solution for SDEs. Furthermore, in the study of the solution for the SDEs, one question arises naturally: Does the uniform L^p-continuity assure the solution for such SDEs? To the best of our knowledge, there are few results on this problem. It is also worth noting that the uniform L^p-continuity of the solution for such SDEs has not been fully investigated, which remains an interesting research topic.

We aim to establish a new result on uniform L^p-continuity of the solution for such SDEs. The coefficients in the system are assumed to satisfy Lipschitz condition and weakened linear growth condition. By some novel technique, some easily verifiable conditions are obtained which ensure the uniform L^p-continuity of the solution for such SDEs.

2. Preliminary and notations

Let $|\cdot|$ denote Euclidean norm in R^n. If A is a vector or a matrix, its transpose is denoted by A^T; if A is a matrix, its trace norm is represented by $|A| = \sqrt{\mathrm{trace}(A^T A)}$. Let t_0 be a positive constant and $(\Omega, \mathcal{F}, P)$, throughout this paper unless otherwise specified, be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq t_0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets). Assume that $B(t)$ is an m-dimensional Brownian motion defined on complete probability space, that is $B(t) = (B_1(t), B_2(t), ..., B_m(t))^T$. We consider the d-dimensional stochastic differential equation of Itô type

$$dx(t) = f(x(t),t)dt + g(x(t),t)dB(t) \quad on\ \ t_0 \leq t \leq T \tag{4}$$

with initial value $x(t_0) = x_0$, where $f : R^d \times [t_0, T] \longrightarrow R^d$, $g : R^d \times [t_0, T] \longrightarrow R^{d\times m}$ be both Borel measurable. By the definition of stochastic differential, this equation is equivalent to the following stochastic integral equation:

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s),s)ds + \int_{t_0}^{t} g(x(s),s)dB(s) \quad on\ \ t_0 \leq t \leq T.$$

To be more precise, we give the definition of the solution of the equation (4) with initial data.

Definition 2.1. An R^d-valued stochastic process $\{x(t)\}_{t_0 \leq t \leq T}$ is called a solution of equation (4) if it the following properties:
(i) $\{x(t)\}$ is continuous and $\mathcal{F}_t$-adapted;
(ii) $\{f(x(t),t)\} \in \mathcal{L}^1([t_0,T]; R^d)$ and $\{g(x(t),t)\} \in \mathcal{L}^2([t_0,T]; R^{d\times m})$;

(iii) equation (6) holds for every $t \in [t_0, T]$with probability 1.
A solution $\{x(t)\}$ is said to be unique if any other solution $\{\bar{x}(t)\}$is indistinguishable from $\{x(t)\}$, that is

$$P\{x(t) = \bar{x} \;\; for \;\; all \;\; t_0 \leq t \leq T\} = 1.$$

□

In order to obtain our main result, we need following assumptions.

Assumption 1. Both $f : R^d \times [t_0, T] \longrightarrow R^d$ and $g : R^d \times [t_0, T] \longrightarrow R^{d\times m}$ satisfy the Lipschitz condition; that is, for all $x, y \in R^d$ and $t \in [t_0, T]$,

$$|f(x,t) - f(y,t)|^2 \vee |g(x,t) - g(y,t)|^2 \leq \overline{K}|x - y|^2. \tag{5}$$

Assumption 2. Both $f : R^d \times [t_0, T] \longrightarrow R^d$ and $g : R^d \times [t_0, T] \longrightarrow R^{d\times m}$ satisfy the linear growth condition; that is, for any $x, y \in R^d$ and $t \in [t_0, T]$,

$$|f(x,t)|^2 \vee |g(x,t)|^2 \leq K(1 + |x|^2). \tag{6}$$

Assumption 3. Let $p \geq 2$. For positive constant α and all $(x, t) \in R^d \times [t_0, T]$,

$$|x^T f(x,t)| \vee \frac{(p-1)}{2}|g(x,t)|^2 \leq \alpha(1 + |x|^2). \tag{7}$$

Assumption 4. Both $f : R^d \times [t_0, T] \longrightarrow R^d$ and $g : R^d \times [t_0, T] \longrightarrow R^{d\times m}$ satisfy the weakened linear growth condition; that is, for any $t \in [t_0, T]$,

$$|f(0,t)|^2 \vee |g(0,t)|^2 \leq K. \tag{8}$$

Assumption 5. Let $p \geq 2$ and $x_0 \in L^p(\Omega; R^d)$. Assume that there are positive constants β, γ such that for all $(x, t) \in R^d \times [t_0, T]$,

$$|x^T f(0,t)| \vee (p-1)|g(0,t)|^2 \leq \beta, \tag{9}$$

$$|x^T(f(x,t) - f(y,t))| \vee (p-1)|g(x,t) - g(y,t)|^2 \leq \gamma|x - y|^2. \tag{10}$$

3. Uniform L^p-continuity

In this section, we will present a new uniform L^p-continuity of the solution for SDEs under Assumption 1-3.

Now, we assume that $x(t), t_0 \leq t \leq T$ is the unique solution of the equation (4) with initial value $x(t_0) = x_0$ under Assumption 1 and Assumption 2 or under Assumption 1 and Assumption 4.

Lemma 3.1. *Let $p \geq 2$ and $x_0 \in L^p(\Omega; R^d)$. Let Assumption 3 hold. Then*

$$E\Big([1 + |x(t)|^2]^{\frac{p}{2}}\Big) \leq 2^{(p-2)/2}[1 + E|x_0|^p]\exp(2p\alpha(T - t_0)). \tag{11}$$

Proof. By Itô's formula, we can derive that for $t \in [t_0, T]$,

$$[1 + |x(t)|^2]^{p/2}$$
$$= [1 + |x(t_0)|^2]^{p/2} + p\int_{t_0}^{t}[1 + |x(s)|^2]^{(p-2)/2}x^T(s)f(x(s), s)ds$$

$$
\begin{aligned}
&+\frac{p}{2}\int_{t_0}^{t}[1+|x(s)|^2]^{(p-2)/2}|g(x(s),s)|^2ds \\
&+\frac{p(p-2)}{2}\int_{t_0}^{t}[1+|x(s)|^2]^{(p-4)/2}|x^T(s)g(x(s),s)|^2ds \\
&+p\int_{t_0}^{t}[1+|x(s)|^2]^{(p-2)/2}x^T(s)g(x(s),s)dB(s) \\
&\leq 2^{\frac{p-2}{2}}[1+|x_0|^p]+p\int_{t_0}^{t}[1+|x(s)|^2]^{\frac{p-2}{2}} \\
&\quad\times\Big[x^T(s)f(x(s),s)+\frac{p-1}{2}|g(x(s),s)|^2\Big]ds \\
&+p\int_{t_0}^{t}[1+|x(s)|^2]^{(p-2)/2}x^T(s)g(x(s),s)dB(s) \\
&\leq 2^{\frac{p-2}{2}}[1+|x_0|^p]+2\alpha p\int_{t_0}^{t}[1+|x(s)|^2]^{\frac{p}{2}}ds \\
&+p\int_{t_0}^{t}[1+|x(s)|^2]^{(p-2)/2}x^T(s)g(x(s),s)dB(s). \qquad (12)
\end{aligned}
$$

For each number $n \geq 1$, define the stopping time

$$\tau_n = T \wedge \inf\{t \in [t_0, T] : |x(t) \ \geq n\}.$$

Obviously, as $n \to \infty, \tau_n \uparrow T$ a.s.. Moreover, it follows from (12) and the property of Itô's integral that

$$
\begin{aligned}
&E\Big([1+|x(t\wedge\tau_n)|^2]^{\frac{p}{2}}\Big) \\
&\leq 2^{\frac{p-2}{2}}[1+E|x_0|^p]+2\alpha pE\int_{t_0}^{t\wedge\tau_n}[1+|x(s)|^2]^{\frac{p}{2}}ds \\
&\leq 2^{\frac{p-2}{2}}[1+E|x_0|^p]+2\alpha p\int_{t_0}^{t}E\Big([1+|x(s\wedge\tau_n)|^2]^{\frac{p}{2}}\Big)ds.
\end{aligned}
$$

The Gronwall inequality yields

$$E\Big([1+|x(t\wedge\tau_n)|^2]^{\frac{p}{2}}\Big) \leq 2^{\frac{p-2}{2}}[1+E|x_0|^p]e^{2\alpha p(t-t_0)}.$$

Letting $n \to \infty$ yields

$$E\Big([1+|x(t)|^2]^{\frac{p}{2}}\Big) \leq 2^{\frac{p-2}{2}}[1+E|x_0|^p]e^{2\alpha p(t-t_0)}$$

and the desired inequality follows. □

Theorem 3.2. *Let Assumptions 1-3 hold. Assume that $x(t), t_0 \leq t \leq T$ is the unique solution of the equation (4). Then $E|x(t)|^p$ is uniformly continuous on $[t_0, T]$.*

Proof. By Itô's formula, we can derive that for $t \in [t_0, T]$,

$$\begin{aligned}
&[|x(t)|^2]^{p/2} \\
&= [|x(t_0)|^2]^{p/2} + p\int_{t_0}^{t}[|x(s)|^2]^{(p-2)/2}x^T(s)f(x(s),s)ds \\
&+\frac{p}{2}\int_{t_0}^{t}[|x(s)|^2]^{(p-2)/2}|g(x(s),s)|^2 ds \\
&+\frac{p(p-2)}{2}\int_{t_0}^{t}[|x(s)|^2]^{(p-4)/2}|x^T(s)g(x(s),s)|^2 ds \\
&+p\int_{t_0}^{t}[|x(s)|^2]^{(p-2)/2}x^T(s)g(x(s),s)dB(s) \\
&\le |x_0|^p + 2\alpha p\int_{t_0}^{t}[|x(s)|^2]^{(p-2)/2}[1+|x(s)|^2]ds \\
&+p\int_{t_0}^{t}[|x(s)|^2]^{(p-2)/2}x^T(s)g(x(s),s)dB(s). \qquad (13)
\end{aligned}$$

For each number $n \ge 1$, define the stopping time

$$\tau_n = T \wedge \inf\{t \in [t_0, T] : |x(t)| \ \ge n\}.$$

Obviously, as $n \to \infty, \tau_n \uparrow T$ a.s.. Moreover, it follows from (13) and the property of Itô's integral that

$$\begin{aligned}
&E\Big(|x(t\wedge\tau_n)|^p\Big) \\
&\le E|x_0|^p + 2\alpha p E\int_{t_0}^{t\wedge\tau_n}[1+|x(s)|^2]^{\frac{p}{2}}ds \\
&\le E|x_0|^p + 2\alpha p\int_{t_0}^{t}E\Big([1+|x(s\wedge\tau_n)|^2]^{\frac{p}{2}}\Big)ds.
\end{aligned}$$

Letting $n \to \infty$ yields

$$E|x(t)|^p \le E|x_0|^p + 2\alpha p\int_{t_0}^{t}E\Big([1+|x(s)|^2]^{\frac{p}{2}}\Big)ds.$$

Hence, we have

$$|E|x(t)|^p - E|x(s)|^p| \le 2\alpha p\int_{s}^{t}E\Big([1+|x(s)|^2]^{\frac{p}{2}}\Big)ds.$$

Applying Lemma 3.1, we get

$$|E|x(t)|^p - E|x(s)|^p| \le \alpha p 2^p[1+E|x_0|^p]e^{2p\alpha(T-t_0)}(t-s).$$

This implies $E|x(t)|^p$ is uniformly continuous on $t_0 \le t \le T$. The proof is complete. □

Lemma 3.3. *Let $p \geq 2$ and $x_0 \in L^p(\Omega; R^d)$. Let Assumption 5 hold. Then*

$$E\Big([\beta/\gamma + |x(t)|^2]^{\frac{p}{2}}\Big) \leq 2^{(p-2)/2}[\beta/\gamma + E|x_0|^p]\exp(2p\gamma(T-t_0)). \tag{14}$$

Proof. By Itô's formula, we can derive that for $t \in [t_0, T]$,

$$\begin{aligned}
&[\beta/\gamma + |x(t)|^2]^{p/2} \\
&= [\alpha/\beta + |x(t_0)|^2]^{p/2} + p\int_{t_0}^{t}[\alpha/\beta + |x(s)|^2]^{(p-2)/2}x^T(s)f(x(s),s)ds \\
&+\frac{p}{2}\int_{t_0}^{t}[\beta/\gamma + |x(s)|^2]^{(p-2)/2}|g(x(s),s)|^2ds \\
&+\frac{p(p-2)}{2}\int_{t_0}^{t}[\beta/\gamma + |x(s)|^2]^{(p-4)/2}|x^T(s)g(x(s),s)|^2ds \\
&+p\int_{t_0}^{t}[\beta/\gamma + |x(s)|^2]^{(p-2)/2}x^T(s)g(x(s),s)dB(s)
\end{aligned}$$

By the Assumption 5, it is easy to see that

$$\begin{aligned}
&[\beta/\gamma + |x(t)|^2]^{\frac{p}{2}} \\
&\leq 2^{\frac{p-2}{2}}[(\beta/\gamma)^{\frac{p}{2}} + |x_0|^p] + p\int_{t_0}^{t}[\beta/\gamma + |x(s)|^2]^{\frac{p-2}{2}} \\
&\quad\times\Big[x^T(s)[f(x(s),s) - f(0,s) + f(0,s)] \\
&\quad+\frac{p-1}{2}|g(x(s),s) - g(0,s) + g(0,s)|^2\Big]ds \\
&+p\int_{t_0}^{t}[\beta/\gamma + |x(s)|^2]^{(p-2)/2}x^T(s)g(x(s),s)dB(s) \\
&\leq 2^{\frac{p-2}{2}}[(\beta/\gamma)^{\frac{p}{2}} + |x_0|^p] + 2\gamma p\int_{t_0}^{t}[\beta/\gamma + |x(s)|^2]^{\frac{p}{2}}ds \\
&+p\int_{t_0}^{t}[\beta/\gamma + |x(s)|^2]^{(p-2)/2}x^T(s)g(x(s),s)dB(s).
\end{aligned} \tag{15}$$

For each number $n \geq 1$, define the stopping time

$$\tau_n = T \wedge \inf\{t \in [t_0, T] : |x(t)| \ \geq n\}.$$

Obviously, as $n \to \infty, \tau_n \uparrow T$ a.s.. Moreover, it follows from (15) and the property of Itô's integral that

$$\begin{aligned}
&E\Big([\beta/\gamma + |x(t\wedge\tau_n)|^2]^{\frac{p}{2}}\Big) \\
&\leq 2^{\frac{p-2}{2}}[(\beta/\gamma)^{\frac{p}{2}} + E|x_0|^p] + 2\gamma pE\int_{t_0}^{t\wedge\tau_n}[\beta/\gamma + |x(s)|^2]^{\frac{p}{2}}ds \\
&\leq 2^{\frac{p-2}{2}}[(\beta/\gamma)^{\frac{p}{2}} + E|x_0|^p] + 2\gamma p\int_{t_0}^{t}E\Big([\beta/\gamma + |x(s\wedge\tau_n)|^2]^{\frac{p}{2}}\Big)ds.
\end{aligned}$$

The Gronwall inequality yields

$$E\Big([\beta/\gamma + |x(t\wedge\tau_n)|^2]^{\frac{p}{2}}\Big) \le 2^{\frac{p-2}{2}}[(\beta/\gamma)^{\frac{p}{2}} + E|x_0|^p]e^{2\gamma p(t-t_0)}.$$

Letting $n \to \infty$ yields

$$E\Big([\beta/\gamma + |x(t)|^2]^{\frac{p}{2}}\Big) \le 2^{\frac{p-2}{2}}[(\beta/\gamma)^{\frac{p}{2}} + E|x_0|^p]e^{2\gamma p(t-t_0)}$$

and the desired inequality follows. □

Theorem 3.4. *Assume that $x(t), t_0 \le t \le T$ is the unique solution of the equation (4) under Assumption 1 and Assumptions 4-5. Then $E|x(t)|^p$ is uniformly continuous on $[t_0, T]$.*

Proof. Applying Lemma 3.3, we can prove this Theorem in a similar way of the proof as Theorem 3.2, so it is omitted. □

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