J. Appl. Math. & Informatics Vol. **31***(2013), No. 3 - 4, pp. 479 - 490*
Website: http://www.kcam.biz

A PRIORI ERROR ESTIMATES AND SUPERCONVERGENCE PROPERTY OF VARIATIONAL DISCRETIZATION FOR NONLINEAR PARABOLIC OPTIMAL CONTROL PROBLEMS†

YUELONG TANG* AND YUCHUN HUA

Abstract. In this paper, we investigate a priori error estimates and superconvergence of varitional discretization for nonlinear parabolic optimal control problems with control constraints. The time discretization is based on the backward Euler method. The state and the adjoint state are approximated by piecewise linear functions and the control is not directly discretized. We derive a priori error estimates for the control and superconvergence between the numerical solution and elliptic projection for the state and the adjoint state and present a numerical example for illustrating our theoretical results.

AMS Mathematics Subject Classification : 49J20, 65M60, 80M10.
Key words and phrases : A priori error estimates, superconvergence, variational discretization, optimal control problems, nonlinear equation.

1. Introduction

There have been extensive studies in convergence of finite element approximation for some classes of nonlinear elliptic optimal control problems, see, for example, [1, 3, 14], although it is impossible to give even a brief review here. Systematic introduction of finite element method for PDEs and optimal control can be found in [7, 13, 15, 19, 20, 21].

The superconvergence property of finite element solutions has also been an active research area in numerical analysis for PDEs and optimal control problems, see, for example, [4, 5, 11, 18, 22, 23, 24, 26, 27, 28]. Recently, superconvergnece of mixed finite element methods for optimal control problems has been studied in [6, 25] and a second-order convergence result of elliptic optimal control problems is proved by Hinze [9]. For parabolic optimal control problems, a priori

Received July 16, 2012. Revised October 26, 2012. Accepted October 27, 2012. *Corresponding author. †This work is supported by the Hunan Provinical Innovation Foundation for Postgraduate (lx2009B120).

error estimates of $h+k$ were established in [2, 8, 16, 17]. But it is more difficult to obtain error estimates and superconvergence for nonlinear parabolic optimal control problems.

In this paper, we are interested in the following nonlinear parabolic optimal control problem:

$$\begin{cases} \min\limits_{u\in U_{ad}} \dfrac{1}{2}\displaystyle\int_0^T \left(\int_\Omega (y-y_d)^2 + \int_\Omega u^2\right) dt, \\ y_t - \mathrm{div}(A\nabla y) + \phi(y) = f + u, \quad \text{in}\, \Omega \times (0,T], \\ y = 0, \quad \text{on}\, \partial\Omega \times (0,T], \\ y(0) = y_0, \quad \text{in}\, \Omega, \end{cases} \tag{1}$$

where Ω be a bounded domain in $\mathbb{R}^n (n \leq 3)$ with a Lipschitz boundary $\partial\Omega$, $0 < T < +\infty$. The coefficient $A = (a_{ij}(x))_{n\times n} \in (W^{1,\infty}(\bar{\Omega}))^{n\times n}$, such that for any $\xi \in \mathbb{R}^n$, $(A(x)\xi)\cdot\xi \geq c \mid \xi \mid^2$ with $c > 0$. Let $y_d, f \in C(0,T;L^2(\Omega))$. We assume that the function $\phi(\cdot) \in W^{2,\infty}(-R,R)$ for any $R > 0$, $\phi'(\cdot) \in L^2(\Omega)$ and $\phi'(\cdot) \geq 0$. Moreover, we suppose that U_{ad} is a nonempty closed convex set in $L^2(0,T;L^2(\Omega))$, defined by

$$U_{ad} = \left\{\ v(x,t) \in L^2(0,T;L^2(\Omega)) : a \leq v(x,t) \leq b, \quad a.e.\ \text{in}\, \Omega \times (0,T]\ \right\},$$

where a and b are constants.

The purpose of this work is to obtain the convergence of h^2+k for linear finite element method and the backward Euler method solving nonlinear parabolic optimal control problems with control constraints.

We adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and seminorm $|\cdot|_{W^{m,q}(\Omega)}$. We set $H_0^1(\Omega) \equiv \left\{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\right\}$ and denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$. We denote by $L^s(J;W^{m,q}(\Omega))$ the Banach space of L^s integrable functions from J into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,q}(\Omega))} = \left(\int_0^T \|v\|^s_{W^{m,q}(\Omega)} dt\right)^{\frac{1}{s}}$ for $s \in [1,\infty)$ and the standard modification for $s = \infty$. Similarly, one can define the space $H^l(J;W^{m,q}(\Omega))$. The details can be found in [13]. In addition, c or C denotes a generic positive constant independent of h and k.

The rest of the paper is organized as follows. In Section 2, we define a variational discretization approximation for the model problem. In Section 3, we derive a priori error estimates for the control. In Section 4, we obtain a superconvergence propery for the state and the adjoint state. We present a numerical example to verify our theoretical results in the last section.

2. Variational discretization for nonlinear parabolic control problems

In this section, we introduce a variational discretization approximation of the model problem. For ease of exposition, we set $V = L^2(0,T;W)$ with $W = H_0^1(\Omega)$

and $X = L^2(0,T;U)$ with $U = L^2(\Omega)$,

$$K = \left\{ v(x) \in L^2(\Omega) : a \le v(x) \le b, \quad a.e.\ \text{in}\, \Omega \right\}.$$

Moreover, we denote $\|\cdot\|_{H^m(\Omega)}$ and $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|_m$ and $\|\cdot\|$, respectively.

Let

$$a(v,w) = \int_\Omega (A\nabla v)\cdot \nabla w, \qquad \forall\, v,\, w \in W,$$
$$(f_1,f_2) = \int_\Omega f_1 \cdot f_2, \qquad \forall\, f_1,\, f_2 \in U.$$

It follows from the assumptions on A that

$$a(v,v) \ge c\|v\|_1^2, \quad |a(v,w)| \le C\|v\|_1\|w\|_1, \qquad \forall\, v,\, w \in W.$$

Thus a possible weak formula for the model problem (1) reads:

$$\begin{cases} \min\limits_{u\in U_{ad}} \dfrac{1}{2}\displaystyle\int_0^T \left(\int_\Omega (y-y_d)^2 + \int_\Omega u^2 \right) dt, \\ (y_t,w) + a(y,w) + (\phi(y),w) = (f+u,w), \qquad \forall\, w \in W,\, t\in(0,T], \\ y(0) = y_0. \end{cases} \tag{2}$$

It is well known (see, e.g., [13]) that the problem (2) has a solution (y,u), and the pair $(y,u) \in \big(H^1(0,T;L^2(\Omega)) \cap V\big) \times U_{ad}$ is the solution of (2) if there is a adjoint state $p \in H^1(0,T;L^2(\Omega)) \cap V$ such that the triplet (y,p,u) satisfies the following optimality conditions:

$$\begin{aligned} (y_t,w) + a(y,w) + (\phi(y),w) &= (f+u,w), \qquad \forall\, w\in W,\, t\in(0,T], \\ y(0) &= y_0, \end{aligned} \tag{3}$$

$$\begin{aligned} -(p_t,q) + a(q,p) + (\phi'(y)p,q) &= (y_d - y, q), \qquad \forall\, q\in W,\, t\in[0,T), \\ p(T) &= 0, \end{aligned} \tag{4}$$

$$(u-p, v-u) \ge 0, \qquad \forall\, v \in K,\, t\in[0,T). \tag{5}$$

Let $\mathcal{T}^h$ be regular triangulations of Ω and $\bar\Omega = \cup_{\tau\in\mathcal{T}^h}\bar\tau$. Let $h = \max_{\tau\in\mathcal{T}^h}\{h_\tau\}$, where h_τ denotes the diameter of the element τ. Moreover, we set

$$W^h = \{\ v_h \in C(\bar\Omega)\ :\ v_h|_\tau \in \mathbb{P}_1,\ \forall\, \tau\in\mathcal{T}^h, v_h|_{\partial\Omega} = 0\ \},$$

where $\mathbb{P}_1$ is the space of polynomials up to order 1.

We discuss a fully discrete variational discretization approximation of the problem (2). Let $0 = t_0 < t_1 < \cdots < t_N = T$, $k_i = t_i - t_{i-1}, i = 1,2,\cdots,N, k = \max\limits_{1\le i\le N} k_i$. Set $\varphi^i = \varphi(x,t_i)$ and

$$d_t\varphi^i = \frac{\varphi^i - \varphi^{i-1}}{k_i},\ i = 1,2,\cdots,N.$$

We define for $1 \le p < \infty$ the discrete time-dependent norms

$$|||\varphi|||_{l^p(0,T;W^{m,q}(\Omega))} := \left(k_i \sum_{i=1-l}^{N-l} \left\|\varphi^i\right\|_{W^{m,q}(\Omega)}^p \right)^{\frac{1}{p}},$$

where $l = 0$ for the control u and the state y and $l = 1$ for the adjoint state p, with the standard modification for $p = \infty$. Just for simplicity, we denote $|||\cdot|||_{l^p(0,T;W^{m,q}(\Omega))}$ by $|||\cdot|||_{l^p(W^{m,q})}$ and let

$$l_D^p(0,T;W^{m,q}(\Omega)) := \left\{ \varphi : |||\varphi|||_{l^p(W^{m,q})} < \infty \right\}, \qquad 1 \le p \le \infty.$$

Then a possible fully discrete variational discretization approximation of (2) is as follows:

$$\begin{cases} \min\limits_{u_h^i \in U_{ad}} \dfrac{1}{2} \sum\limits_{i=1}^{N} k_i \left(\int_\Omega (y_h^i - y_d^i)^2 + \int_\Omega (u_h^i)^2 \right), \\ \left(d_t y_h^i, w_h\right) + a\left(y_h^i, w_h\right) + \left(\phi(y_h^i), w_h\right) = \left(f^i + u_h^i, w_h\right), \\ \forall\, w_h \in W^h,\ i = 1, 2, \cdots, N,\ y_h^0 = y_0^h, \end{cases} \tag{6}$$

where y_0^h is an elliptic projection of y_0.

It follows (see e.g. [26]) that the control problem (6) has a solution $\left(y_h^i, u_h^i\right)$, $i = 1, 2, \cdots, N$, and $\left(y_h^i, u_h^i\right) \in W^h \times K$, $i = 1, 2, \cdots, N$, is the solution of (6) if there is a adjoint state $p_h^{i-1} \in W^h$, $i = 1, 2, \cdots, N$, such that the triplet $\left(y_h^i, p_h^{i-1}, u_h^i\right) \in W^h \times W^h \times K$, $i = 1, 2, \cdots, N$, satisfies the following optimality conditions:

$$\begin{gathered} \left(d_t y_h^i, w_h\right) + a\left(y_h^i, w_h\right) + \left(\phi(y_h^i), w_h\right) = \left(f^i + u_h^i, w_h\right), \\ \forall\, w_h \in W^h,\ i = 1, 2, \cdots, N,\ y_h^0 = y_0^h, \end{gathered} \tag{7}$$

$$\begin{gathered} -\left(d_t p_h^i, q_h\right) + a\left(q_h, p_h^{i-1}\right) + \left(\phi'(y_h^i) p_h^{i-1}, q_h\right) = \left(y_d^i - y_h^i, q_h\right), \\ \forall\, q_h \in W^h,\ i = N, \cdots, 2, 1,\ p_h^N = 0, \end{gathered} \tag{8}$$

$$\left(u_h^i - p_h^{i-1}, v - u_h^i\right) \ge 0, \qquad \forall\, v \in K,\ i = 1, 2, \cdots, N. \tag{9}$$

It should be point out that we minimize over the infinite dimensional set K instead of minimize over a finite dimensional subset of K like in [10].

3. A priori error estimates

In this section, we consider a priori error estimates for the control. We define the following intermediate variables. Let $\left(y_h^i(u), p_h^{i-1}(u)\right) \in W^h \times W^h$, $i = 1, 2, \cdots, N$, satisfies the following system:

$$\begin{gathered} \left(d_t y_h^i(u), w_h\right) + a\left(y_h^i(u), w_h\right) + \left(\phi(y_h^i(u)), w_h\right) = \left(f^i + u^i, w_h\right), \\ \forall\, w_h \in W^h,\ i = 1, 2, \cdots, N,\ y_h^0(u) = y_0^h, \end{gathered} \tag{10}$$

$$
\begin{aligned}
-\left(d_t p_h^i(u), q_h\right) + a\left(q_h, p_h^{i-1}(u)\right) + \left(\phi'(y_h^i(u)) p_h^{i-1}(u), q_h\right) = \left(y_d^i - y_h^i(u), q_h\right), \\
\forall\, q_h \in W^h,\ i = N, \cdots, 2, 1,\ p_h^N(u) = 0.
\end{aligned} \tag{11}
$$

We introduce the elliptic projection operator $P_h:\ W \to W^h$, which satisfies: for any $\phi \in W$

$$
a(\phi - P_h\phi, w_h) = 0, \qquad \forall\, w_h \in W^h. \tag{12}
$$

It has the following approximation properties:

$$
\|\phi - P_h\phi\| \le C h^2 \|\phi\|_2, \qquad \forall\, \phi \in H^2(\Omega). \tag{13}
$$

Lemma 3.1. *Let (y, p, u) and $(y_h(u), p_h(u))$ be the solutions of (3)-(5) and (10)-(11), respectively. Suppose that $y, p \in l_D^2(0, T; H^2(\Omega)) \cap H^1(0, T; H^2(\Omega)) \cap H^2(0, T; L^2(\Omega))$. Then*

$$
|||y_h(u) - y|||_{l^2(L^2)} + |||p_h(u) - p|||_{l^2(L^2)} \le C(h^2 + k). \tag{14}
$$

Proof. From the definition of P_h, (3) and (10), for any $w_h \in W^h$, $i = 1, 2, \cdots, N$, we have

$$
\begin{aligned}
&\left(d_t y_h^i(u) - d_t P_h y^i, w_h\right) + a\left(y_h^i(u) - P_h y^i, w_h\right) \\
=& -\left(d_t P_h y^i, w_h\right) - a(y^i, w_h) + (f^i + u^i, w_h) - (\phi(y_h^i(u)), w_h) \\
=& -\left(d_t P_h y^i - d_t y^i, w_h\right) - (d_t y^i - y_t^i, w_h) + (\phi(y^i) - \phi(y_h^i(u)), w_h).
\end{aligned} \tag{15}
$$

By selecting $w_h = y_h^i(u) - P_h y^i$, we obtain

$$
\begin{aligned}
&\left(d_t y_h^i(u) - d_t P_h y^i, y_h^i(u) - P_h y^i\right) + a\left(y_h^i(u) - P_h y^i, y_h^i(u) - P_h y^i\right) \\
=& -\left(d_t P_h y^i - d_t y^i, y_h^i(u) - P_h y^i\right) - \left(d_t y^i - y_t^i, y_h^i(u) - P_h y^i\right) \\
&+ \left(\phi(P_h y^i) - \phi(y_h^i(u)), y_h^i(u) - P_h y^i\right) + \left(\phi(y^i) - \phi(P_h y^i), y_h^i(u) - P_h y^i\right).
\end{aligned} \tag{16}
$$

Note that $\phi'(\cdot) \ge 0$, $a\left(y_h^i(u) - P_h y^i, y_h^i(u) - P_h y^i\right) \ge 0$, and

$$
\begin{aligned}
&\left(d_t y_h^i(u) - d_t P_h y^i, y_h^i(u) - P_h y^i\right) \\
\ge& \frac{1}{2k_i}\left(\left\|y_h^i(u) - P_h y^i\right\|^2 - \left\|y_h^i(u) - P_h y^i\right\| \left\|y_h^{i-1}(u) - P_h y^{i-1}\right\|\right).
\end{aligned} \tag{17}
$$

It follows from (13), (16)-(17) and Hölder's inequality that

$$
\begin{aligned}
\left\|y_h^i(u) - P_h y^i\right\| \le& \left\|y_h^{i-1}(u) - P_h y^{i-1}\right\| + \left\|(P_h - I)(y^i - y^{i-1})\right\| \\
&+ \left\|y^i - y^{i-1} - k_i y_t^i\right\| + C h^2 k_i \|y^i\|_2.
\end{aligned} \tag{18}
$$

Summing i from 1 to N^* $(1 \leq N^* \leq N)$, we get

$$\begin{aligned}
&\left\| y_h^{N^*}(u) - P_h y^{N^*} \right\| \\
\leq & \sum_{i=1}^{N^*} C \left\| (P_h - I)(y^i - y^{i-1}) \right\| + \sum_{i=1}^{N^*} \left\| y^i - y^{i-1} - k_i y_t^i \right\| + Ch^2 |||y|||_{l^2(H^2)} \\
\leq & \sum_{i=1}^{N^*} Ch^2 \left\| y^i - y^{i-1} \right\|_2 + \sum_{i=1}^{N^*} \int_{t_{i-1}}^{t_i} \|(t_{i-1} - t) y_{tt}\| \, dt + Ch^2 |||y|||_{l^2(H^2)} \\
\leq & Ch^2 \sum_{i=1}^{N^*} \int_{t_{i-1}}^{t_i} \|y_t\|_2 \, dt + k \sum_{i=1}^{N^*} \int_{t_{i-1}}^{t_i} \|y_{tt}\| \, dt + Ch^2 |||y|||_{l^2(H^2)} \\
\leq & Ch^2 \int_0^{t_{N^*}} \|y_t\|_2 \, dt + k \int_0^{t_{N^*}} \|y_{tt}\| \, dt + Ch^2 |||y|||_{l^2(H^2)} \\
\leq & C \left(h^2 \|y_t\|_{L^2(J;H^2(\Omega))} + k \|y_{tt}\|_{L^2(J;L^2(\Omega))} + Ch^2 |||y|||_{l^2(H^2)} \right).
\end{aligned} \tag{19}$$

Thus, we have

$$|||y_h(u) - P_h y|||_{l^\infty(L^2)} \leq C \left(h^2 + k \right). \tag{20}$$

From (13), we derive

$$|||P_h y - y|||_{l^2(L^2)}^2 = \sum_{i=1}^{N} k_i \left\| P_h y^i - y^i \right\|^2 \leq Ch^4 \sum_{i=1}^{N} k_i \left\| y^i \right\|_2^2 = Ch^4 |||y|||_{l^2(H^2)}^2. \tag{21}$$

According to embedding theorem and (20)-(21), we have

$$|||y_h(u) - y|||_{l^2(L^2)} \leq C \left(h^2 + k \right). \tag{22}$$

Similarly, we can prove that

$$|||p_h(u) - p|||_{l^2(L^2)} \leq C \left(h^2 + k \right). \tag{23}$$

Then (14) follows from (22) and (23). □

For ease of exposition, we set

$$\begin{aligned}
J(u) &= \frac{1}{2} \int_0^T \left(\int_\Omega (y - y_d)^2 + \int_\Omega u^2 \right) dt, \\
J_h(u_h) &= \frac{1}{2} \int_0^T \left(\int_\Omega (y_h - y_d)^2 + \int_\Omega u_h^2 \right) dt.
\end{aligned}$$

It can be shown that

$$\begin{aligned}
(J'(u), v) &= \int_0^T (u - p, v) \, dt, \\
(J'_{hk}(u_h), v) &= \sum_{i=1}^{N} k_i \left(u_h^i - p_h^{i-1}(u_h), v \right).
\end{aligned}$$

In many applications, $J(\cdot)$ is uniform convex near the solution u (see, e.g., [14]) that is closely related to the second order sufficient conditions of the control problem. It is assumed in many studies on numerical methods of the problem (see, e.g., [3]). Hence, if h and k are small enough, we can assume that $J_{hk}(\cdot)$ is uniform convex, namely, there is a positive constant c, such that

$$c|||u-v|||^2_{l^2(L^2)} \le \left(J'_{hk}(u) - J'_{hk}(v), u-v\right), \qquad \forall\, u,v \in K. \tag{24}$$

Theorem 3.2. *Let (y,p,u) and (y_h,p_h,u_h) be the solutions of (3)-(5) and (7)-(9), respectively. Assume that all the conditions in Lemma 3.1 are satisfied. Then, we have*

$$|||u-u_h|||_{l^2(L^2)} \le C\left(h^2+k\right). \tag{25}$$

Proof. From (5), (9) and (24), we obtain

$$\begin{aligned} & c|||u-u_h|||^2_{l^2(L^2)} \\ \le & \left(J'_{hk}(u) - J'_{hk}(u_h), u-u_h\right) \\ = & \sum_{i=1}^N k_i\left(u^i - p_h^{i-1}(u), u^i-u_h^i\right) - \sum_{i=1}^N k_i\left(u_h^i - p_h^{i-1}, u^i-u_h^i\right) \\ \le & \sum_{i=1}^N k_i\left(p^i - p_h^{i-1}(u), u^i-u_h^i\right). \end{aligned} \tag{26}$$

By using Hölder's inequality and Young's inequality, we get

$$\begin{aligned} & \sum_{i=1}^N k_i\left(p^i - p_h^{i-1}(u), u^i-u_h^i\right) \\ = & \sum_{i=1}^N k_i\left(p^{i-1} - p_h^{i-1}(u), u^i-u_h^i\right) + \sum_{i=1}^N k_i\left(p^i - p^{i-1}, u^i-u_h^i\right) \\ \le & C(\delta)\sum_{i=1}^N k_i\left\|p^{i-1}-p_h^{i-1}(u)\right\|^2 + C(\delta)\sum_{i=1}^N k_i\left\|p^i-p^{i-1}\right\|^2 + \delta\sum_{i=1}^N k_i\left\|u^i-u_h^i\right\|^2 \\ \le & C(\delta)\left(|||p_h(u)-p|||^2_{l^2(L^2)} + k^2\|p_t\|^2_{L^2(J;L^2(\Omega))}\right) + \delta|||u-u_h|||^2_{l^2(L^2)} \end{aligned} \tag{27}$$

From (14) and (26)-(27), we obtain (25). □

4. Superconvergence analysis

In this section, we discuss the superconvergence properties for the state variable and the adjoint state variable.

Theorem 4.1. *Let (y,p,u) and (y_h,p_h,u_h) be the solutions (3)-(5) and (7)-(9), respectively. Assume that all the conditions in Theorem 3.2 are valid. Then*

$$|||P_hy-y_h|||_{l^2(H^1)} + |||P_hp-p_h|||_{l^2(H^1)} \le C\left(h^2+k\right). \tag{28}$$

Proof. From (3) and (7), for any $w_h \in W^h$ and $i = 1, 2, \cdots, N$, we obtain the following error equation:

$$\left(y_t^i - d_t y_h^i, w_h\right) + a\left(y^i - y_h^i, w_h\right) + \left(\phi(y^i) - \phi(y_h^i), w_h\right) = \left(u^i - u_h^i, w_h\right). \quad (29)$$

By choosing $w_h = P_h y^i - y_h^i$ and using the definition of P_h, we get

$$\begin{aligned} &\left(d_t P_h y^i - d_t y^i + d_t y^i - y_t^i + \phi(P_h y^i) - \phi(y^i) + u^i - u_h^i, P_h y^i - y_h^i\right) \\ =&\left(d_t P_h y^i - d_t y_h^i, P_h y^i - y_h^i\right) + a\left(P_h y^i - y_h^i, P_h y^i - y_h^i\right) \\ &+ (\phi(P_h y^i) - \phi(y_h^i), P_h y^i - y_h^i). \end{aligned} \quad (30)$$

Note that

$$\left(d_t P_h y^i - d_t y_h^i, P_h y^i - y_h^i\right) \geq \frac{1}{2k_i}\left(\left\|P_h y^i - y_h^i\right\|^2 - \left\|P_h y^{i-1} - y_h^{i-1}\right\|^2\right), \quad (31)$$

and

$$\begin{aligned} (d_t P_h y^n - d_t y^n, P_h y^n - y_h^n) \leq& \left\|d_t P_h y^i - d_t y^i\right\| \left\|P_h y^i - y_h^i\right\| \\ \leq& Ch^2 \left\|d_t y^i\right\|_2 \left\|P_h y^i - y_h^i\right\| \\ \leq& Ch^2 k_i^{-1} \int_{t_{i-1}}^{t_i} \|y_t\|_2 \, dt \left\|P_h y^i - y_h^i\right\| \\ \leq& Ch^2 k_i^{-\frac{1}{2}} ||y_t||_{L^2(t_{i-1}, t_i; H^2(\Omega))} \left\|P_h y^i - y_h^i\right\|. \end{aligned} \quad (32)$$

Additionally,

$$\begin{aligned} \left(d_t y^i - y_t^i, P_h y^i - y_h^i\right) =& k_i^{-1}\left(y^i - y^{i-1} - k y_t^k, P_h y^k - y_h^k\right) \\ \leq& k_i^{-1}\left\|y^i - y^{i-1} - k y_t^i\right\| \left\|P_h y^i - y_h^i\right\| \\ =& k_i^{-1}\left\|\int_{t_{i-1}}^{t_i} (t_{i-1} - t) y_{tt} dt\right\| \left\|P_h y^i - y_h^i\right\| \\ \leq& C k_i^{\frac{1}{2}} ||y_{tt}||_{L^2(t_{i-1}, t_i; L^2(\Omega))} \left\|P_h y^i - y_h^i\right\|, \end{aligned} \quad (33)$$

and

$$\begin{aligned} (\phi(P_h y^i) - \phi(y^i), P_h y^i - y_h^i) \leq& C\|P_h y^i - y^i\|\|P_h y^i - y_h^i\| \\ \leq& C(\delta)\|P_h y^i - y^i\|^2 + \delta\|P_h y^i - y_h^i\|^2 \\ \leq& C(\delta)h^4\|y^i\|_2^2 + \delta\|P_h y^i - y_h^i\|^2. \end{aligned} \quad (34)$$

Let δ be small enough. Multiplying both sides of (30) by $2k_i$ and summing i from 1 to N, by using Hölder's inequality and Young's inequality, we have

$$\begin{aligned} &\left\|P_h y^N - y_h^N\right\|^2 + c\sum_{i=1}^N k_i \left\|P_h y^i - y_h^i\right\|_1^2 \\ \leq& C(\delta)\left(h^4||y_t||^2_{L^2(J;H^2(\Omega))} + k^2||y_{tt}||^2_{L^2(J;L^2(\Omega))} + h^4|||y|||^2_{l^2(H^2)} + |||u - u_h|||^2_{l^2(L^2)}\right) \end{aligned} \quad (35)$$

From Theorem 3.2, (25) and (35), we get

$$|||P_h y - y_h|||_{l^2(H^1)} \leq C\left(h^2 + k\right). \quad (36)$$

Similarly, we can prove that

$$|||P_h p - p_h|||_{l^2(H^1)} \le C\left(h^2 + k\right). \tag{37}$$

Then (28) follows from (36)-(37). □

5. Numerical experiments

In this section, we present a numerical example to illustrate our theoretical results. The optimal control problem was dealt numerically with codes developed based on AFEPack. This package is freely available and the details can be found in [12].

We solve the following nonlinear parabolic optimal control problem:

$$\begin{cases} \min\limits_{u\in U_{ad}} \dfrac{1}{2}\displaystyle\int_0^T \left(\int_\Omega (y(x,t)-y_d(x,t))^2 + \int_\Omega (u(x,t))^2\right) dt, \\ y_t(x,t) - \mathrm{div}(A(x)\nabla y(x,t)) + \phi(y(x,t)) = f(x,t) + u(x,t), \quad \text{in}\,\Omega\times(0,T], \\ y(x,t) = 0, \quad \text{on}\,\partial\Omega\times(0,T], \\ y(x,0) = y_0(x), \quad \text{in}\,\Omega, \end{cases} \tag{38}$$

The discretization was described in Section 2. For ease of exposition, we denote $|||\cdot|||_{l^2(H^1)}$ by $|||\cdot|||_1$.

Example 1. Let $\Omega = [0,1]\times[0,1]$, $T = 1$. The data are as follows:

$$\begin{aligned} &\phi(y) = y^3,\ a = -0.5,\ b = 0.5, A(x) = I, \\ &p(x,t) = \sin(2\pi x_1)\sin(2\pi x_2)(1-t), \\ &y(x,t) = \sin(2\pi x_1)\sin(2\pi x_2)t, \\ &u(x,t) = \max(-0.5, \min(0.5, p(x,t))), \\ &f(x,t) = y_t(x,t) - \mathrm{div}(A(x)\nabla y(x,t)) + \phi(y(x,t)) - u(x,t), \\ &y_d(x,t) = y(x,t) - p_t(x,t) - \mathrm{div}(A^*(x)\nabla p(x,t)) + \phi'(y(x,t))p(x,t). \end{aligned}$$

The errors $|||u-u_h|||$, $|||P_h y - y_h|||_1$ and $|||P_h p - p_h|||_1$ on a sequence of uniformly refined meshes are shown in Table 1 where the convergence order is computed by the following formula: $Rate = \frac{log(e_{i+1})-log(e_i)}{log(h_{i+1})-log(h_i)}$. In Figure 1, we show the relationship between $log_{10}(error)$ and $log_{10}(sqrt(dofs))$. It is easy to see $|||u-u_h||| = \mathcal{O}\left(h^2+k\right)$, $|||P_h y - y_h|||_1 = \mathcal{O}\left(h^2+k\right)$ and $|||P_h p - p_h|||_1 = \mathcal{O}\left(h^2+k\right)$ are consistent with our theoretical results. In Figure 2, we plot the profiles of the approximation solution u_h when $t = 0.5$.

TABLE 1. Numerical results, Example 1.

h	k	$\|\|\|u-u_h\|\|\|$	Rate	$\|\|\|P_h y - y_h\|\|\|_1$	Rate	$\|\|\|P_h p - p_h\|\|\|_1$	Rate
$\frac{1}{10}$	$\frac{1}{10}$	4.57845e-02	—	3.31962e-02	—	3.31860e-02	—
$\frac{1}{20}$	$\frac{1}{40}$	1.22388e-02	1.90	8.22965e-03	2.01	8.22864e-03	2.01
$\frac{1}{40}$	$\frac{1}{160}$	3.09192e-03	1.98	2.05253e-03	2.00	2.05242e-03	2.00

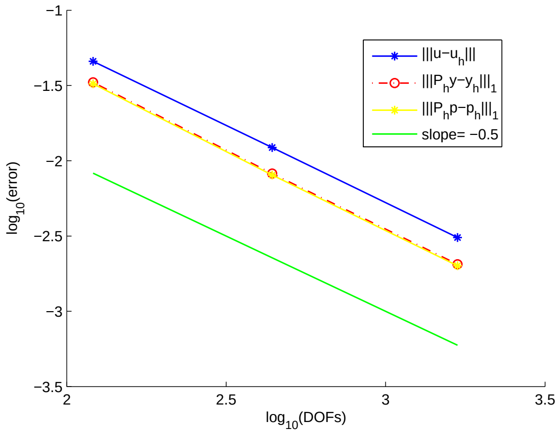

FIGURE 1. The order of convergence.

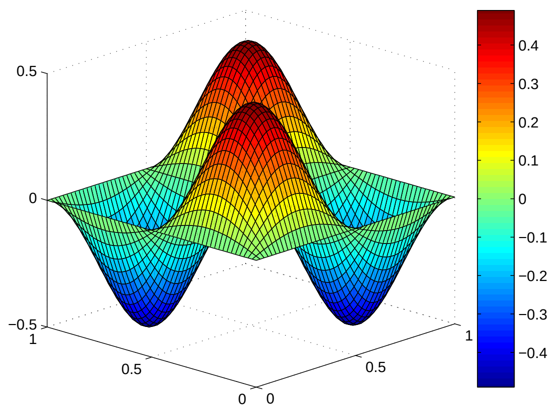

FIGURE 2. The numerical solution u_h when $t = 0.5$.

REFERENCES

1. N. Arada, E. Casas and F. Tröltzsch, *Error estimates for semilinear elliptic control problem*, Comput. Optim. Appl. **23** (2002), 201-229.
2. R. Becker, D. Meidner and B. Vexler, *Efficient numerical solution of parabolic optimization problems by finite element methods*, Optim. Meth. Soft. **00(00)** (2006), 1-28.
3. E. Casas, F. Tröltzsch and A. Unger, *Second order sufficient optimality conditions for some state-constrained control problems of semilinear elliptic equations*, SIAM J. Control Optim. **38(5)** (2000), 1369-1391.

4. Y. Chen and Y. Dai, *Superconvergence for optimal control problems governed by semi-linear elliptic equations*, J. Sci. Comput. **39** (2009), 206-221.
5. Y. Chen, Y. Fu, H. Liu, Y. Dai and H. Wei, *Recovery a posteriori error estimates for general convex elliptic optimal control problems subject to pointwise control constraints*, J. Comput. Math. **27(4)** (2009), 543-560.
6. Y. Chen, Y. Huang, W. Liu, and N. Yan, *Error estimates and superconvergence of mixed finite element methods for convex optimal control problems* , J. Sci. Comput. **42** (2010), 382-403.
7. P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amstterdam, 1978.
8. H. Fu and H. Rui, *A priori error estimates for optimal control problems governed by transient advection-diffusion equationg*, J. Sci. Comput. **38** (2009), 290-315.
9. M. Hinze, *A variational discretization concept in control constrained optimization: the linear-quadratic case*, Comput. Optim. Appl. **30** (2005), 45-63.
10. M. Hinze, N. Yan and Z. Zhou, *Variational discretization for optimal control governed by convection dominated diffusion equations*, J. Comput. Math. **27** (2009), 237-253.
11. Y. Huang and J. Xu, *Superconvergence of quadratic finite elements on mildly structured grids*, Math. Comp. **77** (2008), 1253-1268.
12. R. Li, W. Liu and N. Yan, *A posteriori error estimates of recovery type for distributed convex optimal control problems*, J. Sci. Comput. **33** (2007), 155-182.
13. J. Lions and E. Magenes, *Non Homogeneous Boundary Value Problems and Applications*, Springer-verlag, Berlin, 1972.
14. W. Liu and D. Tiba, *Error estimates for the finite element approximation of a class of nonlinear optimal control problems*, J. Numer. Funct. Optim. **22** (2001), 953-972.
15. W. Liu and N. Yan, *Adaptive Finite Element Methods for Optimal Control Governed by PDEs*, Science Press, Beijing, 2008.
16. D. Meidner and B. Vexler, *A priori error estimates for space-time finite element discretization of parabolic optimal control problems Part I: problems without control constraints*, SIAM J. Control Optim. **47(3)** (2008), 1150-1177.
17. D. Meidner and B. Vexler, *A priori error estimates for space-time finite element discretization of parabolic optimal control problems Part II: problems with control constraints*, SIAM J. Control Optim. **47(3)** (2008), 1301-1329.
18. C. Meyer and A. Rösch, *Superconvergence properties of optimal control problems*, SIAM J. Control Optim. **43** (2004), 970-985.
19. P. Neittaanmaki and D. Tiba, *Optimal Control of Nonlinear Parabolic Systems: Theory, Algorithms and Applications*, Dekker, New York, 1994.
20. C. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992.
21. V. Thomée, *Galerkin finite element methods for parabolic problems*, Springer Series in Comput. Math., Spring-Verlag, 2006.
22. V. Thomée, J. Xu and N. Zhang, *Superconvergence of the gradient in pecewise linear element approximation to a parabolic problems*, SIAM J. Numer. Anal. **26(3)** (1989), 553-573.
23. L. B. Wahlbin, *Superconvergence in Galerkin Finite Element Methods*, Lecture Notes in Math., Spring-Verlag, 1995.
24. M. F. Wheeler and J. R. Whiteman, *Superconvergence recovery of gradient on subdomains from piecewise linear finite element approximations*, Numer. Meth. PDEs. **3** (1987), 357-374.
25. X. Xing and Y. Chen, *Superconvergence of mixed methods for optimal control problems governed by parabolic equations*, Adv. Appl. Math. Mech. **3(4)** (2011), 401-419.
26. D. Yang, Y. Chang and W. Liu, *A priori error estimate and superconvergence annalysis for an opitmal control problem of bilinear type*, J. Comput. Math. **26(4)** (2008), 471-487.

27. O. C. Zienkiwicz and J. Z. Zhu, *The superconvergence patch recovery and a poseriori error estimates*, Int. J. Numer. Meth. Engrg. **33** (1992) 1331-1382.
28. O. C. Zienkiwicz and J. Z. Zhu, *The superconvergence patch recovery (SPR) and adaptive finite element refinement*, Comput. Meth. Appl. Math. **101** (1992), 207-224.

Yuelong Tang received M.Sc. and Ph.D from Xiangtan University. His research interests include numerical optimization and numerical methods for partial differential equations.

Department of Mathematics and Computational Science, Hunan University of Science and Engineering, Yongzhou 425100, Hunan, China.
e-mail: tangyuelonga@163.com

Yuchun Hua received M.Sc. form Yunnan University. His research interests include differential equations and dynamical system.

Department of Mathematics and Computational Science, Hunan University of Science and Engineering, Yongzhou 425100, Hunan, China.
e-mail: huayuchun306@sina.com