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ON SEMILOCAL CONVERGENCE OF A MULTIPOINT THIRD ORDER METHOD WITH R-ORDER $(2+p)$ UNDER A MILD DIFFERENTIABILITY CONDITION

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Abstract. The semilocal convergence of a third order iterative method used for solving nonlinear operator equations in Banach spaces is established by using recurrence relations under the assumption that the second Fréchet derivative of the involved operator satisfies the ω-continuity condition given by $\|F''(x) - F''(y)\| \leq \omega(\|x-y\|),\ x, y \in \Omega$, where, $\omega(x)$ is a nondecreasing continuous real function for $x > 0$, such that $\omega(0) \geq 0$. This condition is milder than the usual Lipschitz/Hölder continuity condition on F''. A family of recurrence relations based on two constants depending on the involved operator is derived. An existence-uniqueness theorem is established to show that the R-order convergence of the method is $(2+p)$, where $p \in (0, 1]$. A priori error bounds for the method are also derived. Two numerical examples are worked out to demonstrate the efficacy of our approach and comparisons are elucidated with a known result.

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1. Introduction

The aim of this paper is to establish the semilocal convergence of a third order iterative method used for solving nonlinear operator equations

$$F(x) = 0, \tag{1}$$

where $F : \Omega \subseteq \mathbb{X} \to \mathbb{Y}$ is a nonlinear operator on an open convex subset Ω of a Banach space $\mathbb{X}$ with values in a Banach space $\mathbb{Y}$. This is done by using recurrence relations under the assumption that the second Fréchet derivative of the involved operator satisfies the ω-continuity condition. The most well known second order iterative methods used to solve (1) are Newton's method and its

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variants. The Kantorovich theorem [15, 19] provides sufficient conditions to ensure convergence of these methods. A lot of research [2, 3, 8, 12, 11] has been carried out to provide improvements in these methods. Third order one point iterative methods such as the Halley method, the Chebyshev method and the Super-Halley method [2, 3, 12] are developed to solve (1). These methods are useful in solving stiff systems of equations [16], where a quick convergence is required. The main difficulties of these one-point third order iterative methods are to evaluate the second order derivative of the operator F which is computationally expensive at times. In fact, a very restrictive condition of one point iteration of order N is that they depend explicitly on the first $N-1$ derivatives of F. This implies that their informational efficiency is less than or equal to unity[20]. All these higher order derivatives are also very difficult to compute. On the other hand multipoint iterative methods[22, 17, 13, 14] which sample F and its derivatives at a number of points have also gained importance recently. The restrictions of one point iterative methods may not hold good for multipoint methods.

The semilocal convergence of these methods by using majorizing sequences and recurrence relations is also established under the assumptions that the First/Second Fréchet derivative of the involved operator satisfies the Lipschitz and the Hölder continuity conditions. Candela and Marquina[2, 3] studied the convergence of the Halley method and the Chebyshev method under the assumption that F'' is Lipschitz continuous by using recurrence relations. Hernández and Salanova[10] and Hernández[13] studied the convergence of the Chebyshev method and the second derivative free version of the Chebyshev method by using recurrence relations under Hölder continuity condition on F''. Ye and Li[22] studied the convergence of the Euler-Halley method under similar conditions. However, the Lipschitz/Hölder continuity condition on the second derivative of F may be violated in many problems.

Example. Consider the following nonlinear integral equation of mixed type [9]:

$$F(x)(s) = x(s) + \sum_{i=1}^{m} \int_a^b k_i(s,t) l_i(x(t))dt - u(s), \ \ s \in [a,b]$$

where $-\infty < a < b < \infty$, u, l_i, and k_i, for $i = 1, 2, \ldots, m$ are known functions and x is a continuous function.

If $l_i''(x(t))$ is L_i-Lipschitz continuous in Ω, $L_i \geq 0$, for $i = 1, 2, \ldots, m$, then F'' does not satisfy any Lipschitz condition, where the sup-norm is considered. In this case

$$\|F''(x) - F''(y)\| = \sum_{i=1}^{m} L_i \|x - y\|, \ \ x, y \in \Omega.$$

Similarly, if $l_i''(x(t))$ is (L_i, p_i)-Hölder continuous in Ω, $L_i \geq 0$, $p_i \in (0,1]$ for $i = 1, 2, \ldots, m$, then we have

$$\|F''(x) - F''(y)\| = \sum_{i=1}^{m} L_i \|x-y\|^{p_i},\ x, y \in \Omega.$$

Here also, F'' is not Hölder continuous, when the sup-norm is used.

Recently, Ezquerro and Hernández [7] and Hernández and Romero [11] considered the more general ω-continuity condition given by

$$\|F''(x) - F''(y)\| \leq \omega(\|x-y\|),\ x, y \in \Omega, \tag{2}$$

where $\omega(x)$ is a nondecreasing continuous real function for $x > 0$, such that $\omega(0) \geq 0$, on F''. This enables us to study the semilocal convergence of the Halley method and a family of third order iterative methods respectively. They proved that R-order convergence of the Halley method is $(2+p),\ p \in (0,1]$.

In this paper, the semilocal convergence of a multipoint third order iterative method used for solving nonlinear equations (1) is discussed. The method is derived from the Halley method by replacing the second derivative with the divided difference containing only the first derivatives. The convergence of the method is established by using recurrence relations under the assumption that F'' satisfies the ω-continuity condition (2). This ω-continuity condition is milder than the usual Lipschitz/Hölder continuity condition. An existence-uniqueness theorem is established to show that the R-order convergence of the method is $(2+p)$, where $p \in (0,1]$. A priori error bounds for the method are also derived. Two numerical examples are worked out to demonstrate the efficacy of our approach. It is observed that our results are better than those obtained by Ezquerro and Hernández[7].

The paper is organized in six sections, with the first one giving a detailed introduction. In Section 2, first we derive a multipoint third order method from Halley's well known method. Next, three scalar sequences are constructed and their properties are studied. The recurrence relations for our method are derived in Section 3. The convergence analysis based on these recurrence relations of the method is given in Section 4. In Section 5, two numerical examples are worked out to demonstrate the efficacy of our approach. It is observed that our results are better than those obtained by Ezquerro and Hernández[7]. Finally, the Section 6 forms the conclusion, where the analysis is studied in detail.

2. Construction of scalar sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ and their properties

In this section, the construction of a multipoint third order method and three real sequences with their properties are described in order to study the convergence of the method. The Halley method given in [1] can be given as

$$\left.\begin{array}{l} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ H(x_n, y_n) = F'(x_n)^{-1}F''(x_n)(y_n - x_n), \\ x_{n+1} = y_n - \frac{1}{2}H(x_n, y_n)[I + \frac{1}{2}H(x_n, y_n)]^{-1}(y_n - x_n),\ n \geq 0. \end{array}\right\} \tag{3}$$

From Taylor's formula, we have

$$F'(z_n) = F'(x_n) + F''(x_n)(z_n - x_n) + \int_{x_n}^{z_n} F'''(x)(z_n - x)dx, \tag{4}$$

where, $z_n = x_n + \theta(y_n - x_n)$ and $\theta \in (0, 1]$. Ignoring the error term in (4), we get

$$F''(x_n)(y_n - x_n) \approx \frac{1}{\theta}[F'(z_n) - F'(x_n)].$$

Using this in (3), we get our two point iteration of order three given by

$$\left.\begin{array}{l} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ z_n = x_n + \theta(y_n - x_n),\ \theta \in (0, 1], \\ H(x_n, y_n) = \frac{1}{\theta}F'(x_n)^{-1}[F'(z_n) - F'(x_n)], \\ Q(x_n, y_n) = -\frac{1}{2}H(x_n, y_n)[I + \frac{1}{2}H(x_n, y_n)]^{-1}, \\ x_{n+1} = y_n + Q(x_n, y_n)(y_n - x_n),\ n \geq 0. \end{array}\right\} \tag{5}$$

One interesting feature of the iteration (5) is that of its simplicity as it avoids the computations of the operator F''. To study its convergence, let F be twice Fréchet differentiable operator in Ω and $\mathcal{BL}(\mathbb{Y}, \mathbb{X})$ be the set of bounded linear operators from $\mathbb{Y}$ into $\mathbb{X}$. It is assumed that $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{BL}(\mathbb{Y}, \mathbb{X})$ exists at some point $x_0 \in \Omega$ and the following conditions hold on F

$$\left.\begin{array}{l} C1.\ \|F'(x_0)^{-1}\| \leq \beta, \\ C2.\ \|F'(x_0)^{-1}F(x_0)\| \leq \eta, \\ C3.\ \|F''(x)\| \leq M,\ \forall\ x \in \Omega, \\ C4.\ \|F''(x) - F''(y)\| \leq \omega(\|x - y\|),\ \forall\ x,\ y \in \Omega, \text{where } \omega : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is a} \\ \qquad \text{continuous and non} - \text{decreasing function such that } \omega(0) \geq 0, \\ C5.\ \text{There exists a continuous and non} - \text{decreasing function } h : [0, 1] \to \mathbb{R}_+, \\ \qquad \text{such that } \omega(tx) \leq h(t)\omega(x),\ \text{with } t \in [0, 1] \text{ and } x \in \mathbb{R}_+. \end{array}\right\} \tag{6}$$

Note that condition C5 of (6) does not involve any restriction, since, as a consequence of ω being a non-decreasing function, there always exists a function h such that $h(t) = 1$. We can consider $h(t) = \sup_{x>0} \omega(tx)/\omega(x)$ to sharpen the error bounds in a given problem. Also, condition $C4$ of (6) is milder than the Lipschitz/Hölder continuity condition as this condition can be reduced to the Lipschitz and the Hölder condition, if we consider $\omega(x) = Nx$ and $\omega(x) = Nx^p$, $p \in (0, 1]$, respectively.

For $n \in \mathbb{Z}_+$, we define three real sequences

$$c_n = f(a_n)g(a_n, b_n),\ a_{n+1} = a_n f(a_n)c_n,\ b_{n+1} = b_n f(a_n)c_n h(c_n) \tag{7}$$

where,

$$f(x) = (2 - x)/(2 - 3x), \tag{8}$$

$$g(x, y) = \Big[\frac{x^2}{(2 - x)^2} + (K_1 + K_2\frac{x}{2 - x})y\Big],\ K_1, K_2 \in \mathbb{R}_+, \tag{9}$$

and $a_0 = M\beta\eta$, $b_0 = \beta\eta\omega(\eta)$, are two parameters. Let r_0 be the smallest positive zero of the polynomial $r(x) = 2x^2 - 3x + 1$, then $r_0 = 0.5$. The following Lemmas establish a number of properties of the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$.

Lemma 2.1. *Let f and g be functions defined by (8) and (9) respectively, then for $x \in (0, r_0]$,*
(i) f is increasing and $f(x) > 1$,
(ii) g is increasing in both arguments for $y > 0$,
(iii) $f(\delta x) < f(x)$ and $g(\delta x, \delta^{p+1} y) < \delta^{p+1} g(x, y)$, for $\delta \in (0,1)$ and $p \in (0,1]$.

Proof. The proof is simple and can be omitted. □

Lemma 2.2. *Let f and g be functions defined by equations (8) and (9) respectively and $h(t) \le 1, \forall t \in [0,1]$. Let us define a function*

$$\Phi(x) = \frac{4(2x^2 - 3x + 1)}{(K_1(2-x) + K_2 x)(2-x)} \tag{10}$$

where K_1, K_2 are positive real numbers. If $0 < a_0 \le r_0$ and $0 \le b_0 \le \Phi(a_0)$, then
(i) $c_n f(a_n) \le 1$,
(ii) $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are decreasing and $a_n < 1, c_n < 1$ $\forall n$.

Proof. Now from definitions of f and g, we have

$$c_n f(a_n) = f(a_n)^2 g(a_n, b_n) \le 1$$

iff

$$\Big(\frac{2-a_n}{2-3a_n}\Big)^2 \Big(\frac{a_n^2}{(2-a_n)^2} + (K_1 + K_2 \frac{a_n}{2-a_n}) b_n\Big) \le 1,$$

or

$$b_n \le \frac{4(2a_n^2 - 3a_n + 1)}{(K_1(2-a_n) + K_2 a_n)(2-a_n)} = \Phi(a_n).$$

We will use induction to prove the Lemma. For $0 < a_0 \le r_0$, $0 \le b_0 \le \Phi(a_0)$, we can easily conclude that $c_0 f(a_0) \le 1$. Thus, from equation (7), we obtain

$$a_1 = a_0 f(a_0) c_0 \le a_0 < 1.$$

As $f(x) > 1$ in $(0, r_0]$ and $h(t) \le 1, \forall t \in [0,1]$, we have

$$b_1 = b_0 f(a_0) c_0 h(c_0) \le b_0 f(a_0) c_0 \le b_0$$

and hence

$$c_1 = f(a_1) g(a_1, b_1) \le f(a_0) g(a_0, b_0) = c_0 < c_0 f(a_0) \le 1.$$

Let our assertion hold for $n = k$. Proceeding similarly, one can easily prove that it holds for $n = k+1$. Thus, by induction hypothesis Lemma 2.2 is proved. □

Lemma 2.3. *Let us suppose $a_0 \in (0, r_0)$, $0 < b_0 < \Phi(a_0)$ and $h(t) \le t^p \le 1, \forall t \in [0,1]$ and $p \in (0,1]$. Define $\gamma = a_1/a_0$, then for $n \ge 1$ we have,*

(i) $a_n \le \gamma^{(2+p)^{n-1}} a_{n-1} \le \gamma^{((2+p)^n - 1)/(1+p)} a_0$, strictly hold for $n \ge 2$,
(ii) $b_n < \left(\gamma^{(2+p)^{n-1}}\right)^{1+p} b_{n-1} < \gamma^{(2+p)^n - 1} b_0$,
(iii) $c_n < \gamma^{(2+p)^n} / f(a_0)$.

Proof. We will prove (i) and (ii) by induction. Since $a_1 = \gamma a_0$ and $a_1 < a_0$ from Lemma 2.2(i), we get $\gamma < 1$. By Lemma 2.1(i) and Lemma 2.2(i) we get

$$b_1 = b_0 f(a_0) c_0 h(c_0) \le b_0 (f(a_0)^2 g(a_0, b_0))(f(a_0) g(a_0, b_0))^p = \Big(\frac{a_1}{a_0}\Big)^{(1+p)} b_0 = \gamma^{(1+p)} b_0.$$

Suppose (i) and (ii) hold for $n = k$, then

$$\begin{aligned} a_{k+1} &= a_k f(a_k)^2 g(a_k, b_k) \\ &< \gamma^{(2+p)^{k-1}} a_{k-1} f(a_{k-1})^2 g\Big(\gamma^{(2+p)^{k-1}} a_{k-1}, \Big(\gamma^{(2+p)^{k-1}}\Big)^{1+p} b_{k-1}\Big) \\ &< \gamma^{(2+p)^{k-1}} a_{k-1} f(a_{k-1})^2 \Big(\gamma^{(2+p)^{k-1}}\Big)^{1+p} g(a_{k-1}, b_{k-1}) = \gamma^{(2+p)^k} a_k. \end{aligned}$$

Also, from $f(x) > 1$ in $(0, r_0)$, we get

$$\begin{aligned} b_{k+1} &= b_k f(a_k) c_k h(c_k) \le b_k (f(a_k)^2 g(a_k, b_k))(f(a_k) g(a_k, b_k))^p \\ &= b_k (f(a_k)^2 g(a_k, b_k))^{1+p} = b_k \Big(\frac{a_{k+1}}{a_k}\Big)^{1+p} < \Big(\gamma^{(2+p)^k}\Big)^{1+p} b_k. \end{aligned}$$

Hence,

$$a_{k+1} < \gamma^{(2+p)^k} a_k < \gamma^{(2+p)^k} \gamma^{(2+p)^{k-1}} \cdots \gamma^{(2+p)^0} a_0 = \gamma^{((2+p)^{k+1}-1)/(1+p)} a_0.$$

and

$$b_{k+1} < \Big(\gamma^{(2+p)^k}\Big)^{1+p} b_k < \Big(\gamma^{(2+p)^k}\Big)^{1+p} \cdots \Big(\gamma^{(2+p)^0}\Big)^{1+p} b_0 = \gamma^{(2+p)^{k+1}-1} b_0.$$

Thus, (i) and (ii) hold by induction. Condition (iii) follows from

$$\begin{aligned} c_n = f(a_n) g(a_n, b_n) &\le f(\gamma^{((2+p)^n-1)/(1+p)} a_0) g(\gamma^{((2+p)^n-1)/(1+p)} a_0, \gamma^{(2+p)^n-1} b_0) \\ &< \gamma^{(2+p)^n} \frac{f(a_0) g(a_0, b_0)}{\gamma} = \gamma^{(2+p)^n} / f(a_0) \end{aligned}$$

as $\gamma = a_1/a_0 = f(a_0)^2 g(a_0, b_0)$. □

3. A Family of Recurrence Relations

In this section, a family of recurrence relations are derived for the method (5) under the assumptions of Section 2. Since, existence of $\Gamma_0 = F'(x_0)^{-1}$ gives existence of y_0, we get $M\|\Gamma_0\|\|y_0 - x_0\| \le M\beta\eta = a_0$. Also,

$$\|\frac{1}{2} H(x_0, y_0)\| \le \frac{1}{2} M \|\Gamma_0\| \|y_0 - x_0\| \le \frac{a_0}{2} < 1.$$

Hence, $\Big(I + \frac{1}{2} H(x_0, y_0)\Big)^{-1}$ exists by Banach theorem [15, p.155], and

$$\Big\|\Big(I + \frac{1}{2} H(x_0, y_0)\Big)^{-1}\Big\| \le \frac{1}{1 - a_0/2} = \frac{2}{2 - a_0}.$$

Thus, $\|Q(x_0, y_0)\| \le a_0/(2 - a_0)$. Hence,

$$\|x_1 - y_0\| = \|Q(x_0, y_0)(y_0 - x_0)\| \le \frac{a_0}{2 - a_0} \|y_0 - x_0\|$$

and

$$\|x_1 - x_0\| \le \|x_1 - y_0\| + \|y_0 - x_0\| \le \frac{2}{2-a_0}\|y_0 - x_0\|.$$

Also,

$$\|\Gamma_0\|\|y_0 - x_0\|\omega(\|y_0 - x_0\|) \le \beta\eta\omega(\eta) = b_0.$$

The following inequalities can now be proved for $n \ge 1$:

$$\left.\begin{array}{l} (I)\ \|\Gamma_n\| = \|F'(x_n)^{-1}\| \le f(a_{n-1})\|\Gamma_{n-1}\|, \\ (II)\ \|y_n - x_n\| = \|\Gamma_n F(x_n)\| \le c_{n-1}\|y_{n-1} - x_{n-1}\|, \\ (III)\ M\|\Gamma_n\|\|y_n - x_n\| \le a_n, \\ (IV)\ \|x_{n+1} - y_n\| \le \frac{a_n}{2-a_n}\|y_n - x_n\|, \\ (V)\ \|x_{n+1} - x_n\| \le \frac{2}{2-a_n}\|y_n - x_n\|, \\ (VI)\ \|\Gamma_n\|\|y_n - x_n\|\omega(\|y_n - x_n\|) \le b_n. \end{array}\right\} \tag{11}$$

Now, induction can be used to prove conditions $(I)-(VI)$. Assume that $x_1 \in \Omega$. We now have

$$\|I - \Gamma_0 F'(x_1)\| \le M\|\Gamma_0\|\|x_0 - x_1\| \le \frac{2}{2-a_0}M\|\Gamma_0\|\|y_0 - x_0\| \le \frac{2a_0}{2-a_0} < 1.$$

Hence, by Banach theorem, $\Gamma_1 = F'(x_1)^{-1}$ exists and

$$\|\Gamma_1\| \le \frac{\|\Gamma_0\|}{1 - M\|\Gamma_0\|\|x_0 - x_1\|} \le \frac{2-a_0}{2-3a_0}\|\Gamma_0\| = f(a_0)\|\Gamma_0\|. \tag{12}$$

Thus, y_1 exists as Γ_1 exists. Note that $x_{n+1} - y_n = Q(x_n, y_n)(y_n - x_n)$ and applying Taylor's method, we can easily obtain

$$\begin{aligned} F(x_{n+1}) = &\tfrac{1}{2}\int_0^1 F''(x_n + t(y_n - x_n))(y_n - x_n)Q(x_n, y_n)(y_n - x_n)dt \\ &+ \int_0^1 [F''(x_n + t(y_n - x_n)) - F''(x_n)](1-t)(y_n - x_n)^2 dt \\ &+ \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)(Q(x_n, y_n)(y_n - x_n))^2 dt \\ &+ \tfrac{1}{2}\int_0^1 [F''(x_n) - F''(x_n + \theta t(y_n - x_n))](y_n - x_n)^2 dt \\ &- \tfrac{1}{2}\int_0^1 [F''(x_n + \theta t(y_n - x_n)) - F''(x_n + t(y_n - x_n))] \\ &\quad \times (y_n - x_n)Q(x_n, y_n)(y_n - x_n)dt. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|F(x_1)\| \le\ & \frac{M}{2}\|y_0 - x_0\|\|x_1 - y_0\| + \int_0^1 \omega(t\|y_0 - x_0\|)(1-t)\|y_0 - x_0\|^2 dt \\ & + \frac{M}{2}\|x_1 - y_0\|^2 + \frac{1}{2}\int_0^1 \omega(\theta t\|y_0 - x_0\|)\|y_0 - x_0\|^2 dt \\ & + \frac{1}{2}\int_0^1 \omega(t(1-\theta)\|y_0 - x_0\|)\|y_0 - x_0\|\|x_1 - y_0\| dt \\ \le\ & \frac{M}{2}\|y_0 - x_0\|\|x_1 - y_0\| + \int_0^1 h(t)(1-t)dt\,\omega(\|y_0 - x_0\|)\|y_0 - x_0\|^2 \\ & + \frac{M}{2}\|x_1 - y_0\|^2 + \frac{1}{2}\int_0^1 h(\theta t)dt\,\omega(\|y_0 - x_0\|)\|y_0 - x_0\|^2 \end{aligned}$$

$$+\frac{1}{2}\int_0^1 h(t(1-\theta))\omega(\|y_0-x_0\|)\|y_0-x_0\|\|x_1-y_0\|dt$$
$$\leq M\frac{a_0}{(2-a_0)^2}\|y_0-x_0\|^2+\Big(K_1+\frac{K_2a_0}{2-a_0}\Big)\omega(\|y_0-x_0\|)\|y_0-x_0\|^2,$$

where $K_1=\int_0^1 h(t)(1-t)dt+\frac{1}{2}\int_0^1 h(\theta t)dt$, $K_2=\frac{1}{2}\int_0^1 h(t(1-\theta))dt$.
This gives

$$\begin{aligned}\|\Gamma_1F(x_1)\| &\leq \|\Gamma_1\|\|F(x_1)\|\\ &\leq f(a_0)\|\Gamma_0\|\Big[M\frac{a_0}{(2-a_0)^2}\|y_0-x_0\|^2\\ &\quad+\Big(K_1+\frac{K_2a_0}{2-a_0}\Big)\omega(\|y_0-x_0\|)\|y_0-x_0\|^2\Big]\\ &\leq f(a_0)\Big[\frac{a_0^2}{(2-a_0)^2}+\Big(K_1+\frac{K_2a_0}{2-a_0}\Big)b_0\Big]\|y_0-x_0\|\\ &= f(a_0)g(a_0,b_0)\|y_0-x_0\|=c_0\|y_0-x_0\|. \qquad (13)\end{aligned}$$

Also,

$$M\|\Gamma_1\|\|y_1-x_1\|\leq M\|\Gamma_0\|f(a_0)c_0\|y_0-x_0\|\leq a_0f(a_0)c_0=a_1. \qquad (14)$$

Thus,

$$\|\frac{1}{2}H(x_1,y_1)\|\leq\frac{1}{2}M\|\Gamma_1\|\|y_1-x_1\|\leq\frac{a_1}{2}<1.$$

Hence, $\Big(I+\frac{1}{2}H(x_1,y_1)\Big)^{-1}$ exists by Banach theorem, and

$$\Big\|\Big(I+\frac{1}{2}H(x_1,y_1)\Big)^{-1}\Big\|\leq\frac{1}{1-a_1/2}=\frac{2}{2-a_1}.$$

Thus, $\|Q(x_1,y_1)\|\leq a_1/(2-a_1)$ and hence,

$$\|x_2-y_1\|=\|Q(x_1,y_1)(y_1-x_1)\|\leq\frac{a_1}{2-a_1}\|y_1-x_1\| \qquad (15)$$

and

$$\|x_2-x_1\|=\|x_2-y_1\|+\|y_1-x_1\|=\frac{2}{2-a_1}\|y_1-x_1\|. \qquad (16)$$

Again,

$$\begin{aligned}\|\Gamma_1\|\|y_1-x_1\|\omega(\|y_1-x_1\|) &\leq \|\Gamma_0\|f(a_0)c_0\|y_0-x_0\|\omega(c_0\|y_0-x_0\|)\\ &\leq f(a_0)c_0h(c_0)\|\Gamma_0\|\|y_0-x_0\|\omega(\|y_0-x_0\|)\\ &\leq b_0f(a_0)c_0h(c_0)=b_1. \qquad (17)\end{aligned}$$

Hence, for $n=1$, the conditions $(I)-(VI)$ follow from equations (12)-(17) respectively. Let these statements hold for $n=k$ and $x_k\in\Omega$. Proceeding similarly as above it can be easily proved that these conditions also hold for $n=k+1$. Hence, by induction it holds for all n.

4. Convergence Analysis

The following theorem will establish the convergence of the sequence $\{x_n\}$ and give a priori error bounds for it. Let us denote $\gamma = a_1/a_0$ and $\Delta = 1/f(a_0)$, $R = \frac{2}{(2-a_0)(1-\gamma\Delta)}$.

Theorem 4.1. *Let $0 < a_0 \leq r_0$ and $0 \leq b_0 \leq \Phi(a_0)$ hold, where r_0 is the smallest positive zero of the polynomial $r(x) = 2x^2 - 3x + 1$ and $\Phi(x)$ be the function defined by equation (10). Also let $h(t) \leq t^p \leq 1, \forall t \in [0,1]$ and $p \in (0,1]$, where $h(t)$ is defined by $C5$ of (6). Under the assumption given in (6) on F and $\bar{\mathcal{B}}(x_0, R\eta) \subseteq \Omega$, the method (5) starting from x_0 generates a sequence $\{x_n\}$ converging to the root x^* of (1) with R-order at least $p+2$. In this case x_n, y_n and x^* lie in $\bar{\mathcal{B}}(x_0, R\eta)$ and the solution x^* is unique in $\mathcal{B}(x_0, 2/(M\beta) - R\eta) \cap \Omega$. Furthermore, the error bounds on x^* is given by*

$$\|x^* - x_n\| \leq \frac{2\gamma^{((2+p)^n-1)/(1+p)}}{(2-\gamma^{((2+p)^n-1)/(1+p)}a_0)} \frac{\Delta^n}{(1-\gamma^{(2+p)^n}\Delta)}\eta. \tag{18}$$

Proof. It is sufficient to show that $\{x_n\}$ is a Cauchy sequence in order to establish the convergence of $\{x_n\}$. Now from (11), we have

$$\|y_n - x_n\| \leq c_{n-1}\|y_{n-1} - x_{n-1}\| \leq \cdots \leq \Big(\prod_{j=0}^{n-1} c_j\Big)\|y_0 - x_0\| \leq \Big(\prod_{j=0}^{n-1} c_j\Big)\eta \tag{19}$$

and

$$\begin{aligned} \|x_{m+n} - x_m\| &\leq \|x_{m+n} - x_{m+n-1}\| + \cdots + \|x_{m+1} - x_m\| \\ &\leq \frac{2}{2-a_{m+n-1}}\|y_{m+n-1} - x_{m+n-1}\| + \cdots + \frac{2}{2-a_m}\|y_m - x_m\| \\ &\leq \frac{2}{2-a_{m+n-1}}\Big(\prod_{j=0}^{m+n-2} c_j\Big)\eta + \cdots + \frac{2}{2-a_m}\Big(\prod_{j=0}^{m-1} c_j\Big)\eta \\ &\leq \frac{2}{2-a_m}\Big[\prod_{j=0}^{m+n-2} c_j + \cdots + \prod_{j=0}^{m-1} c_j\Big]\eta. \end{aligned} \tag{20}$$

Now, for $a_0 = r_0$, we have $b_0 = \Phi(a_0) = 0$. Hence, from Lemma 2.2, we obtain $c_0 f(a_0) = 1$, $a_n = a_{n-1} = \cdots = a_0$, $c_n = c_{n-1} = \cdots = c_0$ and $b_n = b_{n-1} \cdots = b_0 = 0$. Thus,

$$\|y_n - x_n\| \leq c_0^n\eta = \Delta^n\eta$$

and

$$\|x_{m+n} - x_m\| \leq \frac{2}{2-a_0}\Big[\Delta^{m+n-1} + \cdots + \Delta^m\Big]\eta = \frac{2\Delta^m}{2-a_0}\Big(\frac{1-\Delta^n}{1-\Delta}\Big)\eta. \tag{21}$$

Hence, if we take $m = 0$, we have

$$\|x_n - x_0\| \leq \frac{2}{2-a_0}\Big(\frac{1-\Delta^n}{1-\Delta}\Big)\eta. \tag{22}$$

Thus, $x_n \in \bar{\mathcal{B}}(x_0, R\eta)$. Similarly, we can prove that $y_n \in \bar{\mathcal{B}}(x_0, R\eta)$. As $\Delta < 1$, from the above equation (21), we can conclude that $\{x_n\}$ is a Cauchy sequence. Let $0 < a_0 < r_0$ and $0 < b_0 < \Phi(a_0)$. Now, from (19) and Lemma 2.3(iii), for $n \geq 1$, we have

$$\|y_n - x_n\| \leq \Big(\prod_{j=0}^{n-1} c_j\Big)\eta < \prod_{j=0}^{n-1}\Big(\gamma^{(2+p)^j}\Delta\Big)\eta = \gamma^{((2+p)^n-1)/(1+p)}\Delta^n\eta.$$

Hence, from (20), we obtain

$$\begin{aligned}
\|x_{m+n} - x_m\| &\leq \frac{2}{2-a_m}\Big[\Big(\prod_{j=0}^{m+n-2} c_j\Big)\eta + \cdots + \Big(\prod_{j=0}^{m-1} c_j\Big)\eta\Big] \\
&< \frac{2}{2-a_m}\Big[\gamma^{((2+p)^{m+n-1}-1)/(1+p)}\Delta^{m+n-1} + \cdots \\
&\qquad + \gamma^{((2+p)^m-1)/(1+p)}\Delta^m\Big]\eta \\
&= \frac{2\Delta^m}{2-a_m}\Big[\gamma^{((2+p)^{m+n-1}-1)/(1+p)}\Delta^{n-1} + \cdots \\
&\qquad + \gamma^{((2+p)^m-1)/(1+p)}\Big]\eta \\
&< \frac{2\gamma^{((2+p)^m-1)/(1+p)}\Delta^m}{2-\gamma^{((2+p)^m-1)/(1+p)}a_0}\Big[\gamma^{((2+p)^m[(2+p)^{n-1}-1])/(1+p)}\Delta^{n-1} \\
&\qquad + \gamma^{((2+p)^m[(2+p)^{n-2}-1])/(1+p)}\Delta^{n-2} + \cdots \\
&\qquad + \gamma^{((2+p)^m[(2+p)-1])/(1+p)}\Delta + 1\Big]\eta.
\end{aligned}$$

By Bernoulli's inequality, for every real number $x > -1$ and every integer $k \geq 0$, we have $(1+x)^k - 1 \geq kx$. Thus, we get

$$\|x_{m+n} - x_m\| < \frac{2\gamma^{((2+p)^m-1)/(1+p)}\Delta^m}{(2-\gamma^{((2+p)^m-1)/(1+p)}a_0)}\frac{1-\gamma^{(2+p)^m n}\Delta^n}{(1-\gamma^{(2+p)^m}\Delta)}\eta. \tag{23}$$

Thus, for $m = 0$, we get

$$\|x_n - x_0\| < \frac{2}{2-a_0}\frac{1-\gamma^n\Delta^n}{1-\gamma\Delta}\eta < R\eta. \tag{24}$$

Hence, $x_n \in \mathcal{B}(x_0, R\eta)$. Also, $y_n \in \mathcal{B}(x_0, R\eta)$, is evident from the following result.

$$\begin{aligned}
\|y_{n+1} - x_0\| &\leq \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \cdots + \|x_1 - x_0\| \\
&\leq \|y_{n+1} - x_{n+1}\| + \frac{2}{2-a_n}\|y_n - x_n\| + \cdots + \frac{2}{2-a_0}\|y_0 - x_0\| \\
&< \frac{2}{2-a_{n+1}}\|y_{n+1} - x_{n+1}\| + \cdots + \frac{2}{2-a_0}\|y_0 - x_0\| \\
&< \cdots < \frac{2}{2-a_0}\frac{1-\gamma^{n+2}\Delta^{n+2}}{1-\gamma\Delta}\eta < R\eta.
\end{aligned}$$

Taking limit $n \to \infty$ as in (22) and (24), we get $x^* \in \bar{\mathcal{B}}(x_0, R\eta)$. Now, we have to show that x^* is a solution of $F(x) = 0$. For this we have $\|F(x_n)\| \le \|F'(x_n)\|\|\Gamma_n F(x_n)\|$ and the sequence $\{\|F'(x_n)\|\}$ is bounded as

$$\|F'(x_n)\| \le \|F'(x_0)\| + M\|x_n - x_0\| < \|F'(x_0)\| + MR\eta.$$

Now, taking limit $n \to \infty$, we get $F(x^*) = 0$ as F is continuous. To show the uniqueness of the root x^*, let y^* be another root of (1) in $\mathcal{B}(x_0, 2/(M\beta) - R\eta) \cap \Omega$. But, we have

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))dt(y^* - x^*).$$

Thus, we have $y^* = x^*$, if the operator $P = \int_0^1 F'(x^* + t(y^* - x^*))dt$ is invertible. From

$$\begin{aligned}\|I - \Gamma_0 P\| = \|\Gamma_0(F'(x_0) - P)\| &= \|\Gamma_0 \int_0^1 [F'(x^* + t(y^* - x^*)) - F'(x_0)]dt\| \\ &\le M\beta \int_0^1 \|x^* + t(y^* - x^*) - x_0\|dt \\ &\le M\beta \int_0^1 ((1-t)\|x^* - x_0\| + t\|y^* - x_0\|)dt \\ &< \frac{M\beta}{2}\Big(R\eta + \frac{2}{M\beta} - R\eta\Big) = 1,\end{aligned}$$

and by Banach theorem, P is invertible. □

Remark 4.1. Note that if $\omega(x) = Nx^p, N \ge 0, p \in (0, 1]$, then $\omega(tx) \le t^p\omega(x)$ and F'' is (N, p)-Hölder continuous in Ω. So, the R-order of convergence is at least $2 + p$.

Remark 4.2. Also, note that if $\omega(x) = Nx, N \ge 0$ and $p = 1$ then $\omega(tx) \le t\omega(x)$ and F'' is N-Lipschitz continuous in Ω. So, the R-order of convergence is at least 3.

5. Numerical Examples

In this section, two numerical examples are worked out for demonstrating the efficacy of our approach.

Example 1. Let $\mathbb{X} = C[a, b]$ be the space of continuous functions on $[a, b]$ and consider the problem of finding the solutions of nonlinear integral equations $F(x) = 0$ of mixed type [9], given by

$$F(x)(s) = x(s) - f(s) - \lambda \int_a^b G(s,t)[x(t)^{2+p} + x(t)^3]dt, \; p \in (0, 1], \; \lambda \in \mathbb{R} \quad (25)$$

where, f, x are continuous functions such that $f(s) > 0$, $s \in [a, b]$, and the Kernel G is continuous and nonnegative in $[a, b] \times [a, b]$.

Solution- For the solution of the problem, we have taken the norm as sup-norm and $G(s,t)$ as the Green's function

$$G(s,t) = \begin{cases} (b-s)(t-a)/(b-a), & t \le s, \\ (s-a)(b-t)/(b-a), & s \le t, \end{cases}$$

Now, the first and second derivatives of F can easily be obtained and given by

$$F'(x)u(s) = u(s) - \lambda \int_a^b G(s,t)[(2+p)x(t)^{1+p} + 3x(t)^2]u(t)dt, \qquad u \in \Omega,$$

$$F''(x)(uv)(s) = -\lambda \int_a^b G(s,t)[(1+p)(2+p)x(t)^p + 6x(t)](uv)(t)dt, \qquad u, v \in \Omega.$$

For $p \in (0,1)$, we must note here that the second derivative F'' does not satisfy the Lipschitz/Hölder continuity condition, as

$$\begin{aligned} \|F''(x) &- F''(y)\| \\ &= \left\|\lambda \int_a^b G(s,t)[(1+p)(2+p)(x(t)^p - y(t)^p) + 6(x(t)-y(t))]dt\right\| \\ &\le |\lambda| \max_{s\in[a,b]} \left|\int_a^b G(s,t)dt\right| \Big[(1+p)(2+p)\|x(t)^p - y(t)^p\| + 6\|x(t)-y(t)\|\Big] \\ &\le |\lambda|\|l\|[(1+p)(2+p)\|x-y\|^p + 6\|x-y\|], \ \forall x, y \in \Omega, \end{aligned}$$

where

$$\|l\| = \max_{s\in[a,b]} \left|\int_a^b G(s,t)dt\right|.$$

Thus, convergence of our method under Lipschitz/Hölder continuity condition on F'' can not be tested for this problem. However, it satisfies the ω- continuity condition given by

$$\|F''(x) - F''(y)\| \le \omega(\|x-y\|), \ \forall x, y \in \Omega,$$

where, $\omega(x) = |\lambda|\|l\|[(1+p)(2+p)x^p + 6x]$. This leads to $\omega(tx) \le t^p\omega(x)$, for $p \in (0,1]$ and $t \in [0,1]$. Thus, $h(t) = t^p$, and hence $K_1 = \int_0^1 h(t)(1-t)dt + \frac{1}{2}\int_0^1 h(\theta t)dt = \frac{1}{(1+p)(2+p)} + \frac{\theta^p}{2(1+p)}$ and $K_2 = \frac{1}{2}\int_0^1 h(t(1-\theta))dt = \frac{(1-\theta)^p}{2(1+p)}$.
It is easy to compute

$$\|F(x_0)\| \le \|x_0 - f\| + |\lambda|\|l\|[\|x_0\|^{2+p} + \|x_0\|^3]$$

and

$$\|F''(x)\| \le |\lambda|\|l\|\Big[(1+p)(2+p)\|x\|^p + 6\|x\|\Big].$$

This gives $M = |\lambda|\|l\|\Big[(1+p)(2+p)\|x\|^p + 6\|x\|\Big]$. Also,

$$\|I - F'(x_0)\| \le |\lambda|\|l\|[(2+p)\|x_0\|^{1+p} + 3\|x_0\|^2]$$

and if $|\lambda|\|l\|[(2+p)\|x_0\|^{1+p} + 3\|x_0\|^2] < 1$, then by Banach theorem [15], we obtain

$$\|\Gamma_0\| = \|F'(x_0)^{-1}\| \le \frac{1}{1 - |\lambda|\|l\|[(2+p)\|x_0\|^{1+p} + 3\|x_0\|^2]} = \beta$$

and

$$\|\Gamma_0 F(x_0)\| \leq \frac{\|x_0 - f\| + |\lambda|\|l\|[\|x_0\|^{2+p} + \|x_0\|^3]}{1 - |\lambda|\|l\|[(2+p)\|x_0\|^{1+p} + 3\|x_0\|^2]} = \eta.$$

For $a = 0$ and $b = 1$, we get

$$\|l\| = \max_{s\in[0,1]} \Big| \int_0^1 G(s,t)dt \Big| = 1/8.$$

For $\lambda = 1/3$, $p = 1/2$, $f(s) = 1$, $\theta = 0.5$ and initial point $x_0 = x_0(s) = 1$ in $[0,1]$, we get $\|\Gamma_0\| \leq \beta = 1.2973$, $\|\Gamma_0 F(x_0)\| \leq \eta = 0.108108$, $\omega(\eta) = 0.0784017$ and $b_0 = \beta\eta\omega(\eta) = 0.0109957$. Now we look for a domain in the form of $\Omega = \mathcal{B}(x_0, S)$ such that

$$\Omega = \mathcal{B}(x_0, S) \subseteq C[0,1] = \mathbb{X}.$$

Thus, we get $M = M(S) = 0.15625S^p + 0.25S$ and $a_0 = a_0(S) = M(S)\beta\eta = 0.0219138S^p + 0.03506208S$. To calculate S, from the condition of theorem 4.1 it is necessary that $\bar{\mathcal{B}}(x_0, R\eta) \subseteq \Omega$. For this it is sufficient to check $S-(R(S)\eta+1) > 0$ and $\Phi(a_0(S)) - b_0 > 0$. Hence it is necessary that $S \in (1.11237, 11.9938)$ as is evident from fig. 1. In fact fig. 1 gives a sufficient condition on the parameter S such that $S - (R(S)\eta + 1) > 0$ and $\Phi(a_0(S)) - b_0 > 0$. Also $a_0(S) < r_0 = 0.5$, if and only if $S < 12.0875$. Hence if we choose $S = 8$, then we have $\Omega = \mathcal{B}(1, 8)$, $M = 2.44194$, $a_0 = 0.342478$ and $b_0 = 0.0109957 < 0.5472867 = \Phi(0.342478)$. Thus the conditions of the theorem 4.1 are satisfied. Hence a solution of equation (25) exists in the ball $\bar{\mathcal{B}}(1, 0.142266) \subseteq \Omega$ and is unique in the ball $\mathcal{B}(1, 0.489062) \cap \Omega$. On the other hand if we consider the work of Ezquerro and Hernández[7], $A = \int_0^1 (1-t)t^p dt = \frac{1}{(1+p)(2+p)}$. Here we denote the function $\frac{2(x^2-3x+1)}{A(2-x)}$ as $\Psi(x)$. Now in order to calculate S, from the condition of theorem 2.6 of that paper, it is also necessary that $\bar{\mathcal{B}}(x_0, R\eta) \subseteq \Omega$. For this it is sufficient to check $S - (R(S)\eta + 1) > 0$ and $\Psi(a_0(S)) - b_0 > 0$. Hence it is necessary that $S \in (1.11206, 8.98979)$ as is evident from fig. 2. The definitions of R also comes from of that paper. As described above, fig. 2 gives a domain of the paprameter S in which $S - (R(S)\eta + 1) > 0$ and $\Psi(a_0(S)) - b_0 > 0$.

Also $a_0(S) < \frac{3-\sqrt{5}}{2}$, if and only if $S < 9.0172$. Hence if we choose $S = 8$, then we have $\Omega = \mathcal{B}(1, 8)$, $M = 2.44194$, $a_0 = 0.342478$ and $b_0 = 0.0109957 < 0.4065847 = \Psi(0.342478)$. Thus the conditions of the theorem 2.6 are satisfied. Hence a solution of equation (25) exists in the ball $\bar{\mathcal{B}}(1, 0.141602) \subseteq \Omega$ and is unique in the ball $\mathcal{B}(1, 0.489726) \cap \Omega$. From this we conclude that our convergence analysis gives better existence ball than that of Ezquerro and Hernández.

Example 2. Let us consider a integral equation given in [13]

$$F(x)(s) = x(s) - s + \frac{1}{2}\int_0^1 s\cos(x(t))dt, \tag{26}$$

where x is a continuous function and $F : \Omega \subseteq C[0,1] \to C[0,1]$.

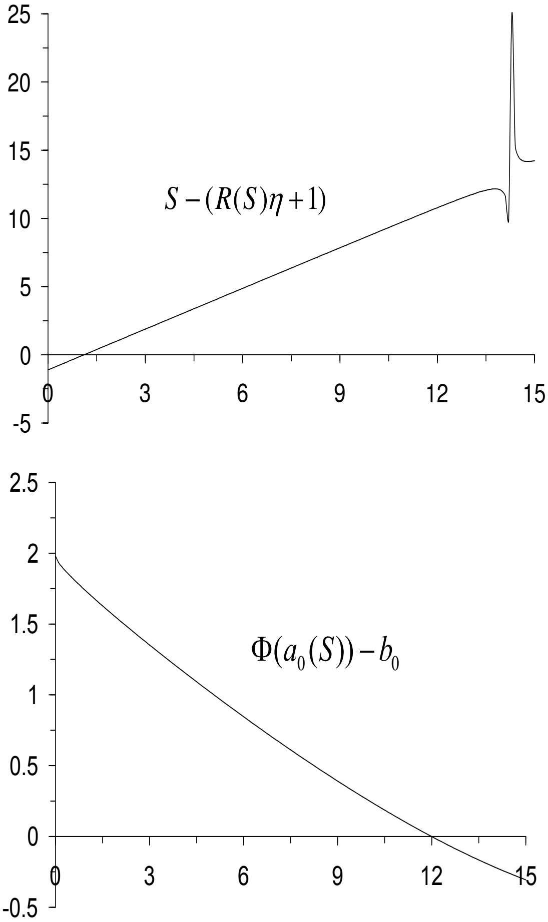

FIGURE 1. Domain for the parameter S for $\Phi(a_0(S)) - b_0 > 0$.

Solution- Here we used sup norm in $C[0, 1]$. Now it is easy to find the first and second derivatives of F as given by

$$F'(x)u(s) = u(s) - \frac{s}{2}\int_0^1 \sin(x(t))u(t)dt,$$

$$F''(x)(uv)(s) = -\frac{s}{2}\int_0^1 \cos(x(t))u(t)v(t)dt.$$

Note that one can easily prove that F'' satisfy the Lipschitz continuity condition as

$$\|F''(x) - F''(y)\| = \|\frac{s}{2}\int_0^1 [\cos(x(t)) - \cos(y(t))]dt\| \leq \frac{1}{2}\|x - y\|.$$

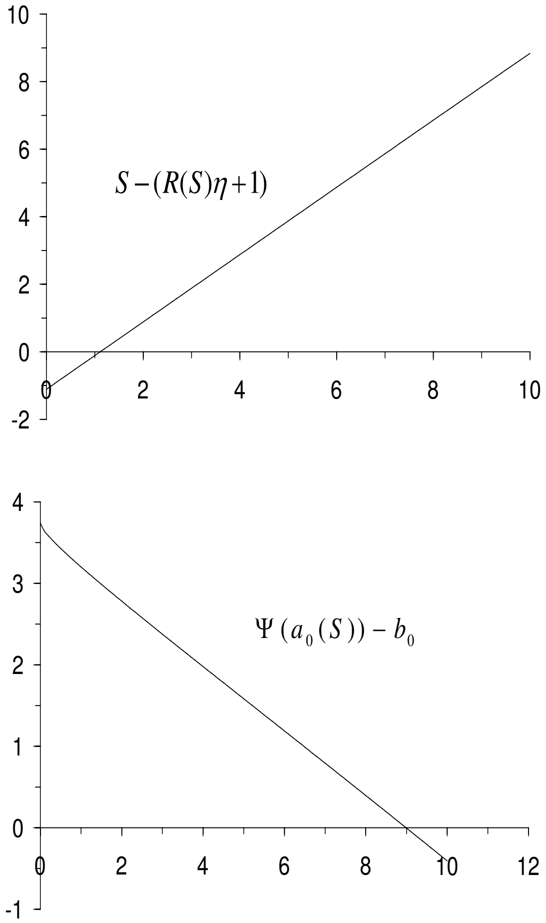

FIGURE 2. Domain for the parameter S for $\Psi(a_0(S)) - b_0 > 0$.

Hence $\omega(x) = Nx$, where $N = \frac{1}{2}$. This leads to $\omega(tx) = t\omega(x)$, for $t \in [0,1]$. Thus, $h(t) = t$, and hence $K_1 = \int_0^1 h(t)(1-t)dt + \frac{1}{2}\int_0^1 h(\theta t)dt = \frac{1}{6} + \frac{\theta}{4}$ and $K_2 = \frac{1}{2}\int_0^1 h(t(1-\theta))dt = \frac{1-\theta}{4}$.
Also we can easily conclude that $\|F''(x)\| \leq \frac{1}{2} = M$.
If we choose $x_0 = x_0(s) = s$, it is easy to find

$$\|\Gamma_0\| \leq \frac{3-\sin 1}{2-\sin 1+\cos 1} = \beta = 1.2705964...,$$

$$\|\Gamma_0 F(x_0)\| \leq \frac{\sin 1}{2-\sin 1+\cos 1} = \eta = 0.4953234...,$$

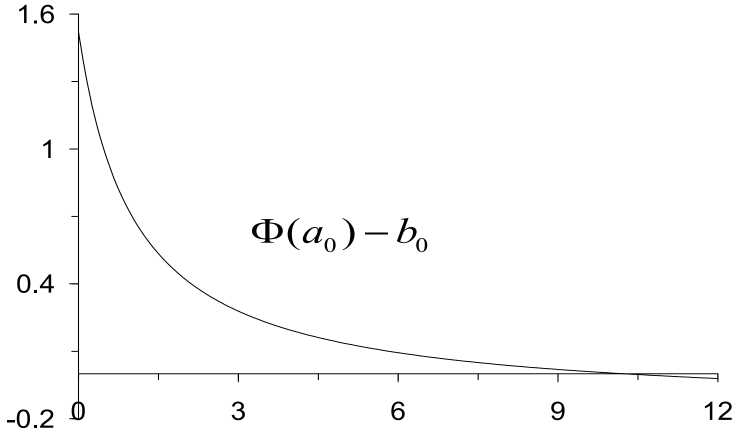

FIGURE 3. θ-domain for $\Phi(a_0) - b_0$

for details see [13]. Hence $a_0 = M\beta\eta = 0.314678... < r_0 = 0.5$ and $b_0 = \beta\eta\omega(\eta) = 0.155867... \leq \Phi(a_0)$, for all $\theta \in (0, 1]$, see figure 3. It is to be mentioned here that the figure 3 gives a domain on θ such that $\Phi(a_0) - b_0$ is positive. Thus the conditions of the theorem 4.1 are satisfied. Hence a solution of equation (26) exists in the ball $\bar{\mathcal{B}}(x_0, 0.674825) \subseteq \Omega$ and is unique in the ball $\mathcal{B}(x_0, 2.4733) \cap \Omega$.

On the other hand for the case of Ezquerro and Hernández[7], $a_0 = M\beta\eta = 0.314678... < \frac{3-\sqrt{5}}{2}$ and $b_0 = \beta\eta\omega(\eta) = 0.155867... \leq \frac{2(a_0^2 - 3a_0 + 1)}{A(2-a_0)} = 1.103561$, where $A = \int_0^1 (1-t)h(t)dt = 1/6$. Thus assumptions of Theorem 2.6 and Theorem 2.8 are satisfied. Hence a solution of equation (26) exists in the ball $\bar{\mathcal{B}}(x_0, 0.656655) \subseteq \Omega$ and unique in the ball $\mathcal{B}(x_0, 2.49147) \cap \Omega$.

From this we conclude that our convergence analysis gives a better existence ball than that of Ezquerro and Hernández.

6. Conclusions

The semilocal convergence of a third order iterative method used for solving nonlinear operator equations in Banach spaces is established by using recurrence relations under the assumption that the second Fréchet derivative of the involved operator satisfies the ω-continuity condition. This $\omega-$continuous condition is milder than the usual Lipschitz/Hölder continuity condition. A family of recurrence relations based on two constants depending on the F is derived. An existence-uniqueness theorem is established to show that the R-order convergence of the method is $(2+p)$, where $p \in (0, 1]$. A priori error bounds for the method is also derived. Two numerical examples are worked out to demonstrate the efficacy of our approach. It is observed that our results are better than those obtained by Ezquerro and Hernández[7].

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