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IMAGE RESTORATION BY THE GLOBAL CONJUGATE GRADIENT LEAST SQUARES METHOD

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ABSTRACT. A variant of the global conjugate gradient method for solving general linear systems with multiple right-hand sides is proposed. This method is called as the global conjugate gradient linear least squares (Gl-CGLS) method since it is based on the conjugate gradient least squares method(CGLS). We present how this method can be implemented for the image deblurring problems with Neumann boundary conditions. Numerical experiments are tested on some blurred images for the purpose of comparing the computational efficiencies of Gl-CGLS with CGLS and Gl-LSQR. The results show that Gl-CGLS method is numerically more efficient than others for the ill-posed problems.

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1. Introduction

The collection of several sparse linear systems with the same coefficient matrix but many different right-hand sides can be written as

$$AX = B, \tag{1}$$

where A is an $N \times N$ nonsingular and nonsymmetric coefficient matrix, and the columns $b^{(i)}$ and $x^{(i)}$ of B and X are the right-hand side and the solution of the sparse linear system $Ax^{(i)} = b^{(i)}, i = 1, 2, \dots, s$ respectively. In practice, using a method for all the systems simultaneously is more efficient than applying iterative methods to each linear system, when s, the number of columns in B and X, is moderately less than N. The Gl-LSQR method, a global version of least squares QR(LSQR) algorithm [13] and a generalization of the classical Krylov subspace methods, has been developed by reducing the coefficient matrix A to a lower bidiagonal matrix form and formulating a simple recurrence relation of the approximate solutions $\{X_k\}$. Their tests in [13] show that the Gl-LSQR method

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has some advantages over some other global methods, such as the global FOM and the global GMRES algorithms based on matrix Krylov subspace methods [5], in the memory storage and computational cost points of view. To solve symmetric positive definite linear system of equations with multiple right-hand sides, Salkuyeh had extracted the global conjugate gradient method (Gl-CG) from Gl-FOM and Gl-GMRES [10]. His numerical tests show that the global CG-type algorithms are often less cost effective than that of the just conjugate gradient algorithm applied to a sequence of right-hand sides.

The regularized deblurring problems, illustrating the general space-invariant imaging system, are often modeled as a linear least squares problems:

$$\min_x(\|Hx - b\|_2^2 + \lambda^2 \|Lx\|_2^2), \tag{2}$$

where H is the $M \times N$ blurring ill-conditioned matrix with some block structures, and b and x represent the observed and the original image respectively. The regularization parameter λ is positive, and the regularization operator L is selected to achieve a solution with desirable properties [3].

The preconditioned conjugate gradient least squares (PCGLS) method or its variants can be applied to the normal equations of (2) with symmetric positive definite coefficient matrix [1]. CGLS is an iterative method with similar qualitative properties as LSQR, but requires less work per iteration and more storages than LSQR [8, 9]. It is known that if the preconditioned and regularized coefficient matrix of (2) is well conditioned, PCGLS is somewhat more efficient than the preconditioned LSQR.

Since the implementation of the image restoration problem typically requires the need of formidable data, the blurred and noisy image can be partitioned into small blocks based on the size of the point spread function and the scheme to process the image in blocks(each block as a column) is often used to reduce the memory storage as well as the execution time to achieve the same effects and results [12]. Collection of image restoration problems for each block image will be transformed into linear system with s multiple right hand sides($N \times s$ matrix B) resulting in minimization problem with respect to the Frobenius norm [2, 7],

$$\min_X\{\|HX - B\|_F^2 + \lambda^2 \|LX\|_F^2\}. \tag{3}$$

In this work, Gl-CGLS method is suggested and implemented as a solver of image deblurring problems with Neumann boundary conditions. Numerical experiments are tested on some blurred images for the purpose of comparing the computational efficiencies of Gl-CGLS with CGLS and Gl-LSQR. The results show that Gl-CGLS method is numerically more efficient than others for the ill-posed problems.

The brief description of the CGLS method is summarized in Section 2. The Gl-CGLS algorithm is proposed to solve general linear systems with multiple right-hand sides in Section 3. Section 4 illustrates how the Gl-CGLS method can be implemented for the image deblurring problems. Numerical experiments and final remarks are described in Section 4.

For X and Y in $\mathbb{R}^{N\times s}$, we define the inner product $< X, Y >_F = tr(X^TY)$, where $tr(Z)$ denotes the trace of the square matrix Z. The Frobenius norm is defined by $\|X\|_F = (< X, X >_F)^{1/2}$.

2. Review of the preconditioned CGLS algorithm

The CGLS method is a special case of Krylov subspace methods and a variant of conjugate gradient method. The method can be applied to the normal equations associated with a least squares problem $\min \|Hx - b\|_2$ in the generated Krylov subspace. But CGLS avoids explicit computation of the cross product H^TH which causes bad performances on ill-conditioned systems. The method performs a sequential linear searches along H^TH-conjugate directions $\{p_0, p_1, \dots, p_{k-1}\}$ that spans the Krylov subspace:

$$\mathcal{K}_k(H^TH, H^Tb) = \text{span}\{H^Tb, (H^TH)H^Tb, \dots, (H^TH)^{k-1}H^Tb\}. \tag{4}$$

The k-th iterate of CGLS solves the least squares problem:

$$x_k = arg \min_{x\in\mathcal{K}_k(H^TH,H^Tb)} \frac{1}{2} \|Hx - b\|_2^2 . \tag{5}$$

The approximation x_k is determined by $x_k = x_{k-1} + \alpha_{k-1}p_{k-1}$, where α_{k-1} solves one dimensional minimization problem:

$$\min_{\alpha} \|H(x_{k-1} + \alpha p_{k-1}) - b\|_2^2 . \tag{6}$$

The search direction vector $p_k = s_k + \beta_{k-1}p_{k-1}$ is updated with the residual error $s_k = H^Tb - H^THx_k$ and the previous direction p_{k-1}, where the parameter β_{k-1} is chosen so that p_k is H^TH-conjugate to all of the previous search directions, that is, $p_k^TH^THp_j = 0, 1 \le j \le k-1$. CGLS can be described as follows:

Algorithm 1. CGLS

1. x_0 is initial guess,
2. Compute $r_0 = b - Hx_0$, $p_0 = s_0 = H^Tr_0$, $\gamma_0 = \|s_0\|_2^2$,
3. For k=0, 1, ..., until convergence do
   - i. $q_k = Hp_k$,
   - ii. Set $\alpha_k = \frac{\gamma_k}{\|q_k\|_2^2}$,
   - iii. $x_{k+1} = x_k + \alpha_kp_k$,
   - iv. $r_{k+1} = r_k - \alpha_kq_k$,
   - v. $s_{k+1} = H^Tr_{k+1}$,
   - vi. $\gamma_{k+1} = \|s_{k+1}\|_2^2$,
   - vii. Set $\beta_k = \frac{\gamma_{k+1}}{\gamma_k}$,
   - viii. $p_{k+1} = s_{k+1} + \beta_kp_k$,
4. Enddo

CGLS requires the storage of four vectors x, p, r, and q which are independent on the number of iterations. Each iteration costs two matrix-vector products such as one with H and one with H^T. It is known that in the absence of rounding errors, the iterates of CGLS will be converged to the exact pseudoinverse solution in the iterations not more than the number of distinct nonzero singular values of H [1]. The rate of convergence of CGLS depends on the condition number and the spectrum of matrix H. LSQR, mathematically equivalent to CGLS, converges faster since the $\|r_k\|$ will often exhibit large oscillation for ill-conditioned matrix H. CGLS, however, can be more efficient for the well conditioned problems. Thus preconditioning can improve the convergence of CGLS by transforming the problem $\min \|Hx-b\|_2$ into

$$\min_y \left\|H\Omega^{-1}y - b\right\|_2, \ \Omega x = y \ , \tag{7}$$

where nonsingular $N \times N$ matrix Ω is chosen so that $H\Omega^{-1}$ is better conditioned and has better spectrum than that of H.

The normal equations in factored form for the preconditioned problem (7) are

$$\Omega^{-T}H^T(H\Omega^{-1}y - b) = \Omega^{-T}H^T(Hx - b) = 0.$$

Multiplying the above equation by Ω^{-1} leads to

$$\Omega^{-1}\Omega^{-T}H^T H\Omega^{-1}y = \Omega^{-1}\Omega^{-T}H^T b.$$

Thus Ω must be chosen so that $\Omega^T\Omega$ is a close approximation to H^TH.

The CGLS algorithm can be used for the regularized least squares problems in the form as

$$\min_x \left\| \begin{pmatrix} H \\ \lambda L \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix} \right\|_2 . \tag{8}$$

3. The Gl-CGLS algorithm

The Gl-CGLS can be considered as a variant of the global CG method suggested by D. K. Salkuyeh [10]. Gl-CGLS solves the matrix equations

$$(H^TH + \lambda^2L^TL)X = H^TB \tag{9}$$

associated with a Tikhonov regularized problem

$$\min_X \left\| \begin{pmatrix} H \\ \lambda L \end{pmatrix} X - \begin{pmatrix} B \\ O \end{pmatrix} \right\|_F . \tag{10}$$

The symmetric coefficient matrix $H^TH + \lambda^2L^TL$ in (9) can be reduced to tridiagonal matrix $T_m = \text{tridiag}(\beta_{\text{i}}, \alpha_i, \beta_{i+1})$ by the global Lanczos algorithm:

Algorithm 2. Global Lanczos algorithm

1. Choose an $N \times s$ matrix V_1 such that $\|V_1\|_F = 1$.
2. Set $\beta_1 = 0$ and $V_0 = 0$.
3. For $j = 1, ..., m$
   i. $W = (H^TH + \lambda^2L^TL)V_j - \beta_jV_{j-1}$,
   ii. $\alpha_j = < V_j, W >_F$,
   iii. $W = W - \alpha_jV_j$,
   iv. $\beta_{j+1} = \|W\|_F$,

v. $V_{j+1} = W/\beta_{j+1}$,

4. Enddo

which constructs an F-orthogonal basis $V_1, V_2, \ldots, V_m$, (i.e., $< V_i, V_j >_F = 0$ for $i \neq j$, $< V_i, V_j >_F = 1$ for $i = j$), of the matrix Krylov subspace $\mathcal{K}_m(H^T H + \lambda^2 L^T L, H^T B)$ with an appropriate V_1 related to $H^T B$ in the algorithm. The Global Lanczos algorithm also provides a tridiagonal matrix of order $(m+1)\times m$, $\tilde{T}_m$ with entries $t_{i,j} =< (H^T H + \lambda^2 L^T L)V_j, V_i >_F$, $i = 1, \ldots, m$, $j = 1, \ldots, m+1$.

Letting $S_0 = H^T B - (H^T H + \lambda^2 L^T L)X_0$ for some X_0, the approximate solution X_m of (9) in $X_0 + \mathcal{K}_m(H^T H + \lambda^2 L^T L, H^T B)$ is given by

$$X_m = X_0 + \mathcal{V}_m * y_m,$$

where $\mathcal{V}_m * y_m = \sum_{k=1}^m y_{m_k} V_k$ with $\mathcal{V}_m = \begin{bmatrix} V_1 & \ldots & V_m \end{bmatrix}$, $V_1 = S_0 / \|S_0\|_F$, and $y_m = T_m^{-1} \|S_0\|_F e_1$. From the LU factorization of T_m,

$$\begin{aligned} T_m &= L_m U_m \\ &= \mathrm{tridiag}(\lambda_\mathrm{i}, 1, 0)\ \mathrm{tridiag}(0, \eta_\mathrm{i}, \beta_{\mathrm{i+1}}) \\ &= \mathrm{lowerbidiag}(\lambda_\mathrm{i}, 1)\ \mathrm{upperbidiag}(\eta_\mathrm{i}, \beta_{\mathrm{i+1}}), \end{aligned}$$

the approximate solution

$$\begin{aligned} X_m &= X_0 + \mathcal{V}_m * (U_m^{-1} L_m^{-1}) \|S_0\|_F e_1 \\ &= X_0 + (\mathcal{V}_m * U_m^{-1}) * (L_m^{-1} \|S_0\|_F e_1) \\ &= X_0 + \mathcal{P}_m * z_m \\ &= X_{m-1} + \zeta_m \mathcal{P}_m, \end{aligned} \tag{11}$$

where $\mathcal{P}_m = \mathcal{V}_m * U_m^{-1} = \begin{bmatrix} P_1 & \ldots & P_m \end{bmatrix}$ and $z_m = L_m^{-1} \|S_0\|_F e_1 = \begin{bmatrix} z_{m-1} \\ \zeta_m \end{bmatrix}$. We can also obtain

$$P_m = \frac{1}{\eta_m} \left[V_m - \beta_m P_{m-1} \right], \tag{12}$$

and

$$\lambda_m = \tfrac{\beta_m}{\eta_{m-1}}, \quad \eta_m = \alpha_m - \lambda_m \beta_m.$$

Proposition 3.1. *For $m = 0, 1, \ldots$, let $S_m = H^T B - (H^T H + \lambda^2 L^T L)X_m$ be the residual matrices of matrix normal equations (9) and P_m be the auxiliary matrices produced by (12). Then residual matrices S_m are F-orthogonal to each other and the auxiliary matrix P_m are $H^T H + \lambda^2 L^T L$-conjugate set with respect to $< \cdot, \cdot >_F$, i.e. $< (H^T H + \lambda^2 L^T L)P_i, P_j >_F = 0$, $i \neq j$.*

Proof. From the tridiagonlization of $H^T H + \lambda^2 L^T L$, we can get

$$(H^T H + \lambda^2 L^T L)\mathcal{V}_m = \mathcal{V}_m * T_m + t_{m+1,m} \begin{bmatrix} O & \ldots & O & V_{m+1} \end{bmatrix}.$$

The residual matrix S_m of matrix normal equations can be written as

$$\begin{aligned} S_m &= S_0 - (H^T H + \lambda^2 L^T L)\mathcal{V}_m * y_m \\ &= S_0 - (\mathcal{V}_m * T_m) * y_m - t_{m+1,m} e_m^T y_m V_{m+1} \\ &= S_0 - \mathcal{V}_m * (T_m y_m) - t_{m+1,m} e_m^T y_m V_{m+1} \\ &= -t_{m+1,m} e_m^T y_m V_{m+1}. \end{aligned}$$

Thus the residual matrices $\{S_m\}_{m=0,1,\dots}$ are F-orthogonal to each other.

From the block matrix $\mathcal{P}_m = \mathcal{V}_m * U_m^{-1} = \begin{bmatrix} P_1 & \dots & P_m \end{bmatrix}$, we obtain $\mathcal{P}_m^T = \sum_{k=1}^{m} (U_m^{-1})^T(:,k) \otimes V_k^T$ and then the following equality holds :

$$\begin{aligned} \mathcal{P}_m^T (H^T H + \lambda^2 L^T L)\mathcal{P}_m &= (\mathcal{V}_m * U_m^{-1})^T (H^T H + \lambda^2 L^T L)(\mathcal{V}_m * U_m^{-1}) \\ &= \left(\sum_{k=1}^{m} (U_m^{-1})^T(:,k) \otimes V_k^T \right) \mathcal{V}_m * T_m * U_m^{-1} \\ &= \left(\sum_{k=1}^{m} (U_m^{-1})^T(:,k) \otimes V_k^T \right) \mathcal{V}_m * T_m U_m^{-1} \\ &= \left(\sum_{k=1}^{m} (U_m^{-1})^T(:,k) \otimes V_k^T \right) \mathcal{V}_m * L_m. \end{aligned} \tag{13}$$

From the detail computations of the last matrix expression of the right hand side of (13), we can drive the F-orthogonality that for $i \neq j$, $< (H^T H + \lambda^2 L^T L)P_i, P_j >_F = tr(P_i^T (H^T H + \lambda^2 L^T L)P_j) = 0$ since $< V_i, V_j >_F = 0$. Hence, the auxiliary matrices $\{P_m\}_{m=0,1,\dots}$ are $(H^T H + \lambda^2 L^T L)$-conjugate set. □

The Gl-CGLS algorithm can be obtained by using this proposition. From the update relation

$$X_{j+1} = X_j + \alpha_j P_j,$$

the residual matrix of normal matrix must satisfy the recurrence

$$S_{j+1} = S_j - \alpha_j (H^T H + \lambda^2 L^T L)P_j,$$

and the next search direction P_{j+1} is a linear combination of S_{j+1} and P_j,

$$P_{j+1} = S_{j+1} + \beta_j P_j.$$

Thus the F-orthogonality of S_j's brings

$$\alpha_j = \frac{< S_j, S_j >_F}{< (H^T H + \lambda^2 L^T L)P_j, P_j >_F}$$

and

$$\beta_j = \frac{< S_{j+1}, S_{j+1} >_F}{< S_j, S_j >_F}.$$

The above relations result in the Gl-CGLS algorithm.

Algorithm 3. Gl-CGLS

1. X_0 is initial guess,

2. Compute $R_0 = \begin{pmatrix} B \\ O \end{pmatrix} - \begin{pmatrix} H \\ \lambda L \end{pmatrix} X_0$, $P_0 = S_0 = \begin{pmatrix} H \\ \lambda L \end{pmatrix}^T R_0$, $\gamma_0 = < S_0, S_0 >_F$,
3. For k=0, 1, ..., until convergence do
   i. $Q_k = \begin{pmatrix} H \\ \lambda L \end{pmatrix} P_k$,
   ii. Set $\alpha_k = \gamma_k / < Q_k, Q_k >_F$,
   iii. $X_{k+1} = X_k + \alpha_k P_k$,
   iv. $R_{k+1} = R_k - \alpha_k Q_k$,
   v. $S_{k+1} = \begin{pmatrix} H \\ \lambda L \end{pmatrix}^T R_{k+1}$,
   vi. $\gamma_{k+1} = < S_{k+1}, S_{k+1} >_F$,
   vii. Set $\beta_k = \gamma_{k+1}/\gamma_k$,
   viii. $P_{k+1} = S_{k+1} + \beta_k P_k$,
4. Enddo

4. Implementations and Numerical Results

4.1. Partitioning and processing of the deblurred image. The deblurring problem (2) is considered as a memory intensive application due to its insurmountable data. Alternative scheme can be used if the issue of storage becomes a problem. One way to reduce memory use is to process the image in small blocks which are its subimages.

We assume that the original image has n^2 pixels. To obtain image restoration problem with multiple right hand sides, we divide the original image to d small blocks with each size of $\eta \times \eta$ ($\eta << n$), where d is n^2/η^2. The divided small blocks are numbered in column-row order. The i-th subimage is denoted by $SIM(i) = \begin{bmatrix} \mathbf{x}^i_{\cdot 1} & \dots & \mathbf{x}^i_{\cdot \eta} \end{bmatrix}_{\eta \times \eta}$, where $\mathbf{x}^i_{\cdot j}$ means the j-th column of i-th subimage. Let X_i be a vector produced by stacking the columns of $SIM(i)$. For each i, let B_i be a vector representation of noisy blurred image corresponding to the i-th block of the original image. Now if we denote two matrices X and B by $X \equiv \begin{bmatrix} X_1 & X_2 & \cdots & X_d \end{bmatrix}_{\eta^2 \times d}$ and $B \equiv \begin{bmatrix} B_1 & B_2 & \cdots & B_d \end{bmatrix}_{\eta^2 \times d}$, respectively, the image restoration problem (2) will be reformulated as Tikhonov regularization problem (3) with respect to F-norm. For computational efficiency and numerical stability, algorithms for computing Tikhonov solutions can be based on the formulation (10) and for certain situations the matrix normal equations (9) is also suited. Since the coefficient matrix of (9) is symmetric positive definite, we turn our attention to the use of Gl-CGLS proposed in the previous Section 3.

The problem (10) is preconditioned by a preconditioner $\hat{\Omega}$:

$$\min_{Y} \left\| \hat{H} \hat{\Omega}^{-1} Y - \hat{B} \right\|_F \tag{14}$$

with $Y = \hat{\Omega} X$, where $\hat{H} = \begin{pmatrix} H \\ \lambda L \end{pmatrix}$ and $\hat{B} = \begin{pmatrix} B \\ O \end{pmatrix}$.

In (3), the matrix H has a special structure depending on the choice of boundary conditions. For the Neumann boundary condition reflecting the image like a mirror with respect to the boundaries, the matrices H and L in (14) will have the structures of block Toeplitz-plus-Hankel with Toeplitz-plus-Hankel blocks which can be diagonalized by two dimensional discrete cosine transformation matrix C [6]. The matrix $\hat{H}\hat{\Omega}^{-1}$ in (14) is to be well conditioned with the preconditioner:

$$\hat{\Omega} = C^*\Lambda C = C^*(|\Lambda_H|^2 + \lambda^2|\Lambda_L|^2)^{1/2}C,$$

where $H \approx C^*\Lambda_H C$ and $L \approx C^*\Lambda_L C$.

4.2. Numerical experiments. This section deals with the efficiency of preconditioned Gl-CGLS algorithm for the image restoration problem where all computations are conducted by Matlab environment.

Gl-CGLS and CGLS method are applied to a practical image deblurring problem with multiple right-hand sides with the Neumann boundary condition. Let $t_{Gl-CGLS}$ and t_{CGLS} denote the CPU time obtained by applying Gl-CGLS for multiple right-hand side linear systems and CGLS for sequential linear systems with single right-hand side.

To show how well the points approximate the true image, we investigate the relative accuracy of the reconstructed image X_{rec} with respect to the exact solution X of system (14), $\frac{\|X-X_{rec}\|_F}{\|X\|_F}$, and the PSNR(peak signal-to-noise ratio) values of the recovered images. Especially, PSNR is most commonly used as a measure of quality of restored image [11]. PSNR for a gray scale image is defined as :

$$\mathrm{PSNR} = 10\log 10\left(\frac{255^2}{\mathrm{MSE}}\right).$$

Here, MSE is the mean square error for two $m \times n$ monochrome images I and J, where one of the images is considered as a noisy approximation of the other. It is defined as $\mathrm{MSE}(I,J) = \frac{\Sigma_{i,j}(I(i,j)-J(i,j))^2}{mn}$. The smaller the relative accuracy and the bigger the PSNR value gets, the better the approximated image becomes.

For a test, we only considered a spatially invariant point spread function whose discrete function was chosen from

$$h_{i-j,\ k-l} = \frac{1}{2\pi\sigma^2}e^{-\frac{(i-j)^2+(k-l)^2}{2\sigma^2}}, \quad -r \le i-j,\ k-l \le r, \tag{15}$$

where σ is the Gaussian variance. Note that this is called by Gaussian PSF, and can be used to model aberrations in a lens with finite aperture [4]. In (15), the Gaussian variance σ was taken as 0.01 and $r = 6$. Our test used only the identity matrix as a regularization operator L and $\lambda = 0.008$ as regularization parameter. To minimize edge effects, each subimage has properly overlapped when the image is divided and patched.

The test images used in our study are the 128×128 *particle* image (Figure 1(a)) and the similar *particle* image with a size of 256×256. The images are divided into $16 \sim 256$ small block images and then by convolving the PSF with exact subimages and adding the error normally distributed with zero mean, the

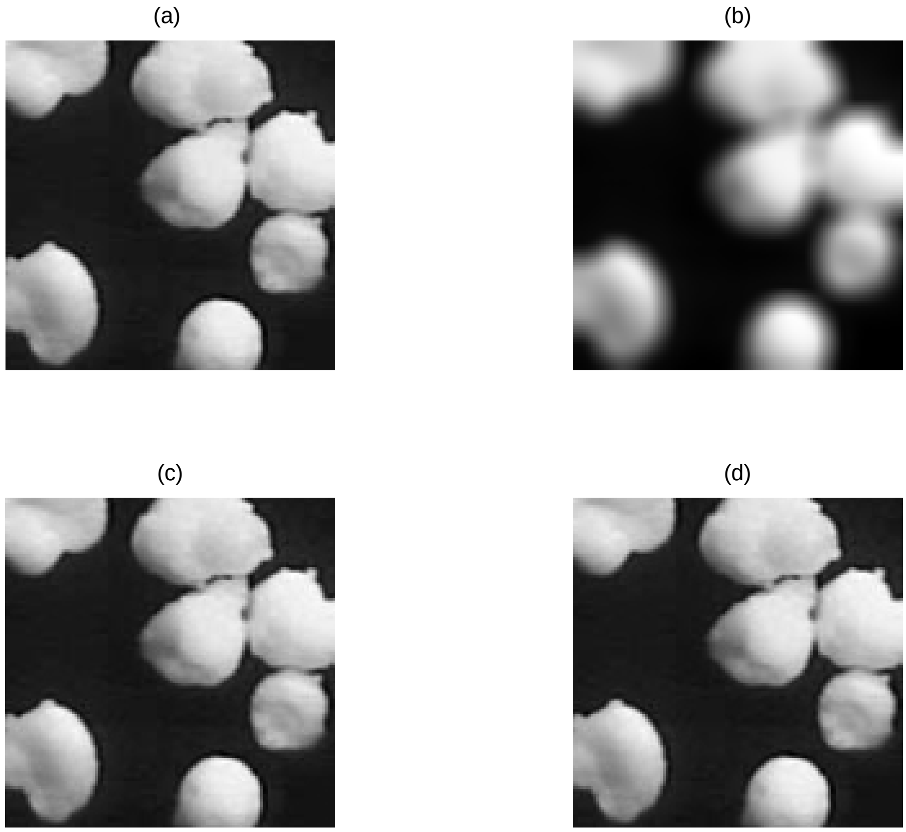

FIGURE 1. Test image of a *particle*. (a) Original image; (b) blurred and noisy image; (c) Restored image by preconditioned Gl-CGLS; (d) Restored image by preconditioned CGLS.

blurred and noisy image of Figure 1(b) can be constructed. The restored images by the preconditioned Gl-CGLS and preconditioned CGLS are shown in Figure 1(c) and Figure 1(d) respectively. For the comparison purpose, the preconditioned Gl-LSQR was also applied to the same images. Each of the blurred images was restored to achieve the relative accuracies ($1.3e-3 \sim 7.1e-3$) and PSNR ($49.85 \sim 64.59$) for the three methods as shown in Table 1. Consequently, the CPU time ratios $t_{CGLS}/t_{Gl-CGLS}$ were improved by $5.41 \sim 17.63$. We can perceive that the Gl-CGLS speeds up for the large number of divided subimages, and restores the degraded images more efficiently than CGLS and even Gl-LSQR.

TABLE 1. Comparison of the restoring results of the degraded image in Figure 1(b). d is the number of small subimages

Image I (128 × 128 *particle* image)

d	Method	CPU time	Relative accuracy	PSNR
16^2	Gl-CGLS	4.97	1.3e-3	64.54
	CGLS	87.63	1.3e-3	64.59
	Gl-LSQR	25.88	1.3e-3	64.54
8^2	Gl-CGLS	9.98	6.9e-3	49.86
	CGLS	116.31	6.6e-3	50.23
	Gl-LSQR	17.07	6.9e-3	49.86
4^2	Gl-CGLS	52.79	5.8e-3	51.37
	CGLS	285.82	5.8e-3	51.48
	Gl-LSQR	38.47	5.8e-3	51.37

Image II (256 × 256 *particle* image)

d	Method	CPU time	Relative accuracy	PSNR
16^2	Gl-CGLS	22.35	7.1e-3	52.86
	Gl-LSQR	53.94	7.1e-3	52.86
8^2	Gl-CGLS	73.34	6.0e-3	54.29
	Gl-LSQR	73.75	6.0e-3	54.29

In this study, we proposed the Gl-CGLS algorithm for deblurring noisy images and compared our approach with CGLS algorithm and Gl-LSQR algorithm. Experimental results show that when the number of divided subimages increases, the Gl-CGLS approach can significantly improve the execution times. It is indicating that Gl-CGLS is more efficient than CGLS for the deblurring image problems.

As final remark, this method can be extended to image mosaicing which has been developed for the detailed reconstruction of large images by successively acquiring image patches in a given row-column or column-row order. Our future work is to investigate whether Gl-CGLS algorithm is also effective in the mosaicing techniques.

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