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THE DRAZIN INVERSES OF THE SUM OF TWO MATRICES AND BLOCK MATRIX†

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ABSTRACT. In this paper, we give a formula of $(P+Q)^D$ under the conditions $P^2Q+QPQ=0$ and $P^3Q=0$. Then applying it to give some results of block matrix $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (A and D are square matrices) with generalized Schur complement is zero under some conditions. Finally, numerical examples are given to illustrate our results.

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1. Introduction

Let $C^{m\times n}$ denote the set of $m\times n$ complex matrices. The Drazin inverse of $A\in C^{n\times n}$ is the unique matrix X, denoted by A^D, satisfying the following equations

$$A^{k+1}X=A^k,\ \ XAX=X,\ \ AX=XA,$$

where $k=ind(A)$ is the index of A, the smallest nonnegative integer for which $rank(A^{k+1})=rank(A^k)$ (see[1]). In particular, when $ind(A)=1$, the Drazin inverse of A is called the group inverse of A and is denoted by A^g. If A is nonsingular, it is clearly $ind(A)=0$ and $A^D=A^{-1}$. Throughout this paper, we denote by $A^\pi=I-AA^D$ and define $A^0=I$, where I is the identity matrix with proper sizes.

The importance of the Drazin inverse and its applications to singular differential equations and difference equations, to Morkov chains and iterative methods, to cryptography, to numerical analysis, to structured matrices and to perturbation bounds for the relative eigenvalue problems can be found in [2, 3-5].

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In 1958, Drazin [6] gave a result of $(P+Q)^D$ with P and Q are square matrices and proved that

$$(P+Q)^D = P^D + Q^D \text{ when } PQ = QP = 0.$$

In 2001, Hartwig et al. [7] gave a result of $(P+Q)^D$ when $PQ = 0$. In 2005, Castro-Gonzlez [8] gave a result of $(P+Q)^D$ when $P^DQ = 0$, $PQ^D = 0$ and $Q^\pi PQP^\pi = 0$. In 2008, Castro-Gonzlez et al. [9] gave the representation of $(P+Q)^D$ when $P^2Q = 0$ and $PQ^2 = 0$. In 2009, Martnez-Serrano and Castro-Gonzlez [10] gave a result of $(P+Q)^D$ when $P^2Q = 0$ and $Q^2 = 0$. In 2011, Yang and Liu [10] gave the result of $(P+Q)^D$ when $PQ^2 = 0$ and $PQP = 0$, and in 2012, Bu et al. [12] gave the representation of $(P+Q)^D$ when $P^2Q = 0$, $Q^2P = 0$ and $P^3Q = 0$, $QPQ = 0$, $QP^2Q = 0$ respectively. Other results have been studied in [4,13-15,16-18,19].

On the other hand, a related topic is to discuss a representation of the Drazin inverse of block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A and D are square matrices. Campbell and Meyer [2] first proposed an open problem to find an explicit formula of the Drazin inverse of block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, (where A and D are square matrices) in terms of A, B, C and D. To find the Drazin inverse of $(P+Q)$ and M in terms of P, Q, P^D, Q^D and A, B, C, D, respectively, without side condition are very difficult and it has not been solved till now.

The generalized Schur complement of A in M which is stated as $S = D - CA^DB$, is very important to find the Drazin inverse of M. When the generalized Schur complement is either zero or nonsingular, the Drazin inverse of M have been studied in [10,20], when generalized Schur complement is equal to zero and also has been studied in [10,21,22], when the generalized Schur complement is nonsingular. Some representations for the Drazin inverse of M when the generalized Schur complement is zero, including a generalizations of the above mentioned results, will derived in section 4 under some conditions.

This paper is organized as follows. In section 2, some helpful lemmas will be given. In section 3, we give the formula of $(P+Q)^D$ under the conditions $P^2Q + QPQ = 0$, $P^3Q = 0$ and also give a numerical example to illustrate our result. In section 4, we use our result to find the Drazin inverse of block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, when the generalized Schur complement is equal to zero, which can be regarded as the generalizations of some results given in [5,20]. Finally, in section 5, we give a numerical example to illustrate our result of block matrix.

2. Some Lemmas

In order to prove the main results, first we need the following lemmas.

Lemma 2.1 ([1]). *Let $A \in C^{m\times n}$, $B \in C^{n\times m}$. Then $(AB)^D = A((BA)^2)^DB$.*

Lemma 2.2 ([7]). *Let $P, Q \in C^{n\times n}$, if $PQ = 0$, then*

$$(P+Q)^D = Q^\pi \sum_{i=0}^{t-1} Q^i (P^D)^{i+1} + \sum_{i=0}^{t-1} (Q^D)^{i+1} P^i P^\pi,$$

where $t = max\{ind(P), ind(Q)\}$.

Lemma 2.3 ([23])**.** *Let* $M_1 = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}$, $M_2 = \begin{pmatrix} B & C \\ 0 & A \end{pmatrix}$, *where* A *and* B *are square matrices with* $ind(A) = r$ *and* $ind(B) = s$, *then*

$$M_1^D = \begin{pmatrix} A^D & 0 \\ X & B^D \end{pmatrix}, \quad M_2^D = \begin{pmatrix} B^D & X \\ 0 & A^D \end{pmatrix},$$

where $X = \sum_{i=0}^{r-1} (B^D)^{i+2} C A^i A^\pi + \sum_{i=0}^{s-1} B^\pi B^i C (A^D)^{i+2} - B^D C A^D$.

Lemma 2.4 ([20])**.** *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in C^{n\times n}$ *(A and D are square matrices), if* $S = D - CA^D B = 0$, $A^\pi B = 0$ *and* $CA^\pi = 0$, *then*

$$M^D = \begin{pmatrix} I \\ CA^D \end{pmatrix} \left[(AW)^D\right]^2 A \begin{pmatrix} I & A^D B \end{pmatrix}, \quad W = AA^D + A^D BCA^D.$$

3. Main results

In this section, we first give the formula for the Drazin inverse of $P+Q$ under some conditions.

Theorem 3.1. *Let* $P, Q \in C^{n\times n}$, *if* $P^2Q + QPQ = 0$ *and* $P^3Q = 0$ *then*

$$\begin{aligned}(P+Q)^D = {} & \begin{pmatrix} I & Q \end{pmatrix} \sum_{i=0}^{t-1} \begin{pmatrix} (PQ)^\pi & PQX_2 + (P^2Q+PQ^2)(PQ)^D \\ 0 & (PQ)^\pi \end{pmatrix} \\ & \times \begin{pmatrix} PQ & P^2Q+PQ^2 \\ 0 & PQ \end{pmatrix}^i \begin{pmatrix} (P^D)^2 & 0 \\ X_1 & (Q^D)^2 \end{pmatrix}^{i+1} \begin{pmatrix} P \\ I \end{pmatrix} \\ & + \begin{pmatrix} I & Q \end{pmatrix} \sum_{i=0}^{t-1} \begin{pmatrix} (PQ)^D & X_2 \\ 0 & (PQ)^D \end{pmatrix}^{i+1} \begin{pmatrix} P^2 & 0 \\ P+Q & Q^2 \end{pmatrix}^i \\ & \times \begin{pmatrix} P^\pi & 0 \\ P^D + Q(P^D)^2 + Q^2 X_1 & Q^\pi \end{pmatrix} \begin{pmatrix} P \\ I \end{pmatrix},\end{aligned} \tag{1}$$

where

$$X_1 = \sum_{i=0}^{t-1} (Q^D)^{2i+4}(P+Q)P^{2i}P^\pi + \sum_{i=0}^{t-1} Q^\pi Q^{2i}(P+Q)(P^D)^{2i+4} - (Q^D)^2(P+Q)(P^D)^2,$$

$$\begin{aligned}X_2 = {} & \sum_{i=0}^{t-1} ((PQ)^D)^{i+2}(P^2Q+PQ^2)(PQ)^i(PQ)^\pi + \sum_{i=0}^{t-1} (PQ)^\pi (PQ)^i (P^2Q+PQ^2)((PQ)^D)^{i+2} \\ & - (PQ)^D(P^2Q+PQ^2)(PQ)^D\end{aligned}$$

and

$$t = max\{ind(P^2),\ ind(Q^2),\ ind(PQ)\}.$$

Proof. Using Lemma 2.1, we have

$$\begin{aligned}(P+Q)^D &= \left(\begin{pmatrix} I & Q \end{pmatrix}\begin{pmatrix} P \\ I \end{pmatrix}\right)^D = \begin{pmatrix} I & Q \end{pmatrix}\left(\begin{pmatrix} P & PQ \\ I & Q \end{pmatrix}^2\right)^D \begin{pmatrix} P \\ I \end{pmatrix} \\ &= \begin{pmatrix} I & Q \end{pmatrix}\begin{pmatrix} P^2+PQ & P^2Q+PQ^2 \\ P+Q & Q^2+PQ \end{pmatrix}^D \begin{pmatrix} P \\ I \end{pmatrix}.\end{aligned} \tag{2}$$

Let

$$M = \begin{pmatrix} P^2+PQ & P^2Q+PQ^2 \\ P+Q & Q^2+PQ \end{pmatrix} = E+F,$$

where $E = \begin{pmatrix} P^2 & 0 \\ P+Q & Q^2 \end{pmatrix}$, $F = \begin{pmatrix} PQ & P^2Q+PQ^2 \\ 0 & PQ \end{pmatrix}$.
From $P^2Q + QPQ = 0$ and $P^3Q = 0$, we get $EF = 0$. Then applying Lemma 2.2, we have

$$M^D = \sum_{i=0}^{t-1} F^\pi F^i \left(E^D\right)^{i+1} + \sum_{i=0}^{t-1} \left(F^D\right)^{i+1} E^i E^\pi, \tag{3}$$

where $t = max\{ind(E),\ ind(F)\}$.
Applying Lemma 2.3, we have

$$E^D = \begin{pmatrix} (P^D)^2 & 0 \\ X_1 & (Q^D)^2 \end{pmatrix}, \quad F^D = \begin{pmatrix} (PQ)^D & X_2 \\ 0 & (PQ)^D \end{pmatrix}, \tag{4}$$

where

$$\begin{aligned} X_1 &= \sum_{i=0}^{t-1} (Q^D)^{2i+4}(P+Q)P^{2i}P^\pi + \sum_{i=0}^{t-1} Q^\pi Q^{2i}(P+Q)(P^D)^{2i+4} - (Q^D)^2(P+Q)(P^D)^2, \\ X_2 &= \sum_{i=0}^{t-1} ((PQ)^D)^{i+2}(P^2Q+PQ^2)(PQ)^i(PQ)^\pi + \sum_{i=0}^{t-1} (PQ)^\pi (PQ)^i (P^2Q+PQ^2)((PQ)^D)^{i+2} \\ &\quad - (PQ)^D(P^2Q+PQ^2)(PQ)^D \end{aligned}$$

and

$$t = max\{ind(P^2),\ ind(Q^2),\ ind(PQ)\}.$$

Substituting (4) into (3), then substituting (3) into (2), we get the result. $\square$

Similarly, we give a symmetrical form of Theorem 3.1.

Theorem 3.2. *Let $P, Q \in C^{n\times n}$, if $PQ^2 + PQP = 0$ and $PQ^3 = 0$ then*

$$
\begin{aligned}
(P+Q)^D = &\begin{pmatrix} I & Q \end{pmatrix} \sum_{i=0}^{t-1} \begin{pmatrix} P^{\pi} & 0 \\ P^D + Q(P^D)^2 + Q^2 X_1 & Q^{\pi} \end{pmatrix} \\
&\times \begin{pmatrix} P^2 & 0 \\ P+Q & Q^2 \end{pmatrix}^{i} \begin{pmatrix} (PQ)^D & X_2 \\ 0 & (PQ)^D \end{pmatrix}^{i+1} \begin{pmatrix} P \\ I \end{pmatrix} \\
&+ \begin{pmatrix} I & Q \end{pmatrix} \sum_{i=0}^{t-1} \begin{pmatrix} (P^D)^2 & 0 \\ X_1 & (Q^D)^2 \end{pmatrix} i+1 \begin{pmatrix} PQ & P^2Q + PQ^2 \\ 0 & PQ \end{pmatrix}^{i} \\
&\times \begin{pmatrix} (PQ)^{\pi} & PQX_2 + (P^2Q+PQ^2)(PQ)^D \\ 0 & (PQ)^{\pi} \end{pmatrix} \begin{pmatrix} P \\ I \end{pmatrix},
\end{aligned} \tag{5}
$$

where

$$
\begin{aligned}
X_1 = &\sum_{i=0}^{t-1} (Q^D)^{2i+4}(P+Q)P^{2i}P^{\pi} + \sum_{i=0}^{t-1} Q^{\pi}Q^{2i}(P+Q)(P^D)^{2i+4} - (Q^D)^2(P+Q)(P^D)^2, \\
X_2 = &\sum_{i=0}^{t-1} ((PQ)^D)^{i+2}(P^2Q+PQ^2)(PQ)^i(PQ)^{\pi} + \sum_{i=0}^{t-1} (PQ)^{\pi}(PQ)^i(P^2Q+PQ^2)((PQ)^D)^{i+2} \\
&- (PQ)^D(P^2Q+PQ^2)(PQ)^D,
\end{aligned} \tag{6}
$$

and

$$t = max\{ind(P^2),\ ind(Q^2),\ ind(PQ)\}.$$

Next, we give a numerical example of Theorem 3.1 which does not satisfy the conditions $P^2Q = 0$, $Q^2 = 0$ in Theorem 2.2 in Ref. [10], but it satisfies the conditions of our theorem 3.1.
Numerical example : Consider the matrices $P, Q \in C^{4\times 4}$, where

$$
P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
$$

Since $P^2Q = 0$ and $Q^2 \neq 0$, the result is not valid to apply in Theorem 2.2 in Ref. [10]. But it satisfies $P^2Q + QPQ = 0$, $P^3Q = 0$, also we have

$$ind(P^2) = 1, \ \ ind(Q^2) = 1, \ \ ind(PQ) = 2$$

and

$$(PQ)^D = 0, \quad X_2 = 0,$$

$$
X_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad P^D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q^D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

so applying Theorem 3.1, we get

$$(P+Q)^D = P^D + PQP^D + QX_1P + Q^D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

4. Drazin inverse of a block matrix

In this section, we use our formula to give the representations for the Drazin inverse of block matrix. Now we consider the generalized Schur complement is equal to zero.

Theorem 4.1. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in C^{n\times n}$ *(A and D are square matrices), if* $S = D - CA^DB = 0$, $BCA^\pi A = 0$ *and* $BCA^\pi B = 0$, *then*

$$\begin{aligned} M^D =& P^D + \begin{pmatrix} 0 & 0 \\ CAA^\pi & CA^\pi B \end{pmatrix} \left(P^D\right)^3 + \begin{pmatrix} 0 & 0 \\ CA^2A^\pi & CAA^\pi B \end{pmatrix} X_1 P^D \\ &+ \begin{pmatrix} AA^\pi & A^\pi B \\ 0 & 0 \end{pmatrix} X_1 \begin{pmatrix} A^2A^D & AA^DB \\ C & CA^DB \end{pmatrix}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} (P^D)^i =& (P_1^D)^i + (P_1^D)^{i+1} \begin{pmatrix} 0 & 0 \\ CA^\pi & 0 \end{pmatrix}, \quad \text{for} \quad i \geq 1, \\ (P_1^D)^i =& \begin{pmatrix} I \\ CA^D \end{pmatrix} \left[(AW)^D\right]^{i+1} A \begin{pmatrix} I & A^DB \end{pmatrix}, \\ W =& AA^D + A^DBCA^D, \quad \text{for} \quad i \geq 1, \\ X_1 =& \begin{pmatrix} A & B \\ C & D \end{pmatrix} (P^D)^4 + \sum_{i=1}^{t-1} \begin{pmatrix} A^{2i}A^\pi & A^{2i-1}A^\pi B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} (P^D)^{2i+4}, \end{aligned} \tag{8}$$

and

$$t = max\{ind(A^2), \ \ ind((AW)^2)\}.$$

Proof. Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A^2A^D & AA^DB \\ C & CA^DB \end{pmatrix}, \quad Q = \begin{pmatrix} AA^\pi & A^\pi B \\ 0 & 0 \end{pmatrix}.$$

From $BCA^\pi A = 0$ and $BCA^\pi B = 0$, we have $P^2Q + QPQ = 0$ and $P^3Q = 0$. We can see that Q is $t+1-nilpotent$, where $t = ind(A)$, so we get $Q^D = 0$ and $Q^\pi = I$. Moreover $(PQ)^2 = 0$, so $(PQ)^D = 0$. Applying Theorem 3.1, we have

$$M^D = P^D + PQ(P^D)^3 + PQ^2 X_1 P^D + QX_1 P. \tag{9}$$

where X_1 is given in (4.2).

Let $P = P_1 + P_2$, where $P_1 = \begin{pmatrix} A^2A^D & AA^DB \\ CAA^D & CA^DB \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 & 0 \\ CA^\pi & 0 \end{pmatrix}$. Now we have, $P_2P_1 = 0$ and $P_2^2 = 0$. By Lemma 2.2, we have

$$(P^D)^i = (P_1^D)^i + (P_1^D)^{i+1}P_2, \quad \text{for} \quad i \geq 1. \tag{10}$$

Let S_1 be the generalized Schur complement of P_1, then we have

$$S_1 = CA^DB - CAA^D(A^2A^D)^DAA^DB = 0,$$

$$(A^2A^D)^\pi AA^DB = 0, \;\; CAA^D(A^2A^D)^\pi = 0.$$

Using Lemma 2.4, we get

$$\begin{aligned} (P_1^D)^i &= \begin{pmatrix} I \\ CA^D \end{pmatrix} [(AW)^D]^{i+1} A \begin{pmatrix} I & A^DB \end{pmatrix}, \\ W &= AA^D + A^DBCA^D, \qquad \text{for} \quad i \geq 1, \end{aligned} \tag{11}$$

Substituting (11) into (10), then substituting (10) into (9), we get the result. $\square$

Similarly, we consider another splitting of the block matrix and state another theorem.

Theorem 4.2. *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in C^{n\times n}$ (A and D are square matrices), if $S = D - CA^DB = 0$, $A^\pi BCA^\pi = 0$ and $ABCA^\pi = 0$, then*

$$\begin{aligned} M^D =& P^D + \begin{pmatrix} BCA^\pi & 0 \\ 0 & 0 \end{pmatrix} (P^D)^3 + \begin{pmatrix} BCA^\pi A & 0 \\ 0 & 0 \end{pmatrix} X_1P^D \\ &+ \begin{pmatrix} AA^\pi & 0 \\ CA^\pi & 0 \end{pmatrix} X_1 \begin{pmatrix} A^2A^D & B \\ CAA^D & CA^DB \end{pmatrix}, \end{aligned} \tag{12}$$

where

$$(P^D)^i = (P_1^D)^i + \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix} P_1^D)^{i+1}, \;\; \text{for} \;\; i \geq 1$$

$$(P_1^D)^i = \begin{pmatrix} I \\ CA^D \end{pmatrix} [(AW)^D]^{i+1} A \begin{pmatrix} I & A^DB \end{pmatrix}, \; W = AA^D + A^DBCA^D, \; \text{for } i \geq 1$$

$$X_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} (P^D)^4 + \sum_{i=1}^{t-1} \begin{pmatrix} A^{2i}A^\pi & 0 \\ CA^{2i-1}A^\pi & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} (P^D)^{2i+4}$$

and

$$t = max\{ind(A^2), \; ind((AW)^2)\}.$$

Proof. Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A^2A^D & B \\ CAA^D & CA^DB \end{pmatrix}, \quad Q = \begin{pmatrix} AA^\pi & 0 \\ CA^\pi & 0 \end{pmatrix}.$$

The remaining proof follows directly from Theorem 4.1. □

5. Numerical example

In this section, we give a numerical example to illustrate our result, when the Schur complement is equal to zero and our other conditions of Theorem 4.1 are also satisfied.

Example 5.1.

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

By computing we obtain $S = D - CA^DB = 0$ and $BCA^\pi A = 0$, $BCA^\pi B = 0$. Also we have

$$ind(A^2) = 1, \ \ ind((AW)^2) = 1,$$

$$A^D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (AW)^D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A^\pi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then applying Theorem 4.1, we get

$$M^D = \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remark 1. Our above example shows that the conditions given in our Theorem 4.1 are satisfied but the conditions given in Theorem 3.6 in Ref. [10] are not satisfied.

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