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ON APPROXIMATIONS FOR $GI/G/c$ RETRIAL QUEUES[†]

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Abstract. The effects of the moments of the interarrival time and service time on the system performance measures such as blocking probability, mean and standard deviation of the number of customers in service facility and orbit are numerically investigated. The results reveal the performance measures are more sensitive with respect to the interarrival time than the service time. Approximation for $GI/G/c$ retrial queues using $PH/PH/c$ retrial queue is presented.

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1. Introduction

Consider the retrial queueing system that consists of a service facility with finite capacity and an orbit of an infinite size. An arriving customer enters the service facility if the service facility is not full upon arrival. Otherwise, the customer joins orbit and repeats its request after random amount of time. The time interval between two consecutive attempts of each customer in orbit is called a retrial time.

Even the ordinary $GI/G/c$ queue is known to be very difficult to analyze and one has to resort to approximations. There are several approaches to obtain approximate numerical results in ordinary $GI/G/c$ queue such as (1) approximate the service time distribution by phase type (PH) distribution in $M/G/c$ queue (2) approximate the continuous time model by a discrete time model and (3) use two-moment approximations based on the exact solutions of $GI/D/c$ queue and $GI/M/c$ queue, e.g. see [6, 11, 14] and for more detailed descriptions and related references for approximations of ordinary $GI/G/c$ queue see [18]. As the case of ordinary queue, it seems to be very difficult to obtain analytical results

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or approximations which guarantee the predetermined accuracy for the general retrial queue with general interarrival time and general service time. Since neither analytical nor algorithmic solution for retrial queue with general interarrival process is available, the approximation methods (2) and (3) mentioned above for ordinary queue can not be applicable to the retrial queue.

A distribution function $F(x)$ on $(0, \infty)$ is said to be of phase type with representation $(\boldsymbol{\alpha}, T)$ and denote it by $\mathrm{PH}(\boldsymbol{\alpha}, T)$ if $F(x) = 1 - \boldsymbol{\alpha}\exp(Tx)\boldsymbol{e}$, where $\boldsymbol{e}$ is the column m-vector whose components are all 1, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ is a probability distribution and $T = (t_{ij})$ is the $m \times m$ matrix with $t_{ii} < 0$, $1 \le i \le m$ and $t_{ij} \ge 0$, $i \ne j$, and $T\boldsymbol{e} \le \mathbf{0}$. For more details about phase type (PH) distribution, see Neuts [15, Chapter 2]. It is well known that the set of PH-distributions is dense (in the sense of weak convergence) in the set of all probability distributions on $(0, \infty)$ (e.g. see Asmussen [2, page 84]). There are many moment matching methods for fitting the general distribution by the PH distributions cf. Bobbio et al. [3], Johnson and Taaffe [12] and Whitt [19] and references therein. Richness of PH family, compounded with the nonuniqueness of PH-distribution representations, makes the PH-distribution-selection problem elusive. This suggests that a reasonable approach to the selection problem is to restrict selection to a PH-family subset that is not too restrictive nor unnecessarily general. The realization of the full utility of the PH family depends on the availability of methods for efficiently selecting PH distributions that are suitable for the intended computational analysis and adequately reflect the randomness being modelled [12]. However, it is not easy for the authors to find a concrete method for choosing PH distribution for approximating the distributions of interarrival time and/or service time in a given specific model.

For the retrial queues with multiple servers with Markovian arrival process (MAP), phase-type (PH) distribution of service and exponential retrial time, algorithmic solutions using matrix analytic method are presented by several authors e.g. [4, 7, 1]. For the references about matrix analytic method in retrial queue, see [9] and more details of retrial queues, refer the monographs Artalejo and Gómes-Correl [1], Falin and Templeton [1] and references therein. One can use $PH/PH/c$ retrial queue for $GI/G/c$ retrial queue by approximating the distribution of interarrival time and service time with PH distributions.

Random elements that affect the system performances in $GI/G/c$ retrial queue are interarrival time, service time and retrial time. Shin and Moon [16, 17] investigate the sensitivity of $M/M/c$ retrial queue with respect to retrial time and $M/G/c$ retrial queue with respect to service time, respectively. The experiments in [16, 17] are focused only on one element retrial time or service time for fixed other two random elements and do not investigate the random effects of other elements. It is necessary to investigate the effects of moments of interarrival time and service time when one approximate the retrial queue with general distributions of interarrival time and service time by the phase type distributions matching the moments of original distributions. In this paper, the effects of the moments of the interarrival time and service time to the performance measures such as the blocking probability, standard deviation of the busy servers, mean and standard deviation of the number of customers in orbit in $GI/G/c$ retrial queues are investigated numerically and present an approximation for $GI/G/c$ retrial queues using $PH/PH/c$ retrial queue.

In Section 2, we briefly introduce moment matching method for nonnegative random variables and present the sensitivity of ordinary $M/G/1$ and $GI/M/1$ queues. Sensitivity of some performance measures with respect to the moments of interarrival time and service time in $GI/G/c$ retrial queue is investigated in Section 3 and 4. Approximation examples for $GI/G/c$ retrial queue are presented in Section 5. Concluding remarks are presented in Section 6.

2. Preliminaries

2.1. Moment matching methods. There are some moment matching methods for fitting the general distribution of a positive random variable X with the first three moments $m_i = \mathbb{E}[X^i]$, $i = 1, 2, 3$ by PH distributions. Denote by $C_X^2 = \frac{m_2}{m_1^2} - 1$ the squared coefficient of variation of X. In this section we briefly introduce moment matching methods to be used for interrival time and service time.

Hyperexponential distribution : The hyperexponential distribution of order 2, denoted by $\mathrm{H}_2(p;\gamma_1,\gamma_2)$ or simply H_2, has the probability density function of the form $f(t) = p\gamma_1 e^{-\gamma_1 t} + (1-p)\gamma_2 e^{-\gamma_2 t}$, $t \geq 0$ and the phase type representation $PH(\alpha, T)$ with $\boldsymbol{\alpha} = (p,\ 1-p)$ and

$$T = \begin{pmatrix} -\gamma_1 & 0 \\ 0 & -\gamma_2 \end{pmatrix}.$$

The parameters p, γ_1 and γ_2 can be determined by the first two moments m_1 and C_X^2 as follows

$$p = \frac{1}{2}\left(1 + \sqrt{\frac{C_X^2 - 1}{C_X^2 + 1}}\right),\ \ \gamma_1 = \frac{2p}{m_1},\ \ \gamma_2 = \frac{2(1-p)}{m_1}. \tag{1}$$

The H_2 distribution can also be used for fitting the three moments of nonnegative random variables satisfying $C_X^2 > 1$ and

$$H = \frac{m_1 m_3}{1.5 m_2^2} > 1. \tag{2}$$

In this case, the distribution $H_2(p;\gamma_1,\gamma_2)$ with the preassigned moments m_i, $i = 1, 2, 3$ is uniquely determined by the parameters, see [18, 19]

$$\gamma_{1,2} = \frac{1}{2}\left(a_1 \pm \sqrt{a_1^2 - 4a_2}\right), \quad p = \frac{\gamma_1(1-\gamma_2 m_1)}{\gamma_1 - \gamma_2}, \tag{3}$$

where

$$a_1 = \frac{1}{m_1}(1 + \frac{1}{2}m_2 a_2), \quad a_2 = \frac{6m_1^2 - 3m_2}{\frac{3}{2}m_2^2 - m_1 m_3}.$$

The requirement (2) holds for the gamma distribution, lognormal distribution and Weibul distribution with $C_X^2 > 1$.

Erlang distribution of order k denoted by $\mathrm{E}_k(\mu)$ has the probability density function

$$f(t) = \frac{\mu^k}{(k-1)!} t^{k-1} e^{-\mu t}, \ t > 0$$

and phase type representation $PH(\boldsymbol{\alpha}, T)$ with $\boldsymbol{\alpha} = (1, 0, \cdots, 0)$ and

$$T = \begin{pmatrix} -\mu & \mu & & \\ & \ddots & \ddots & \\ & & -\mu & \mu \\ & & & -\mu \end{pmatrix}.$$

The mean and the squared variation of $\mathrm{E}_k(\mu)$ are $m_1 = \frac{k}{\mu}$ and $C_X^2 = \frac{1}{k}$.

Coxian distribution with Erlang node denote by $\mathrm{CE}_{k,j}(p; \mu_1, \mu_2)$ is the composition of the mixture of $\mathrm{E}_k(\mu_1)$ and $\mathrm{E}_j(\mu_2)$ whose Laplace transform $f^*(s)$ is given by

$$f^*(s) = p \left(\frac{\mu_1}{\mu_1 + s} \right)^k \left(\frac{\mu_2}{\mu_2 + s} \right)^j + (1-p) \left(\frac{\mu_2}{\mu_2 + s} \right)^j, \ s \geq 0$$

and the phase type representation $PH(\boldsymbol{\alpha}, T)$ with $\boldsymbol{\alpha} = p\boldsymbol{e}_{k+j,1} + (1-p)\boldsymbol{e}_{k+j,k+1}$, where $\boldsymbol{e}_{n,i}$ is the n dimensional vector whose ith component is one and others are all 0 and

$$T = \begin{pmatrix} T(k, \mu_1) & T^0(k, \mu_1) \\ O & T(j, \mu_2) \end{pmatrix}$$

where $T(n, \mu)$ is the matrix corresponding to the $E_n(\mu)$ distribution and $T^0(k, \mu_1)$ is the $k \times j$ matrix whose $(k, 1)$ component is μ_1 and others are all zero. If the nth moment of $\mathrm{CE}_{k,j}(p; \mu_1, \mu_2)$ is m_n, then the nth moment of $\mathrm{CE}_{k,j}(p; \frac{\mu_1}{m}, \frac{\mu_2}{m})$ is $m^n m_n$. Bobbio et al. [3] present explicit method to fit the first three moments of a positive random variable by $\mathrm{CE}_{1,j}(p; \mu_1, \mu_2)$ and $\mathrm{CE}_{k,1}(p; \mu_1, \mu_2)$ and the formulae for determining the parameters are so complicated and are omitted here.

Mixture of Erlang distributions of common order : Johnson and Taaffe [12] provide a method that a mixture $\mathrm{E}_{k,k}(p; \mu_1, \mu_2)$ of two Erlang distributions $\mathrm{E}_k(\mu_1)$ and $\mathrm{E}_k(\mu_2)$ with probability density function

$$f(t) = p\mu_1 \frac{(\mu_1 t)^{k-1}}{(k-1)!} e^{-\mu_1 t} + (1-p)\mu_2 \frac{(\mu_2 t)^{k-1}}{(k-1)!} e^{-\mu_2 t}$$

can fit the first three moments m_1, m_2 and m_3 of a positive random variable X. The parameters are given by

$$\mu_{1,2}^{-1} = \frac{1}{2a} \left(-b \pm \sqrt{b^2 - 4ac} \right), \quad p = \frac{\mu_1 - \mu_1 \mu_2 m_1 / k}{\mu_2 - \mu_1},$$

TABLE 1. Three-moment matching for $\mathrm{LN}(\mu,\sigma^2)$ with $m_1 = 1.0$

C_X^2	PH	p	γ_1	γ_2	m_4(PH)	m_4(LN)
0.5	$\mathrm{CE}_{1,3}(p;\gamma_1,\gamma_2)$	0.116747	0.950128	3.420263	11.2631	11.3906
1.0	$\mathrm{CE}_{1,2}(p;\gamma_1,\gamma_2)$	0.089641	0.509162	2.427350	54.3036	64.0
2.0	$\mathrm{H}_2(p;\gamma_1,\gamma_2)$	0.971405	1.138071	0.195262	486.0	729.0
	$\mathrm{CE}_{1,2}(p;\gamma_1,\gamma_2)$	0.054538	0.250255	2.557321	412.68	
5.0	$\mathrm{H}_2(p;\gamma_1,\gamma_2)$	0.990751	1.158270	0.063952	13284.0	46656.0
	$\mathrm{CE}_{1,2}(p;\gamma_1,\gamma_2)$	0.011275	0.069447	2.387636	12344.4	

TABLE 2. Three-moment matching for $\mathrm{Weib}(a,b)$ with $m_1 = 1.0$

C_X^2	PH	p	γ_1	γ_2	m_4(PH)	m_4(Weib)
0.5	$\mathrm{CE}_{2,1}(p;\gamma_1,\gamma_2)$	0.751282	2.880980	2.090070	6.8722	6.7948
2.0	$\mathrm{H}_2(p;\gamma_1,\gamma_2)$	0.658728	2.036487	0.504439	127.41	136.43
	$\mathrm{CE}_{1,2}(p;\gamma_1,\gamma_2)$	0.311072	0.523413	4.929922	124.63	
5.0	$\mathrm{H}_2(p;\gamma_1,\gamma_2)$	0.908248	1.816497	0.183503	1944.0	2520.0
	$\mathrm{CE}_{1,2}(p;\gamma_1,\gamma_2)$	0.090648	0.188615	3.850595	1901.1	

where

$$\begin{aligned} a &= k(k+2)m_1 y,\ b = -\left(kx + \frac{k(k+2)}{k+1}y^2 + (k+2)m_1^2 y\right),\ c = m_1 x, \\ y &= m_2 - \left(\frac{k+1}{k}\right)m_1^2,\ x = m_1 m_3 - \left(\frac{k+2}{k+1}\right)m_2^2. \end{aligned}$$

For other two-moment matching methods, see Tijms [18, Appendix B].

The lognormal distribution $\mathrm{LN}(\mu,\sigma^2)$ and Weibul distribution $\mathrm{Weib}(a,b)$ with probability density functions

$$\begin{aligned} f_{\mathrm{LN}}(x) &= \frac{1}{\sqrt{2\pi}\sigma x}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right),\ x>0, \\ f_{\mathrm{Weib}}(x) &= \frac{a}{b}\left(\frac{x}{b}\right)^{a-1}\exp\left[-\left(\frac{x}{b}\right)^a\right],\ x>0 \end{aligned}$$

are frequently used in application area. Some examples for fitting the first three moments of $\mathrm{LN}(\mu,\sigma^2)$ and $\mathrm{Weib}(a,b)$ with $m_1 = 1.0$ are listed in Tables 1-2, where m_4(PH), m_4(LN) and m_4(Weib)are the fourth moments of PH, lognormal and Weibul distributions, respectively.

2.2. Moment formulae for $M/G/1$ and $GI/M/1$ queues. It can be obtained from Pollaczek-Khintchine transform equation in ordinary $M/G/1$ queue that the mean L_q and variance V_q of the number of customers in queue are given by

$$L_q = \frac{\rho^2(1+C_s^2)}{2(1-\rho)},$$

$$V_q = \frac{\lambda^3 m_{s,3}}{3(1-\rho)} + L_q + L_q^2,$$

where $\rho = \frac{m_{s,1}}{m_{a,1}}$ is the traffic intensity and C_s^2 is the squared coefficient of variation of service time and $m_{a,k}$ and $m_{s,k}$ are kth moments of interarrival time and service time, respectively [10].

It is known that the mean L_{Orbit} and the variance V_{Orbit} of the number of customers in orbit in $M/G/1$ retrial queue with exponential retrial time with rate γ are given by

$$L_{\text{Orbit}} = L_q + \frac{\lambda}{\gamma}\frac{\rho}{1-\rho},$$
$$V_{\text{Orbit}} = V_q + \frac{\lambda}{\gamma}\left(\frac{\rho}{1-\rho} + L_q\right),$$

where λ is the arrival rate [8]. From this formula we can see that L_{Orbit} (V_{Orbit}) depend only on the first two (three) moments of service time.

It is known that the mean $L_q(GI/M/1)$ and variance $V_q(GI/M/1)$ of the number of customers in queue in ordinary $GI/M/1$ queue with service rate μ are given by

$$L_q(GI/M/1) = \rho\frac{r}{1-r},$$
$$V_q(GI/M/1) = \rho\frac{2r}{(1-r)^2} - L_q(GI/M/1) - L_q(GI/M/1)^2,$$

where r is the unique solution of the equation $A^*(\mu - \mu z) = z$ in $0 < z < 1$ and $A^*(s)$ is the Laplace-Stieltjes transform of interarrival time distribution [10]. In Tables 3-4, $L_q(GI/M/1)$ and $\sigma_q = \sqrt{V_q(GI/M/1)}$ for lognormal distribution (LN) of interarrival time are listed and the results are compared with those for H_2 and $CE_{1,2}$ whose first three moments $m_{a,i}$, $i = 1, 2, 3$ are the same as those of lognormal distribution. The H_2^b in the tables means the hyperexponential distribution whose parameters are determined by $m_{a,1} = 1.0$, $C_a^2 = 2.0$ and the formula (1) and by Dev(%) denotes the deviation from lognormal results (LN), for example, the deviation of H_2 from LN is calculated by Dev(%) $= |H_2 - LN| \times 100/LN$. No analytical results for $GI/M/1$ retrial queue are known. However, we can expect from the results of ordinary $GI/M/1$ queue that the mean and variance of the number of customers in orbit in $GI/M/1$ retrial queue may depend on the fourth or higher moments of interarrival time. In the following sections, the effects of interarrival times and service times in $GI/G/c$ retrial queue with exponential retrial time are investigated numerically.

3. Sensitivity with respect to interarrival time

We consider $GI/G/c$ retrial queue with exponential retrial time with rate γ. Let $m_{a,k}$ and $m_{s,k}$ ($k = 1, 2, 3$) be the kth moment of interarrival time and service time, respectively and $C_a^2 = \frac{m_{a,2}}{m_{a,1}^2} - 1$, $C_s^2 = \frac{m_{s,2}}{m_{s,1}^2} - 1$ the squared coefficient of

TABLE 3. L_q in $GI/M/1$ queue with $m_{a,1} = 1.0$ and $C_a^2 = 2.0$

ρ	0.3		0.6		0.9	
G	L_q	Dev(%)	L_q	Dev(%)	L_q	Dev(%)
LN	0.1379		1.1455		11.540	
H_2	0.1480	7.32	1.1062	3.43	11.394	1.26
$CE_{1,2}$	0.0946	31.4	0.9466	17.4	11.280	2.26
H_2^b	0.1806	31.0	1.3212	15.3	12.136	5.16

TABLE 4. σ_q in $GI/M/1$ queue with $m_{a,1} = 1.0$ and $C_a^2 = 2.0$

ρ	0.3		0.6		0.9	
G	σ_q	Dev(%)	σ_q	Dev(%)	σ_q	Dev(%)
LN	0.9057		2.8420		17.763	
H_2	0.9316	2.86	2.7718	2.47	17.551	1.19
$CE_{1,2}$	0.7944	12.3	2.4866	12.5	17.384	2.13
H_2^b	1.0149	12.1	3.1557	11.0	18.628	4.87

variation and $\rho = \frac{m_{s,1}}{c\, m_{a,1}}$. Let X_0 and X_1 be the number of customers in service facility and orbit, respectively in stationary state and $P_B = P(X_0 = c)$ the blocking probability, $L_i = \mathbb{E}[X_i]$ and $\sigma(X_i)$ be the mean and standard deviation of X_i, $i = 0, 1$. It can be seen from Little's law that $L_0 = \frac{m_{s1}}{m_{a1}}$. As we have seen in Tables $3 - 4$, L_1 and σ_1 can be affected by the fourth or higher moments of interarrival time. For investigating the effects of interarrival time in retrial queue, we consider the two retrial queues $CE_{1,2}/G/3$ retrial queue and $H_2/G/3$ retrial queue with common $m_{a,1} = 1.0$, $C_a^2 = 2.0$ and $m_{a,3} = 27.0$ and different fourth moments $m_{a,4}(CE_{1,2}) = 412.68$, $m_{a,4}(H_2) = 486.0$. The deviations Dev(%) $= |CE_{1,2} - H_2| \times 100/H_2$ for P_B, L_1 and σ_1 between $CE_{1,2}/G/3$ retrial queue and $H_2/G/3$ retrial queue are listed in Tables $5 - 7$, where the service time distributions are selected as Erlang distribution (E_2) of order 2 for $C_s^2 = 0.5$, exponential distribution (M) for $C_s^2 = 1.0$ and hyperexponential distribution (H_2) whose parameters are determined by (1) for $C_s^2 = 2.0$. We can see from the tables that (1) Dev(%) increases as γ increases and the deviations for ordinary queue ($\gamma = \infty$) are upper bound of retrial queue except the case for L_1 and $\sigma(X_1)$ with small value of retrial rate $\gamma = 0.1$ (2) Dev(%) increases as ρ decreases and can be very large for light traffic case and (3) Dev(%) increases as C_s^2 decreases.

The deviations Dev(%) between $CE_{1,2}/M/c$ and $H_2/M/c$ queues without retrials (ordinary queues) are listed in Table 8 for various c. From the table Dev(%) increases rapidly as c increases for the light traffic system. It should be noted that in the light traffic system the values of P_B, L_1 and $\sigma(X_1)$ are very small and the absolute deviation is very small even though Dev(%) is very large. For example, the values of L_1 in $CE_{1,2}/M/7$ and $H_2/M/7$ queues with $\rho = 0.3$

TABLE 5. P_B in $GI/G/3$ retrial queues ($m_{a,1} = 1.0$, $C_a^2 = 2.0$, $m_{a,3} = 27.0$)

ρ	γ	Serv. Arr.	E_2 P_B	Dev(%)	M P_B	Dev(%)	H_2 P_B	Dev(%)
0.3	0.1	$CE_{1,2}$	0.0510		0.0552		0.0575	
		H_2	0.0643	20.8	0.0642	14.1	0.0642	10.4
	1.0	$CE_{1,2}$	0.0529		0.0582		0.0619	
		H_2	0.0683	22.5	0.0692	15.9	0.0704	12.2
	10.0	$CE_{1,2}$	0.0565		0.0626		0.0667	
		H_2	0.0759	25.6	0.0769	18.6	0.0778	14.3
	∞^*	$CE_{1,2}$	0.0583		0.0644		0.0683	
		H_2	0.0794	26.6	0.0799	19.4	0.0805	15.1
0.5	0.1	$CE_{1,2}$	0.1987		0.2019		0.2032	
		H_2	0.2048	2.97	0.2049	1.43	0.2057	1.20
	1.0	$CE_{1,2}$	0.2155		0.2220		0.2265	
		H_2	0.2249	4.21	0.2280	2.63	0.2320	2.35
	10.0	$CE_{1,2}$	0.2348		0.2425		0.2462	
		H_2	0.2507	6.33	0.2529	4.10	0.2551	3.51
	∞^*	$CE_{1,2}$	0.2419		0.2490		0.2517	
		H_2	0.2593	6.73	0.2604	4.39	0.2617	3.79
0.8	0.1	$CE_{1,2}$	0.5746		0.5766		0.5797	
		H_2	0.5740	0.11	0.5757	0.16	0.5790	0.12
	1.0	$CE_{1,2}$	0.6282		0.6298		0.6325	
		H_2	0.6244	0.61	0.6274	0.40	0.6316	0.14
	10.0	$CE_{1,2}$	0.6715		0.6690		0.6670	
		H_2	0.6668	0.71	0.6664	0.39	0.6668	0.03
	∞^*	$CE_{1,2}$	0.6825		0.6784		0.6750	
		H_2	0.6775	0.73	0.6757	0.39	0.6749	0.01

* ordinary $GI/G/3$ queue

are $L_1(CE_{1,2}) = 0.0015$ and $L_1(H_2) = 0.0045$, respectively and the absolute deviation is $|L_1(CE_{1,2}) - L_1(H_2)| = 0.0030$ even though Dev(%) = 65.4.

4. Sensitivity with respect to service time

For investigation of the influence of the moments of service time, we choose the service time distributions $CE_{1,2}$ and H_2 with common $m_{s,1} = 1.0$, $C_s^2 = 2.0$ and $m_{s,3} = 27.0$ in Table 1 and the numerical results for P_B and L_1 are listed in Tables 9 − 10. The numerical results show that P_B and L_1 are affected weakly by the fourth or the higher moments of the service time which is expected from the results for the system with $c = 1$.

TABLE 6. L_1 in $GI/G/3$ retrial queues ($m_{a,1} = 1.0$, $C_a^2 = 2.0$, $m_{a,3} = 27.0$)

ρ	γ	Serv. / Arr.	E_2 / L_1	Dev(%)	M / L_1	Dev(%)	H_2 / L_1	Dev(%)
0.3	0.1	$CE_{1,2}$	0.3927		0.4571		0.5049	
		H_2	0.7888	50.2	0.7944	42.5	0.8061	37.4
	1.0	$CE_{1,2}$	0.0530		0.0682		0.0873	
		H_2	0.1108	52.2	0.1202	43.2	0.1344	35.1
	10.0	$CE_{1,2}$	0.0171		0.0258		0.0385	
		H_2	0.0405	57.7	0.0485	46.8	0.0597	35.5
	∞^*	$CE_{1,2}$	0.0126		0.0203		0.0320	
		H_2	0.0317	60.2	0.0394	48.5	0.0500	35.9
0.5	0.1	$CE_{1,2}$	2.375		2.531		2.682	
		H_2	3.047	22.0	3.102	18.4	3.218	16.7
	1.0	$CE_{1,2}$	0.415		0.506		0.639	
		H_2	0.551	24.6	0.623	18.7	0.742	13.9
	10.0	$CE_{1,2}$	0.191		0.268		0.384	
		H_2	0.280	31.9	0.347	22.8	0.452	15.1
	∞^*	$CE_{1,2}$	0.162		0.236		0.350	
		H_2	0.246	34.2	0.311	24.1	0.414	15.7
0.8	0.1	$CE_{1,2}$	16.02		16.75		18.11	
		H_2	16.84	4.87	17.51	4.34	18.87	4.02
	1.0	$CE_{1,2}$	4.361		5.095		6.428	
		H_2	4.489	2.85	5.207	2.15	6.537	1.67
	10.0	$CE_{1,2}$	2.881		3.651		4.980	
		H_2	3.048	5.46	3.776	3.31	5.083	2.03
	∞^*	$CE_{1,2}$	2.686		3.463		4.792	
		H_2	2.868	6.35	3.597	3.73	4.901	2.21

* ordinary $GI/G/3$ queue

5. Approximation of $GI/G/c$ retrial queue

In this section we describe the approximation procedure for $LN/Weib/3$ retrial queue with $m_{s,1} = 1.0$ by using $PH/PH/3$ retrial queue and make some numerical comparisons.

Once the interarrival time and service time are approximated by PH distributions, the $PH/PH/c$ retrial queue can be easily modeled by a level dependent quasi-birth-and-death process (LDQBD) (e.g. see Artalejo and Gómes-Correl [1]) and one can use the algorithm in [5] for computing the stationary distribution of LDQBD process. There may be several PH distributions that match the first three moments of service time. It is recommended to use the PH distribution among them as small number of phases as possible for saving the computer memory and computing time and the difference of fourth moment is as small as possible for accuracy of approximation. For an approximation, we first choose an appropriate PH distribution by fitting the first three moments of interarrival

TABLE 7. $\sigma(X_1)$ in $GI/G/3$ retrial queues ($m_{a,1} = 1.0$, $C_a^2 = 2.0$, $m_{a,3} = 27.0$)

		Serv.	E_2		M		H_2	
ρ	γ	Arr.	$\sigma(X_1)$	Dev(%)	$\sigma(X_1)$	Dev(%)	$\sigma(X_1)$	Dev(%)
0.3	0.1	$CE_{1,2}$	0.7009		0.7997		0.9296	
		H_2	1.0767	34.9	1.1281	29.1	1.2203	23.8
	1.0	$CE_{1,2}$	0.2592		0.3149		0.4049	
		H_2	0.4110	36.9	0.4523	30.4	0.5245	22.8
	10.0	$CE_{1,2}$	0.1507		0.2006		0.2851	
		H_2	0.2581	41.6	0.3018	33.5	0.3731	23.6
	∞^*	$CE_{1,2}$	0.1311		0.1809		0.2658	
		H_2	0.2322	43.6	0.2771	34.7	0.3493	23.9
0.5	0.1	$CE_{1,2}$	2.114		2.398		2.872	
		H_2	2.570	17.7	2.787	13.9	3.201	10.3
	1.0	$CE_{1,2}$	0.887		1.105		1.504	
		H_2	1.122	21.0	1.308	15.5	1.662	9.52
	10.0	$CE_{1,2}$	0.617		0.835		1.229	
		H_2	0.832	25.8	1.022	18.3	1.373	10.4
	∞^*	$CE_{1,2}$	0.575		0.793		1.188	
		H_2	0.788	27.1	0.980	19.0	1.331	10.7
0.8	0.1	$CE_{1,2}$	9.784		11.01		13.38	
		H_2	10.09	2.98	11.27	2.26	13.58	1.52
	1.0	$CE_{1,2}$	5.002		6.133		8.355	
		H_2	5.220	4.17	6.296	2.59	8.459	1.23
	10.0	$CE_{1,2}$	4.114		5.292		7.552	
		H_2	4.386	6.20	5.486	3.54	7.670	1.54
	∞^*	$CE_{1,2}$	3.990		5.179		7.447	
		H_2	4.274	6.66	5.381	3.74	7.569	1.63

* ordinary $GI/G/3$ queue

TABLE 8. Dev(%) between ordinary $H_2/M/c$ and $CE_{1,2}/M/c$ queues ($m_{a,1} = 1.0$, $C_a^2 = 2.0$, $m_{a,3} = 27.0$, $m_{a,4}(H_2) = 486.0$, $m_{a,4}(CE_{1,2}) = 412.7$)

	P_B			L_1			$\sigma(X_1)$		
ρ	$c=3$	$c=5$	$c=7$	$c=3$	$c=5$	$c=7$	$c=3$	$c=5$	$c=7$
0.3	19.4	34.0	46.0	48.5	57.8	65.4	34.7	41.1	46.7
0.5	4.39	8.36	12.1	24.1	27.3	30.2	19.0	20.8	22.5
0.7	0.01	0.32	0.73	8.80	9.10	9.47	7.99	8.15	8.34
0.8	0.39	0.46	0.45	3.73	3.67	3.68	3.74	3.72	3.72
0.9	0.24	0.31	0.34	0.77	0.69	0.66	0.93	0.91	0.90

Table 9. P_B in $GI/G/3$ retrial queues ($m_{s,1} = 1.0$, $C_s^2 = 2.0$, $m_{s,3} = 27.0$)

		Arr.	E_2		M		H_2	
ρ	γ	Serv.	P_B	Dev(%)	P_B	Dev(%)	P_B	Dev(%)
0.3	0.1	$CE_{1,2}$	0.0402		0.0581		0.0712	
		H_2	0.0422	4.58	0.0580	0.15	0.0715	0.38
	1.0	$CE_{1,2}$	0.0425		0.0627		0.0789	
		H_2	0.0444	4.20	0.0627	0.12	0.0791	0.24
	10.0	$CE_{1,2}$	0.0448		0.0686		0.0881	
		H_2	0.0469	4.59	0.0687	0.12	0.0882	0.05
	∞^*	$CE_{1,2}$	0.0457		0.0709		0.0917	
		H_2	0.0479	4.58	0.0709	0.11	0.0913	0.36
0.5	0.1	$CE_{1,2}$	0.1686		0.1918		0.2104	
		H_2	0.1725	2.28	0.1916	0.11	0.2114	0.49
	1.0	$CE_{1,2}$	0.1799		0.2089		0.2360	
		H_2	0.1836	2.03	0.2087	0.10	0.2370	0.42
	10.0	$CE_{1,2}$	0.1933		0.2313		0.2671	
		H_2	0.1982	2.50	0.2315	0.07	0.2682	0.44
	∞^*	$CE_{1,2}$	0.1990		0.2405		0.2801	
		H_2	0.2040	2.46	0.2400	0.19	0.2779	0.77
0.8	0.1	$CE_{1,2}$	0.5549		0.5625		0.5696	
		H_2	0.5563	0.26	0.5624	0.02	0.5701	0.09
	1.0	$CE_{1,2}$	0.5756		0.5907		0.6105	
		H_2	0.5778	0.37	0.5908	0.02	0.6112	0.13
	10.0	$CE_{1,2}$	0.6084		0.6336		0.6628	
		H_2	0.6127	0.70	0.6340	0.07	0.6647	0.28
	∞^*	$CE_{1,2}$	0.6242		0.6520		0.6844	
		H_2	0.6283	0.64	0.6514	0.10	0.6811	0.49

* ordinary $GI/G/3$ queue

time and the service time and then compute the performance characteristics of the approximating system. In order to fit the first three moments of lognormal distribution with $C_a^2 = 0.5$ and $C_a^2 = 2.0$, we adopt $CE_{1,3}$ and H_2, respectively described in Table 1 and the Weibul distributions with $C_s^2 = 0.5$ and $C_s^2 = 2.0$ are fitted by $CE_{2,1}$ and H_2 described in Table 2, respectively. Approximation results (App) are compared with simulation (Sim) in Tables 11 − 12, where Dev = $App - Sim$. Simulation models are developed with ARENA [13]. Simulation run time is set to 80,000 unit times including 20,000 unit times of warm-up period, where the expected value of service time is one unit time. Ten replications are conducted for each case and the average value and the half length of 95% confidence interval (c.i.) are obtained. Tables 11 − 12 show that approximations of $GI/G/c$ retrial queue using $PH/PH/c$ retrial queue are quite similar to simulation within the confidence interval.

TABLE 10. L_1 in $GI/G/3$ retrial queues ($m_{s,1} = 1.0$, $C_s^2 = 2.0$, $m_{s,3} = 27.0$)

ρ	γ	Arr. Serv.	E_2 L_1	Dev(%)	M L_1	Dev(%)	H_2 L_1	Dev(%)
0.3	0.1	$CE_{1,2}$	0.2267		0.5911		0.9339	
		H_2	0.2457	7.71	0.5910	0.01	0.9397	0.61
	1.0	$CE_{1,2}$	0.0378		0.0944		0.1588	
		H_2	0.0404	6.39	0.0960	1.65	0.1564	1.52
	10.0	$CE_{1,2}$	0.0154		0.0393		0.0727	
		H_2	0.0169	8.87	0.0418	5.83	0.0685	6.20
	∞^*	$CE_{1,2}$	0.0126		0.0324		0.0571	
		H_2	0.0140	9.91	0.0349	7.27	0.0616	7.29
0.5	0.1	$CE_{1,2}$	2.3512		3.862		5.246	
		H_2	2.4654	4.63	3.859	0.07	5.297	0.96
	1.0	$CE_{1,2}$	0.4323		0.692		1.016	
		H_2	0.4467	3.23	0.698	0.77	1.012	0.47
	10.0	$CE_{1,2}$	0.1988		0.333		0.541	
		H_2	0.2097	5.16	0.345	3.41	0.524	3.26
	∞^*	$CE_{1,2}$	0.1674		0.286		0.458	
		H_2	0.1782	6.06	0.299	4.30	0.477	4.04
0.8	0.1	$CE_{1,2}$	29.47		34.31		38.67	
		H_2	29.79	1.09	34.30	0.03	38.80	0.34
	1.0	$CE_{1,2}$	5.774		6.993		8.898	
		H_2	5.813	0.67	7.004	0.15	8.897	0.01
	10.0	$CE_{1,2}$	3.130		4.016		5.631	
		H_2	3.174	1.40	4.055	0.97	5.586	0.79
	∞^*	$CE_{1,2}$	2.780		3.633		5.150	
		H_2	2.831	1.81	3.680	1.28	5.205	1.06

* ordinary $GI/G/3$ queue

6. Conclusions

It is well known to be very difficult to analyze $GI/G/c$ retrial queue and there are few results even for approximation of the system. A promising approach to approximate $GI/G/c$ retrial queue is to use $PH/PH/c$ retrial queue by fitting the first three moments of interarrival time and service time with PH distributions. In this paper, we have investigated numerically the effects of the moments of the interarrival time and the service time to the performance measures related with the number X_0 of busy servers and the number X_1 of customers in orbit in $GI/G/c$ retrial queue. Numerical results show that the performance measures are more sensitive with respect to interarrival time than service time, in particular the case that traffic intensity ρ is small. More specifically speaking, the sensitivity of performance measures with respect to interarrival time increases as γ increases or ρ decreases. However, in case of small ρ, the values of P_B, L_1 and $\sigma(X_1)$ are very small and the absolute deviation is very small even though

TABLE 11. Appr. of $LN/Weib/3$ retrial queue by $CE_{13}/H_2/3$ retrial queue ($m_{s1} = 1.0$, $C_a^2 = 0.5$, $C_s^2 = 2.0$)

	ρ	γ	0.1	1.0	10.0
P_B	0.4	Sim (c.i.)	0.0978 (±0.0013)	0.1038 (±0.0015)	0.1113 (±0.0014)
		App (Dev)	0.0967 (−0.0011)	0.1030 (−0.0008)	0.1104 (−0.0009)
	0.8	Sim (c.i.)	0.5604 (±0.0043)	0.5848 (±0.0045)	0.6209 (±0.0044)
		App (Dev)	0.5572 (−0.0032)	0.5814 (−0.0034)	0.6181 (−0.0028)
$\sigma(X_0)$	0.4	Sim (c.i.)	0.9274 (±0.0019)	0.9392 (±0.0021)	0.9514 (±0.0018)
		App (Dev)	0.9247 (−0.0027)	0.9367 (−0.0025)	0.9487 (−0.0027)
	0.8	Sim (c.i.)	0.7720 (±0.0044)	0.8138 (±0.0041)	0.8720 (±0.0032)
		App (Dev)	0.7730 (+0.0010)	0.8155 (+0.0017)	0.8734 (+0.0014)
L_1	0.4	Sim (c.i.)	0.8624 (±0.0207)	0.1585 (±0.0042)	0.0759 (±0.0021)
		App (Dev)	0.8479 (−0.0145)	0.1563 (−0.0022)	0.0748 (−0.0011)
	0.8	Sim (c.i.)	29.920 (±0.686)	5.9595 (±0.1593)	3.3297 (±0.1058)
		App (Dev)	29.358 (−0.562)	5.8402 (−0.1193)	3.2563 (−0.0734)
$\sigma(X_1)$	0.4	Sim (c.i.)	1.2482 (±0.0230)	0.5634 (±0.0142)	0.4127 (±0.0134)
		App (Dev)	1.2473 (−0.0009)	0.5658 (+0.0024)	0.4143 (+0.0016)
	0.8	Sim (c.i.)	13.991 (±0.718)	6.6082 (±0.2438)	5.1572 (±0.2111)
		App (Dev)	13.817 (−0.174)	6.4646 (−0.1436)	5.0701 (−0.0871)

TABLE 12. Appr. of $LN/Weib/3$ retrial queue by $H_2/CE_{21}/3$ retrial queue ($m_{s1} = 1.0$, $C_a^2 = 2.0$, $C_s^2 = 0.5$)

	ρ	γ	0.1	1.0	10.0
P_B	0.4	Sim (c.i.)	0.1281 (±0.0018)	0.1378 (±0.0015)	0.1546 (±0.0016)
		App (Dev)	0.1243 (−0.0038)	0.1336 (−0.0042)	0.1498 (−0.0048)
	0.8	Sim (c.i.)	0.5658 (±0.0044)	0.6031 (±0.0035)	0.6589 (± 0.0038)
		App (Dev)	0.5657 (−0.0001)	0.6017 (−0.0014)	0.6555 (−0.0034)
$\sigma(X_0)$	0.4	Sim (c.i.)	1.0091 (±0.0015)	1.0310 (±0.0017)	1.0601 (±0.0013)
		App (Dev)	0.9977 (−0.0114)	1.0187 (−0.0123)	1.0464 (−0.0137)
	0.8	Sim (c.i.)	0.7930 (±0.0059)	0.8616 (±0.0031)	0.9540 (±0.0051)
		App (Dev)	0.7895 (−0.0035)	0.8614 (−0.0002)	0.9520 (−0.0020)
L_1	0.4	Sim (c.i.)	1.9265 (±0.0433)	0.2958 (±0.0057)	0.1208 (±0.0032)
		App (Dev)	1.9653 (+0.0388)	0.2963 (+0.0005)	0.1211 (+0.0003)
	0.8	Sim (c.i.)	35.040 (±0.731)	6.5583 (±0.1326)	3.4209 (±0.0713)
		App (Dev)	35.314 (−0.274)	6.5321 (−0.0262)	3.2936 (−0.1273)
$\sigma(X_1)$	0.4	Sim (c.i.)	1.8144 (±0.0231)	0.7289 (±0.0081)	0.4840 (±0.0104)
		App (Dev)	1.8585 (+0.0441)	0.7369 (+0.0080)	0.4911 (+0.0071)
	0.8	Sim (c.i.)	14.581 (±0.365)	6.2761 (±0.1827)	4.6943 (±0.1175)
		App (Dev)	14.749 (−0.168)	6.2445 (−0.0316)	4.5273 (−0.1670)

Dev(%) is large. The approximation procedure for $LN/Weib/3$ retrial queue by using $PH/PH/3$ retrial queue has been described as an example. Approximation results are very similar to those of simulation within the confidence interval. So the simulation can be replaced by the method proposed in this paper for practical purpose.

Now we describe the restrictions of the method proposed and further research area for improving the accuracy of approximation. The method to approximate the multi server retrial queue by fitting the general distribution with PH distributions is needed to compute the stationary distribution of level dependent quasi-birth-and-death process which may require relatively long computation times and large size of memory when both of the number of phases of PH distribution and the number of servers are large. So the method is limited to small values of c and to the PH distribution of lower order.

When one approximates retrial queue with general distribution of interarrival and service times using the phase type distribution, it should be careful to choose PH distribution especially for interarrival time distribution. There may be the case, for example the light traffic system with large number of servers that it is not sufficient to fit only the first three moments of general distribution with PH distribution. However, there are few results for general and effective method of matching the first four or higher moments of nonnegative random variable by PH distributions which is necessary to be developed for more accurate approximation.

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