FRACTIONAL CHEBYSHEV FINITE DIFFERENCE METHOD FOR SOLVING THE FRACTIONAL BVPS

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ABSTRACT. In this paper, we introduce a new numerical technique which we call fractional Chebyshev finite difference method (FChFD). The algorithm is based on a combination of the useful properties of Chebyshev polynomials approximation and finite difference method. We tested this technique to solve numerically fractional BVPs. The proposed technique is based on using matrix operator expressions which applies to the differential terms. The operational matrix method is derived in our approach in order to approximate the fractional derivatives. This operational matrix method can be regarded as a non-uniform finite difference scheme. The error bound for the fractional derivatives is introduced. The fractional derivatives are presented in terms of Caputo sense. The application of the method to fractional BVPs leads to algebraic systems which can be solved by an appropriate method. Several numerical examples are provided to confirm the accuracy and the effectiveness of the proposed method.

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1. Introduction

Fractional differential equations have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, biology, physics and engineering [2]. Consequently, considerable attention has been given to the solutions of FDEs and integral equations of physical interest. Most FDEs do not have exact analytical solutions, so approximate and numerical techniques [7, 19, 23] must be used. Several numerical methods to solve FDEs have been given such as homotopy perturbation method [21, 22], Adomian decomposition method [13], homotopy analysis method [12] and collocation method

[14, 15, 16, 17]. Representation of a function in terms of a series expansion using orthogonal polynomials is a fundamental concept in approximation theory and form the basis of the solution of differential equations [3, 10, 11, 16, 24, 25]. Chebyshev polynomials are widely used in numerical computation. One of the advantages of using Chebyshev polynomials as a tool for expansion functions is the good representation of smooth functions by finite Chebyshev expansion provided that the function $y(x)$ is infinitely differentiable. The coefficients in Chebyshev expansion approach zero faster than any inverse power in n as n goes to infinity. Clenshaw and Curtis [4] give a procedure for the numerical integration of a non-singular function $y(x)$ by expanding the function in a series of Chebyshev polynomials and integration term by term. Elbarbary introduced Chebyshev finite difference approximation for the boundary value problems of integer derivatives [5, 6, 17]. A new formula expressing explicitly the derivatives of shifted Chebyshev polynomials of any degree and for any fractional order in terms of Chebyshev polynomials themselves is stated and proved in [7].

The purpose of this paper is to present an alternative operational matrix for the fractional differentiation. The fractional derivatives of the function $y(x)$ at the point x_k, $0 \leq k \leq N$ are expanded as a linear combination from the values of the function $y(x)$ at the shifted Gauss-Lobatto points $x_r = \frac{L}{2} - \frac{L}{2}\cos(\frac{\pi r}{N})$, $r = 0, 1, ..., N$ associated with the interval $[0, L]$. The main characteristic of this new technique is that it gives a straight forward algorithm in converting fractional BVPs to a system of algebraic equations. The first and last rows of the coefficients matrix of the algebraic system are replaced by suitable formulation of the boundary conditions. The suggested method is more accurate in comparison to the finite difference and finite element methods as the approximation of the fractional derivatives is defined over the whole domain. This algorithm has several advantages such as being non-differentiable, non-integral and easily implemented on a computer, because its structure is dependent on matrix operations only. The main aim of the presented paper is concerned with the application of this approach to obtain the numerical solution of fractional BVPs.

The structure of this paper is arranged in the following way: In section 2, we introduce some basic definitions about Caputo fractional derivatives and properties the shifted Chebyshev polynomials. In section 3, the basic formulation of the new operational matrix method using FChFD method. In section 4, an error bound of the fractional derivatives is introduced. In section 5, numerical examples are given to solve FBVPs and show the accuracy of the presented method. Finally, in section 6, the report ends with a brief conclusion and some remarks.

2. Preliminaries and notations

In this section, we present some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

2.1. The fractional derivative in the Caputo sense.

Definition 2.1. The Caputo fractional derivative operator D^ν of order ν is defined in the following form:

$$D^\nu f(x) = \frac{1}{\Gamma(m-\nu)} \int_0^x \frac{f^{(m)}(\xi)}{(x-\xi)^{\nu-m+1}} d\xi, \qquad \nu > 0,$$

where $m-1 < \nu \le m$, $m \in \mathbb{N}$, $x > 0$.

Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation:

$$D^\nu \left(\lambda p(x) + \mu q(x)\right) = \lambda \, D^\nu p(x) + \mu \, D^\nu q(x), \tag{1}$$

where λ and μ are constants.
For the Caputo's derivative we have:

$$D^\nu C = 0, \qquad \text{C is a constant}, \tag{2}$$

$$D^\nu x^n = \begin{cases} 0, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n < \lceil \nu \rceil; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\nu)} x^{n-\nu}, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n \ge \lceil \nu \rceil. \end{cases} \tag{3}$$

We use the ceiling function $\lceil \nu \rceil$ to denote the smallest integer greater than or equal to ν and $\mathbb{N}_0 = \{0, 1, 2, ...\}$. Recall that for $\nu \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.
For more details on fractional derivatives definitions and its properties see [1, 10, 20].

2.2. The definition and properties of the shifted Chebyshev polynomials.
The well known Chebyshev polynomials are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formula:

$$T_{n+1}(z) = 2z \, T_n(z) - T_{n-1}(z), \quad T_0(z) = 1, \ T_1(z) = z \quad n = 1, 2, ...$$

It is well known that $T_i(1) = 1$, and $T_i(-1) = (-1)^n$. The analytic form of the Chebyshev polynomials $T_n(z)$ of degree n is given by

$$T_n(z) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \, 2^{n-2\,i-1} \, \frac{n\,(n-i-1)!}{(i)!\,(n-2\,i)!} \, z^{n-2\,i}, \tag{4}$$

where $\lfloor n/2 \rfloor$ denotes the integer part of $n/2$. The orthogonality condition is

$$\int_{-1}^1 \frac{T_i(z)\,T_j(z)}{\sqrt{1-z^2}} \, dz = \begin{cases} \pi, & \text{for} \quad i = j = 0; \\ \frac{\pi}{2}, & \text{for} \quad i = j \ne 0; \\ 0, & \text{for} \quad i \ne j. \end{cases} \tag{5}$$

In order to use these polynomials on the interval $[0, L]$ we define the so called shifted Chebyshev polynomials by introducing the change of variable $z = \frac{2x}{L} - 1$. The shifted Chebyshev polynomials are defined as:

$$T_n^*(x) = T_n(\frac{2x}{L} - 1), \ \text{where} \quad T_0^*(x) = 1, \quad T_1^*(x) = \frac{2x}{L} - 1.$$

The analytic form of the shifted Chebyshev polynomial $T_n^*(x)$ of degree n is given by:

$$T_n^*(x) = n \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k-1)! 2^{2k}}{(n-k)!(2k)! L^k} x^k, \tag{6}$$

where, $T_n^*(0) = (-1)^n$, and $T_n^*(L) = 1$. The orthogonality condition of these polynomials is:

$$\int_0^L T_j^*(x) T_k^*(x) w(x) dx = \delta_{jk} h_k, \tag{7}$$

where, the weight function $w(x) = \frac{1}{\sqrt{Lx - x^2}}$, $h_k = \frac{b_k}{2}\pi$, with $b_0 = 2, b_k = 1, \ k \geq 1$.

The function $y(x)$ which belongs to the space of square integrable in $[0, L]$, may be expressed in terms of shifted Chebyshev polynomials as:

$$y(x) = \sum_{n=0}^{\infty} c_n T_n^*(x),$$

where the coefficients c_n are given by:

$$c_n = \frac{1}{h_n} \int_0^L y(x) T_n^*(x) w(x) dx, \quad n = 0, 1, 2, \dots. \tag{8}$$

3. Basic formulation of the new operational matrix method using FChFD method

The well known shifted Chebyshev polynomials of the first kind of degree n are defined on the interval $[0, L]$ as in Eq.(6). We choose the grid (interpolation) points to be the Chebyshev- Gauss Lobatto points associated with the interval $[0, L]$, $x_r = \frac{L}{2} - \frac{L}{2} \cos(\frac{\pi r}{N})$, $r = 0, 1, ..., N$. These grids can be written as $L = x_N < x_{N-1} < ... < x_1 < x_0 = 0$.

Clenshaw and Curtis [4] introduced an approximation of the function $y(x)$, we reformulate it to be used on the shifted Chebyshev polynomials as follows,

$$y(x) = \sum_{n=0}^{N} {}'' a_n T_n^*(x), \quad a_n = \frac{2}{N} \sum_{r=0}^{N} {}'' y(x_r) T_n^*(x_r). \tag{9}$$

The summation symbol with double primes denotes a sum with both first and last terms halved.

The fractional derivative of the function $y(x)$ at the point x_s is given in the following theorem.

Theorem 3.1. *The fractional derivative of order ν in the Caputo sense for the function $y(x)$ at the point x_s is given by*

$$y^{(\nu)}(x_s) = \sum_{r=0}^{N} d_{s,r}^{(\nu)} y(x_r), \quad \nu > 0, \tag{10}$$

such that

$$d_{s,r}^{(\nu)} = \frac{4\theta_r}{N} \sum_{n=\lceil\nu\rceil}^{N} \sum_{j=0}^{N} \sum_{k=\lceil\nu\rceil}^{n} \frac{n\theta_n}{b_j} \frac{(-1)^{n-k}\,(n+k-1)!\,\Gamma(k-\nu+\frac{1}{2})}{L^\nu\,\Gamma(k+\frac{1}{2})\,(n-k)!\,\Gamma(k-\nu-j+1)\,\Gamma(k-\nu+j+1)}\,T_n^*(x_r)\,T_j^*(x_s),$$

where, $s, r = 0, 1, 2, ..., N$ with $\theta_0 = \theta_N = \frac{1}{2}$, $\theta_i = 1\ \forall\ i = 1, 2, ..., N-1$.

Proof. The fractional derivative of the approximate formula for the function $y(x)$ in Eq.(9) is given by

$$D^{(\nu)}\,y(x) = \sum_{n=0}^{N}{}'' a_n\,D^{(\nu)}T_n^*(x). \tag{11}$$

Employing Eqs.(2) and (3) we have:

$$D^{(\nu)}T_n^*(x) = 0, \quad n = 0, 1, ..., \lceil\nu\rceil - 1,$$

then,

$$D^{(\nu)}\,y(x) = \sum_{n=\lceil\nu\rceil}^{N}{}'' a_n\,D^{(\nu)}T_n^*(x), \quad a_n = \frac{2}{N}\sum_{r=0}^{N}{}'' y(x_r)\,T_n^*(x_r). \tag{12}$$

Also, for $n = \lceil\nu\rceil, ..., N$ and by using Eqs.(2)-(3), we get

$$
\begin{aligned}
D^{(\nu)}\,T_n^*(x) &= n\sum_{k=0}^{n}(-1)^{n-k}\frac{(n+k-1)!\,2^{2k}}{(n-k)!\,(2k)!\,L^k}\,D^{(\nu)}x^k \\
&= n\sum_{k=\lceil\nu\rceil}^{n}(-1)^{n-k}\frac{(n+k-1)!\,2^{2k}\,k!}{(n-k)!\,(2k)!\,L^k\Gamma(k-\nu+1)}\,x^{k-\nu}.
\end{aligned}
\tag{13}
$$

Now, $x^{k-\nu}$ can be expressed approximately in terms of shifted Chebyshev series, so we have:

$$x^{k-\nu} \cong \sum_{j=0}^{N} c_{kj}\,T_j^*(x), \tag{14}$$

where, c_{kj} is obtained from (8) with $y(x) = x^{k-\nu}$ [7]. If only the first $(N+1)$-terms from shifted Chebyshev polynomials in Eq.(9) are considered, the approximate formula for the fractional derivative of the shifted Chebyshev polynomials introduced by Doha [7] as follows:

$$D^{(\nu)}\,T_n^*(x) = \sum_{j=0}^{N} \sum_{k=\lceil\nu\rceil}^{n} \frac{(-1)^{n-k}\,2n\,(n+k-1)!\,\Gamma(k-\nu+\frac{1}{2})}{b_j\,L^\nu\,\Gamma(k+\frac{1}{2})\,(n-k)!\,\Gamma(k-\nu-j+1)\,\Gamma(k-\nu+j+1)}\,T_j^*(x). \tag{15}$$

From Eqs.(12) and (15), we have:

$$D^{(\nu)}\,y(x) = \frac{4}{N}\sum_{n=\lceil\nu\rceil}^{N}{}''\sum_{r=0}^{N}{}''\sum_{j=0}^{N}\sum_{k=\lceil\nu\rceil}^{n}\frac{(-1)^{n-k}n(n+k-1)!\Gamma(k-\nu+\frac{1}{2})y(x_r)T_n^*(x_r)T_j^*(x)}{b_j\,L^\nu\Gamma(k+\frac{1}{2})\,(n-k)!\Gamma(k-\nu-j+1)\Gamma(k-\nu+j+1)}. \tag{16}$$

From Eq.(16), the fractional derivative of order ν for the function $y(x)$ at the point x_s leads to the desired result. $\qquad\square$

The coefficients $d_{s,r}^{(\nu)}$ which are defined in Theorem 1 are the elements of the s-th row of the matrix D_ν which is defined in the following relation:

$$[y^{(\nu)}] = D_\nu[y],$$

where, D_ν is a square matrix of order $(N+1)$ and the column matrices $[y^{(\nu)}]$ and $[y]$ are given by $y_r^{(\nu)} = y^{(\nu)}(x_r)$ and $y_r = y(x_r)$.

4. Error bound for the fractional derivatives of order ν

We approximate the derivatives of a function $y(x)$ by interpolating the function with a polynomial at the shifted Chebyshev Gauss-Lobatto nodes x_k, differentiating the polynomial and then evaluating the polynomial at the same nodes. with $y_k = y(x_k)$, construct a global Nth order Chebyshev interpolating polynomial [9]

$$(P_N y)(x) = \sum_{r=0}^{N} y_r \, \varphi_r(x), \quad \varphi_r(x) = \frac{2\theta_r}{N} \sum_{k=0}^{N} \theta_k \, T_k^*(x_r) T_k^*(x), \quad \theta_0 = \theta_N = \frac{1}{2} \text{ and } \theta_i = 1, \, i \geq 1, \quad (17)$$

The projection operator $(P_N y)(x)$ is a unique N-th degree interpolating polynomial defined as

$$(P_N y)(x_r) = y(x_r), \quad r = 0, 1, ..., N. \quad (18)$$

Alternatively, the interpolating polynomial $(P_N y)(x)$ can be expressed in terms of series expansion of the shifted Chebyshev polynomials of the first kind as follows

$$(P_N y)(x) = \sum_{n=0}^{N} a_n \, T_n^*(x), \quad a_n = \frac{2\theta_n}{N} \sum_{r=0}^{N} \theta_r y_r \, T_n^*(x_r). \quad (19)$$

We use the shifted Chebyshev Gauss-Lobatto nodes $x_r = \frac{L}{2} - \frac{L}{2} \cos(\frac{\pi r}{N})$, $r = 0, 1, ..., N$ as interpolated points. The fractional derivatives of order ν for $y(x)$ can be estimated at the points x_r by differentiating Eq.(17) and evaluating the resulting expression. This yields

$$D^{(\nu)}(P_N y)(x) = \sum_{r=0}^{N} y_r \, D^{(\nu)} \varphi_r(x)$$

$$= \sum_{r=0}^{N} \frac{4\theta_r}{N} \sum_{n=\lceil \nu \rceil}^{N} \sum_{j=0}^{N} \sum_{k=\lceil \nu \rceil}^{n} \frac{n\theta_n}{b_j} \frac{(-1)^{n-k} \, (n+k-1)! \, \Gamma(k-\nu+\frac{1}{2}) T_n^*(x_r) \, T_j^*(x)}{L^\nu \, \Gamma(k+\frac{1}{2}) \, (n-k)! \, \Gamma(k-\nu-j+1) \, \Gamma(k-\nu+j+1)}.$$

Setting $\mathbf{y} = [y(x_0), y(x_1), \ldots, y(x_N)]^T$ and $\mathbf{y}^{(\nu)} = [y^{(\nu)}(x_0), y^{(\nu)}(x_1), \ldots, y^{(\nu)}(x_N)]^T$. We approximate the derivatives of $y(x)$ at the points x_r, $r = 0, 1, \ldots, N$, by the equation

$$\mathbf{y}^{(\nu)} = D_\nu \mathbf{y}. \quad (20)$$

The entries of the matrix D_ν are given in Eq.(10) and can be replaced by

$$d_{s,r}^{(\nu)} = \frac{4\theta_r}{N} \sum_{n=\lceil \nu \rceil}^{N} \sum_{j=0}^{N} \sum_{k=\lceil \nu \rceil}^{n} \frac{n\theta_n}{b_j} \frac{(-1)^{n-k} \, (n+k-1)! \, \Gamma(k-\nu+\frac{1}{2})}{L^\nu \, \Gamma(k+\frac{1}{2}) \, (n-k)! \, \Gamma(k-\nu-j+1) \, \Gamma(k-\nu+j+1)}$$

$$\times (\frac{L}{2} - \frac{L}{2} \cos(\frac{n\pi r}{N})) \, (\frac{L}{2} - \frac{L}{2} \cos(\frac{j\pi s}{N})),$$

$$d_{s,r}^{(\nu)} = \frac{4\theta_r}{N} \sum_{n=\lceil\nu\rceil}^{N} \sum_{j=0}^{N} \sum_{k=\lceil\nu\rceil}^{n} \frac{n\theta_n}{b_j} \frac{(-1)^{n-k}(n+k-1)!\,\Gamma(k-\nu+\frac{1}{2})}{L^\nu\,\Gamma(k+\frac{1}{2})\,(n-k)!\,\Gamma(k-\nu-j+1)\,\Gamma(k-\nu+j+1)}$$

$$\times \frac{L^2}{4}\left(1 + \cos(\frac{n\pi r}{N})\cos(\frac{j\pi s}{N}) - 2\cos(\frac{(nr+js)\pi}{N})\cos(\frac{(nr-js)\pi}{N})\right). \tag{21}$$

Now, we investigate the effect of roundoff error on the elements $d_{s,r}^{(\nu)}$. In finite precision arithmetic, however, we have $x_r^* = x_r + \delta_r$, where δ_r denotes a small error, with $|\delta_r|$ approximately equal to machine precision ϵ and $\delta = max_r|\delta_r|$ we use the notation x_r^* for the exact value whereas x_r for the computed value. The absolute errors of the quantities $x_r x_n$ still being on the order of machine precision [9].

$$|x_r^* x_n^* - x_r x_n| = (\delta_r + \delta_n) - O\left(\frac{1}{N^2}\delta_r\right) - O\left(\frac{1}{N^2}\delta_n\right).$$

In order to evaluate the error bound for the fractional derivatives of any arbitrary order, we introduce the following theorem.

Theorem 4.1. *The effect of roundoff error on the elements $d_{s,r}^{(\nu)}$ is bounded by the following formula*

$$d_{s,r}^{(\nu)*} - d_{s,r}^{(\nu)} \leq \sum_{j=0}^{N} \sum_{k=\lceil\nu\rceil}^{n} \frac{n\theta_n}{b_j} \frac{(-1)^{n-k}(n+k-1)!\,\Gamma(k-\nu+\frac{1}{2})}{L^\nu\,\Gamma(k+\frac{1}{2})\,(n-k)!\,\Gamma(k-\nu-j+1)\,\Gamma(k-\nu+j+1)}$$

$$\times \frac{\theta_r L^2}{N}\left(1 - \left(\delta - O\left(\frac{1}{N^2}\delta\right)\right)\right). \tag{22}$$

Proof. Using the periodic properties of the cosine functions in Eq.(21), we have

$$d_{s,r}^{(\nu)} = \frac{4\theta_r}{N} \sum_{n=\lceil\nu\rceil}^{N} \sum_{j=0}^{N} \sum_{k=\lceil\nu\rceil}^{n} \frac{n\theta_n}{b_j} \frac{(-1)^{n-k}(n+k-1)!\,\Gamma(k-\nu+\frac{1}{2})}{L^\nu\,\Gamma(k+\frac{1}{2})\,(n-k)!\,\Gamma(k-\nu-j+1)\,\Gamma(k-\nu+j+1)}$$

$$\times \frac{L^2}{4}\left[1 + (-1)^{\lfloor\frac{nr}{N}\rfloor + \lfloor\frac{js}{N}\rfloor} x_{nr-N\lfloor\frac{nr}{N}\rfloor} x_{js-N\lfloor\frac{js}{N}\rfloor}\right.$$

$$\left. - 2(-1)^{\lfloor\frac{nr+js}{N}\rfloor + \lfloor\frac{nr-js}{N}\rfloor} x_{nr+js-N\lfloor\frac{nr+js}{N}\rfloor} x_{nr-js-N\lfloor\frac{nr-js}{N}\rfloor}\right]. \tag{23}$$

$$d_{s,r}^{(\nu)} = \frac{\theta_r L^2}{N} \sum_{n=\lceil\nu\rceil}^{N} \sum_{j=0}^{N} \sum_{k=\lceil\nu\rceil}^{n} \left(\frac{n\theta_n}{b_j} \frac{(-1)^{n-k}(n+k-1)!\,\Gamma(k-\nu+\frac{1}{2})}{L^\nu\,\Gamma(k+\frac{1}{2})\,(n-k)!\,\Gamma(k-\nu-j+1)\,\Gamma(k-\nu+j+1)}\right.$$

$$\times \left[1 + (-1)^{\lfloor\frac{nr}{N}\rfloor + \lfloor\frac{js}{N}\rfloor}\left((\delta_{nr-N\lfloor\frac{nr}{N}\rfloor} + \delta_{js-N\lfloor\frac{js}{N}\rfloor}) - O\left(\frac{1}{N^2}\delta_{nr-N\lfloor\frac{nr}{N}\rfloor}\right) - O\left(\frac{1}{N^2}\delta_{js-N\lfloor\frac{js}{N}\rfloor}\right)\right)\right.$$

$$-2(-1)^{\lfloor\frac{nr+js}{N}\rfloor + \lfloor\frac{nr-js}{N}\rfloor}\left((\delta_{nr+js-N\lfloor\frac{nr+js}{N}\rfloor} + \delta_{nr-js-N\lfloor\frac{nr-js}{N}\rfloor}) - O\left(\frac{1}{N^2}\delta_{nr+js-N\lfloor\frac{nr+js}{N}\rfloor}\right)\right.$$

$$\left.\left.\left.-O\left(\frac{1}{N^2}\delta_{nr-js-N\lfloor\frac{nr-js}{N}\rfloor}\right)\right)\right]\right). \tag{24}$$

As $\delta = \max_r|\delta_r|$ for any r, Eq.(23) leads to the desired result and completes the proof of the theorem. $\qquad\square$

5. Numerical implementation

In order to illustrate the effectiveness of the proposed method, we implement it to solve the following ordinary fractional differential equations.

Example 1. Consider the following fractional Bagley-Torvik equation [18]:

$$D^2 y(x) + D^{\frac{3}{2}} y(x) + y(x) = g(x), \quad 0 \le x \le 5, \tag{25}$$

with the following boundary conditions: $y(0) = 0, \quad y(5) = 25$.
Where $g(x) = x^2 + 2 + 4\sqrt{x/\pi}$ and the exact solution at this problem is $y(x) = x^2$.
In order to solve Eq.(25) by the proposed FChFD method, we use Eq.(9) to approximate $y(x)$. A collocation scheme is defined by substituting Eqs.(9) and (10) into Eq.(25) and evaluating the results at the shifted Gauss-Lobatto nodes x_s, $s = 1, 2, ..., N - 1$. This gives

$$\sum_{r=0}^{N} d_{s,r}^{(2)} y(x_r) + \sum_{r=0}^{N} d_{s,r}^{(\frac{3}{2})} y(x_r) + y(x_s) = g(x_s), \quad s = 1, 2, ..., N - 1, \tag{26}$$

where $d_{s,r}^{(2)}$ and $d_{s,r}^{(\frac{3}{2})}$ are defined in Theorem 1. By using boundary conditions we have $y(x_0) = 0$ and $y(x_N) = 25$. Eq.(26) gives $N - 1$ algebraic equations which can be solved for the unknown coefficients $y(x_1), y(x_2), ..., y(x_{N-1})$. Consequently $y(x)$ given in Eq.(9) can be calculated. For $N = 3$ and using the boundary conditions, we have a system of two linear algebraic equations:

$$1.1990\, y(x_1) - 1.2341\, y(x_2) = -15.4807, \tag{27}$$

$$0.0395\, y(x_1) + 0.3195\, y(x_2) = 4.5545. \tag{28}$$

After solving this system using the conjugate gradient method, we obtain:

$$y(x_0) = 0, \quad y(x_1) = 1.5625, \quad y(x_2) = 14.0625, \quad y(x_3) = 25.$$

So, we obtain the approximate solution

$$y(x) \cong \frac{2}{3} \sum_{n=0}^{3} {}'' \sum_{n=0}^{3} {}'' y(x_r) T_n^*(x_r) T_n^*(x) = x^2,$$

which coincides with the exact solution of this problem.

Example 2. Consider the following non-linear fractional boundary value problem [8]:

$$D^3 y(x) + D^{\frac{5}{2}} y(x) + y^2(x) = x^4, \quad 0 \le x \le 1, \tag{29}$$

with the following boundary conditions: $y(0) = 0, \quad y(1) = 1$.
In order to solve Eq.(29) by the proposed FChFD method, we use Eq.(9) to approximate $y(x)$. A collocation scheme is defined by substituting Eqs.(9) and (10) into Eq.(29) and evaluating the results at the shifted Gauss-Lobatto nodes x_s, $s = 1, 2, ..., N - 1$. This gives

$$\sum_{r=0}^{N} d_{s,r}^{(3)} y(x_r) + \sum_{r=0}^{N} d_{s,r}^{(\frac{5}{2})} y(x_r) + y^2(x_s) = x_s^4, \quad s = 1, 2, ..., N - 1, \tag{30}$$

where $d_{s,r}^{(3)}$ and $d_{s,r}^{\left(\frac{5}{2}\right)}$ are defined in Theorem 1. By using boundary conditions we have $y(x_0) = 0$ and $y(x_N) = 1$. Eq.(30) gives $N-1$ algebraic equations which can be solved for the unknown coefficients $y(x_1), y(x_2), ..., y(x_{N-1})$. Consequently $y(x)$ given in Eq.(9) can be calculated. For $N = 3$ and using the boundary conditions, we have a system of two non-linear algebraic equations:

$$50.0541 + 100.108\, y(x_1) - 100.108\, y(x_2) + y^2(x_1) = 0.00391, \qquad (31)$$

$$63.2706 + 126.541\, y(x_1) - 126.541\, y(x_2) + y^2(x_2) = 0.3164. \qquad (32)$$

After solving this system and using the Newton iteration method, we obtain:

$$y(x_0) = 0, \quad y(x_1) = 0.0625, \quad y(x_2) = 0.5625, \quad y(x_3) = 1.$$

So, we obtain the approximate solution

$$y(x) \cong \frac{2}{3} \sum_{n=0}^{3}{}'' \sum_{n=0}^{3}{}'' y(x_r)\, T_n^*(x_r)\, T_n^*(x) = x^2$$

which coincides with the exact solution $y(x) = x^2$ of this problem.

6. Conclusion and remarks

In this article, we introduced an efficient method for solving fractional boundary value problems. Our approach was based on a basic formulation of the new operational matrix method. In this work, the fractional derivatives of a non-singular function at any point from the Gauss-Lobatto points are expanded as a linear combination from the values of the function at these points. The coefficients of this linear combination are the elements of the suggested matrix D_ν. This new proposed method is non-differentiable, non-integral, straightforward and well adapted to the computer implementation. The solution is expressed as a truncated FChFD series and so it can be easily evaluated for arbitrary values of x using any computer program without any computational effort. From illustrative examples, it can be seen that this new numerical approach can obtain very accurate and satisfactory results. All computational calculations are made by Mathematica.

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