

## BLOWUP PROPERTIES FOR PARABOLIC EQUATIONS COUPLED VIA NON-STANDARD GROWTH SOURCES<sup>†</sup>

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ABSTRACT. This paper deals with parabolic equations coupled via non-standard growth sources, subject to homogeneous Dirichlet boundary conditions. Three kinds of necessary and sufficient conditions are obtained, which determine the complete classifications for non-simultaneous and simultaneous blowup phenomena. Moreover, blowup rates are given.

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### 1. Introduction

In this paper, we consider the following nonlocal parabolic problem

$$\begin{cases} u_t = \Delta u + \int_{\Omega} u^{m(x)}(x, t)e^{p(x)v(x, t)} dx, & (x, t) \in \Omega \times (0, T), \\ v_t = \Delta v + \int_{\Omega} u^{q(x)}(x, t)e^{n(x)v(x, t)} dx, & (x, t) \in \Omega \times (0, T), \\ u(x, t) = v(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x) \geq, \neq 0, & x \in \Omega, \end{cases} \quad (1)$$

where  $\Omega \subset \mathbb{R}^N$  with smooth boundary  $\partial\Omega$ ;  $m(x), n(x), p(x), q(x) \geq 0$  are continuous in  $\Omega$ ; initial data  $u_0(x), v_0(x)$  satisfy the compatibility conditions on  $\partial\Omega$ ;  $T$  denotes the maximal existence time of the solutions. The local existence of classical solutions to (1) is well-known (see, for example, [1, 2]). For the uniqueness of classical solutions, we assume  $m(x), q(x) > 1$ . The nonlinear parabolic problems like (1) come from several branches of applied mathematics and physics, such as, flows of electrorheological or thermo-rheological fluids [3, 4, 5], and the

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processing of digital images [6, 7, 8]. For more details, we refer the readers to books [9, 10].

Li, Huang, and Xie [11] considered the nonlocal parabolic equations

$$u_t = \Delta u + \int_{\Omega} u^m(x, t)v^n(x, t)dx, \quad v_t = \Delta v + \int_{\Omega} u^p(x, t)v^q(x, t)dx \quad (2)$$

in  $\Omega \times (0, T)$ , with null Dirichlet boundary conditions, where constants  $m, n, p, q \geq 0$ . They obtained that the solutions of system (2) blow up under large initial data if  $m > 1$ , or  $q > 1$ , or  $pn > (1 - m)(1 - q)$ , and also determined blow-up rates of solutions. There are many other results for parabolic equations with nonlocal nonlinearities (see, for example, [12, 13, 14, 15]).

Pinasco [16] in 2009 studied the homogeneous Dirichlet problem of

$$u_t = \Delta u + a(x) \int_{\Omega} u^{p(x)}(x, t)dx, \quad (x, t) \in \Omega \times (0, T), \quad (3)$$

subject to null Dirichlet boundary conditions, where the variable exponent  $p(x)$  and  $a(x)$  satisfy  $1 < p_- \leq p(x) \leq p_+ < +\infty$  and  $0 < c_- \leq a(x) \leq c_+ < +\infty$ . Here,  $p_+ = \sup_{x \in \Omega} p(x)$  and  $p_- = \inf_{x \in \Omega} p(x)$ . The authors [16] obtained  $u$  blows up at finite time  $T$  in the sense of  $\|u(\cdot, t)\|_{L^\infty(\Omega)} \rightarrow +\infty$  as  $t \rightarrow T$  for large initial data. Moreover, blowup solution for homogeneous Dirichlet problem of  $u_t = \Delta u + a(x)u^{p(x)}$  is studied.

Antontsev and Shmarev [17] discussed the evolution  $p(x)$ -Laplace parabolic equation

$$u_t = \operatorname{div}(a(x, t)|\nabla u|^{p(x)-2}\nabla u) + b(x, t)|u|^{\sigma(x, t)-2}u, \quad (x, t) \in \Omega \times (0, T),$$

subject to null Dirichlet boundary condition, with the variable functions  $p(x)$ ,  $\sigma(x, t) \in (1, +\infty)$ . If  $p(x) \equiv 2$ ,  $a(x, t) \equiv 1$ , and  $b(x, t) \geq b^- > 0$  (i.e. the semilinear equation), blow-up happens if the initial data are sufficiently large and either  $\min_{x \in \Omega} \sigma(x, t) = \sigma^-(t) > 2$  for all  $t > 0$ , or  $\sigma^-(t) \geq 2$ ,  $\sigma^-(t) \searrow 2$  as  $t \rightarrow \infty$  and  $\int_1^\infty e^{s(2-\sigma^-(s))}ds < \infty$ . For the Laplace equation with the exponents  $p(x)$  and  $\sigma(x)$ , they proved that every solution, corresponding to large initial data, exhibits blow-up if  $b(x, t) \geq b^- > 0$ ,  $a_t(x, t) \leq 0$ ,  $b_t(x, t) \geq 0$ ,  $\min_{x \in \Omega} \sigma(x) > 2$ ,  $\max_{x \in \Omega} p(x) \leq \min_{x \in \Omega} \sigma(x)$ .

In work [18], Ferreira, Pablo, Pérez-Llanos, and Rossi discussed the homogeneous Dirichlet problem of  $u_t = \Delta u + u^{p(x)}$  and also its corresponding Cauchy problem in  $\mathbf{R}^N$ . They obtained some interesting results for nonnegative  $p(x)$  as follows, for  $\Omega = \mathbf{R}^n$  or bounded  $\Omega$ , if  $p_+ > 1$ , there exist blow-up solutions, while if  $p_+ \leq 1$ , then every solution is global. For the Cauchy problem, if  $p_- > 1 + 2/N$ , there exist global nontrivial solutions; If  $1 < p_- < p_+ \leq 1 + 2/N$ , all solutions blow up; If  $p_- < 1 + 2/N < p_+$ , there are functions  $p(x)$  such that the problem possesses global nontrivial solutions and functions  $p(x)$  such that all solutions blow up. Two more results of global solutions were obtained: If  $\Omega \subset B_r(x_0)$  for some  $x_0 \in \mathbf{R}^N$  and  $r < \sqrt{2N}$ , then the problem possesses global nontrivial solutions, regardless of the exponent  $p(x)$ ; If  $p_- > 1$ , then there are global solutions,

regardless of the size of  $\Omega$ . The authors of [18] found out some new phenomena in bounded domains, which are quite different from the corresponding parabolic problems without variable exponents: there are suitable functions  $p(x)$  and suitable bounded domains  $\Omega$  such that positive solutions blow up in finite time for any initial data. By the way, the homogeneous Dirichlet problem of parabolic equations

$$u_t = \Delta u + v^{p(x)}, \quad v_t = \Delta v + u^{q(x)}, \quad (x, t) \in \Omega \times (0, T),$$

have been firstly obtained by Bai and Zheng [19]. Some criteria are established for distinguishing global and non-global solutions of the problem, depending or independent on initial data. Especially, they extended the Fujita-type result of [18] to the coupled equations case.

For the nonlocal non-standard growth problem (1), how to classify blowup solutions by using variable exponents and how to represent their blowup rates are worthy of being studied. In this paper, we firstly deal with blowup criteria of (1), and then identify simultaneous and non-simultaneous blowup under suitable assumptions on the initial data and the variable exponents. Finally, we discuss blowup rates for all kinds of blowup solutions. The present paper is arranged as follow. In the next section, we show the main results of the present paper. At sections 3 and 4, the classification for blowup solutions and blowup rates are proved, respectively.

### 2. Main results

The following proposition shows the criteria for blowup solutions. Denote  $m_+ = \sup_{x \in \Omega} m(x)$ ,  $n_+ = \sup_{x \in \Omega} n(x)$ ,  $p_+ = \sup_{x \in \Omega} p(x)$ ,  $q_+ = \sup_{x \in \Omega} q(x)$ .

**Proposition 2.1.** *Under the assumptions  $m_+ > 1$ ,  $n_+ > 0$ ,  $p_+ > 0$  and  $q_+ > 1$ , the classical solutions of (1) blow up in finite time for large initial data.*

In the sequel, we deal with the blowup solutions under the assumption

$$\Delta u_0 + (1 - \varepsilon\varphi) \int_{\Omega} u_0^{m(x)} e^{p(x)v_0} dx, \quad \Delta v_0 + (1 - \varepsilon\varphi) \int_{\Omega} u_0^{q(x)} e^{n(x)v_0} dx \geq 0, \quad (4)$$

where constant  $\varepsilon \in (0, 1)$ ,  $x \in \Omega$ ,  $\varphi$  and  $\lambda$  are the first eigenfunction and the first eigenvalue respectively of

$$-\Delta\varphi = \lambda\varphi \text{ in } \Omega, \quad \varphi = 0 \text{ on } \partial\Omega, \quad (5)$$

normalized by  $\int_{\Omega} \varphi(x)dx = 1$  and  $\varphi > 0$  in  $\Omega$ . By the comparison principle,  $u_t, v_t \geq 0$ . We assume  $u(x, t)$ ,  $v(x, t)$  and  $m(x), n(x), p(x), q(x)$  attain their maxima at the same point  $x_0 \in \Omega$  and the measure of the sub-domain of  $\Omega$  where variable exponents reach their maxima is not zero. It can be found that, if the classical solution  $(u, v)$  and the variable exponents are radially symmetric and non-increasing in  $r = |x| \in (0, R)$ , then the above assumption can be met.

Now, we state the complete classifications for non-simultaneous and simultaneous blowup by three theorems. The first one determines the existence of non-simultaneous blowup.

**Theorem 2.2.** (i) *There exists initial data such that  $u$  blows up alone if and only if  $m_+ > q_+ + 1$ .* (ii) *There exists initial data such that  $v$  blows up alone if and only if  $n_+ > p_+$ .*

**Corollary 2.3.** *Any blowup must be simultaneous if and only if  $m_+ \leq q_+ + 1$  and  $n_+ \leq p_+$ .*

The second shows the exponent regions for non-simultaneous blow-up only.

**Theorem 2.4.** (i) *Any blowup must be  $u$  blowing up alone if and only if  $m_+ > q_+ + 1$  and  $n_+ \leq p_+$ .* (ii) *Any blowup must be  $v$  blowing up alone if and only if  $m_+ \leq q_+ + 1$  and  $n_+ > p_+$ .*

The third one presents an interesting exponent region where three kinds of blowup phenomena may occur.

**Theorem 2.5.** *Both simultaneous and non-simultaneous blowup may occur if and only if  $m_+ > q_+ + 1$  and  $n_+ > p_+$ .*

Theorems 2.2–2.5 yield the optimal classification for blowup solutions of (1). In the coexistence region  $\{m_+ > q_+ + 1, n_+ > p_+\}$ , both simultaneous and non-simultaneous blow-up may occur, sensitively depending on the choosing of initial data: roughly speaking, larger  $u_0$  ( $v_0$ ) and smaller  $v_0$  ( $u_0$ ) lead to the single component blowup of  $u$  ( $v$ ), and simultaneous blowup occurs under somewhat balanced  $u_0$  and  $v_0$ .

The next theorem covers all possible simultaneous blowup of solutions in the two related regions  $\{m_+ \leq q_+ + 1, n_+ \leq p_+\}$  and  $\{m_+ > q_+ + 1, n_+ > p_+\}$ .

**Theorem 2.6.** *The following blowup rates hold on any compact subset of  $\Omega$ .*

(i) *If  $m_+ < q_+ + 1$  and  $n_+ < p_+$ , or  $m_+ > q_+ + 1, n_+ > p_+$  and simultaneous blowup occurs, then*

$$c \leq u(x, t)(T - t)^{\frac{p_+ - n_+}{p_+ q_+ - n_+ (m_+ - 1)}} \leq C, \quad c \leq \frac{v(x, t)}{|\log(T - t)|} \leq C.$$

*In particular,  $c \leq e^{v(x_0, t)}(T - t)^{\frac{q_+ + 1 - m_+}{p_+ q_+ - n_+ (m_+ - 1)}} \leq C$ .*

(ii) *If  $m_+ < q_+ + 1$  and  $n_+ = p_+$ , then*

$$c \leq \frac{u^{q_+ + 1 - m_+}(x, t)}{|\log(T - t)|} \leq C, \quad c \leq \frac{v(x, t)}{|\log(T - t)|} \leq C. \tag{6}$$

*In particular,*

$$c \leq e^{n_+ v(x_0, t)} v^{\frac{q_+}{q_+ + 1 - m_+}}(x_0, t)(T - t) \leq C. \tag{7}$$

(iii) *If  $m_+ = q_+ + 1$  and  $n_+ < p_+$ , then*

$$c \leq \frac{\log u(x, t)}{|\log(T - t)|} \leq C, \quad c \leq \frac{v(x, t)}{\log |\log(T - t)|} \leq C.$$

In particular,

$$c \leq u^{m-1}(x_0, t)(\log u(x_0, t))^{\frac{p_+}{p_+ - n_+}}(T - t) \leq C, \quad c \leq \frac{e^{(p_+ - n_+)v(x_0, t)}}{|\log(T - t)|} \leq C.$$

(iv) If  $m_+ = q_+ + 1$  and  $n_+ = p_+$ , then

$$c \leq \frac{\log u(x, t)}{|\log(T - t)|} \leq C, \quad c \leq \frac{v(x, t)}{|\log(T - t)|} \leq C.$$

### 3. Proofs of Proposition 2.1 and Theorems 2.2–2.5

*Proof of Proposition 2.1.* Similarly to the proof for problem (3) in [16], if  $m_+ > 1$  or  $n_+ > 0$ , the classical solution of (1) blows up for large initial data.

Denote a positive constant  $\beta = \min\{\min_{x \in \bar{\Omega}} p(x) + 1, \min_{x \in \bar{\Omega}} q(x)\} > 1$ . Introduce two functions  $\zeta(t) = \int_{\Omega} \varphi(x)u(x, t)dx$ ,  $\xi(t) = \int_{\Omega} \varphi(x)v(x, t)dx$ . Define two subsets of  $\Omega$  as follows,

$$\Omega_{\{\geq 1\}} = \{x \in \Omega : u \geq 1\}, \quad \Omega_{\{< 1\}} = \{x \in \Omega : u < 1\}.$$

We have

$$\begin{aligned} \zeta'(t) &\geq -\lambda\zeta(t) + \int_{\Omega} \varphi(x) \int_{\Omega} u^{m(y)}(y, t)e^{p(y)v(y, t)} dy dx \\ &\geq -\lambda\zeta(t) + \int_{\Omega} \varphi(x) \int_{\Omega_{\{\geq 1\}}} u^{m-}(y, t)e^{p-v(y, t)} dy dx \\ &\geq -\lambda\zeta(t) + \int_{\Omega} \varphi(x) \int_{\Omega_{\{\geq 1\}}} e^{p-v(y, t)} dy dx \\ &\quad + \int_{\Omega} \varphi(x) \int_{\Omega_{\{< 1\}}} e^{p-v(y, t)} dy dx - \int_{\Omega} \varphi(x) \int_{\Omega_{\{< 1\}}} e^{p-v(y, t)} dy dx \\ &\geq -\lambda\zeta(t) + c \int_{\Omega} \varphi(x) \int_{\Omega} v^{p-+1}(y, t) dy dx - c \\ &\geq -\lambda\zeta(t) + c \int_{\Omega} \varphi(y)v^{p-+1}(y, t) dy - c. \end{aligned}$$

By applying the Jensen's inequality, we have

$$\zeta'(t) \geq -\lambda\zeta(t) + c\xi^{\beta}(t), \tag{8}$$

similarly, there is the inequality

$$\xi'(t) \geq -\lambda\xi(t) + c\zeta^{\beta}(t). \tag{9}$$

Define  $K(t) = \zeta(t) + \xi(t)$ . Combining (8) with (9), we have

$$K'(t) \geq -\lambda K(t) + c(\zeta^{\beta}(t) + \xi^{\beta}(t)) \geq -\lambda K(t) + cK^{\beta}(t).$$

Hence  $K(t)$  blows up in finite time for large initial data, which deduces  $(\|u(\cdot, t)\|_{\infty} + \|v(\cdot, t)\|_{\infty})$  blows up.  $\square$

In order to prove Theorem 2.2, we introduce the following lemma. Let  $\phi$  solve

$$\begin{cases} \phi_t = \Delta\phi, & (x, t) \in \Omega \times (0, T), \\ \phi = 0, & (x, t) \in \partial\Omega \times (0, T), \\ \phi(x, 0) = \varphi(x), & x \in \Omega, \end{cases} \quad (10)$$

where  $\varphi$  is the normalized first eigenfunction of (5).

**Lemma 3.1.** *Under the condition (4),*

$$u_t \geq \varepsilon\phi \int_{\Omega} u^{m(x)}(x, t)e^{p(x)v(x,t)} dx, \quad (x, t) \in \Omega \times [0, T], \quad (11)$$

$$v_t \geq \varepsilon\phi \int_{\Omega} u^{q(x)}(x, t)e^{n(x)v(x,t)} dx, \quad (x, t) \in \Omega \times [0, T]. \quad (12)$$

*Proof.* Construct functions

$$J = u_t - \varepsilon\phi \int_{\Omega} u^{m(x)}(x, t)e^{p(x)v(x,t)} dx, \quad (x, t) \in \Omega \times [0, T].$$

$$K = v_t - \varepsilon\phi \int_{\Omega} u^{q(x)}(x, t)e^{n(x)v(x,t)} dx, \quad (x, t) \in \Omega \times [0, T].$$

Noticing  $u_t, v_t \geq 0$  by the comparison principle with (4), we know

$$J_t - \Delta J \geq (1 - \varepsilon\phi) \int_{\Omega} \left( m(x)u^{m(x)-1}e^{p(x)v}u_t + p(x)u^{m(x)}e^{p(x)v}v_t \right) dx \geq 0,$$

$$\begin{aligned} K_t - \Delta K &\geq (1 - \varepsilon\phi) \int_{\Omega} \left( q(x)u^{q(x)-1}e^{n(x)v}u_t + n(x)u^{q(x)}e^{n(x)v}v_t \right) dx \\ &\geq 0, \quad (x, t) \in \Omega \times (0, T), \end{aligned}$$

$J(x, t) = K(x, t) = 0$ ,  $(x, t) \in \partial\Omega \times (0, T)$ , and  $J(x, t) = K(x, t) = 0$ ,  $(x, t) \in \partial\Omega \times (0, T)$ , which yield (11) and (12) by the comparison principle.  $\square$

**Remark 3.1.** Let  $\xi = \phi_t$ . It follows from (10) that  $\xi_t = \Delta\xi$  in  $\Omega \times (0, T)$ ,  $\xi = 0$  on  $\partial\Omega \times (0, T)$ ,  $\xi(x, 0) = -\lambda\varphi(x) < 0$  in  $\Omega$ , and thus  $\xi = \phi_t \leq 0$  for  $(x, t) \in \Omega \times (0, T)$ . By (11) with  $\phi(x_0, t) \geq \phi(x_0, T)$ ,  $v(x_0, t) \geq v_0(x_0)$ , we have

$$u_t(x_0, t) \geq \varepsilon|\sigma|\phi(x_0, T)e^{p+v_0(x_0)}u^{m_+}(x_0, t), \quad t \in [0, T],$$

where  $\sigma \subset \Omega$  denotes a set in which the variable exponents take their maxima and  $|\sigma|$  is the measure, and hence

$$u(x_0, t) \leq \left[ \varepsilon(m_+ - 1)|\sigma|\phi(x_0, T)e^{p+v_0(x_0)} \right]^{-\frac{1}{m_+-1}} (T - t)^{-\frac{1}{m_+-1}}, \quad (13)$$

for  $m_+ > 1$ ,  $t \in [0, T)$ . Similarly,

$$e^{v(x_0, t)} \leq \left[ \varepsilon n_+|\sigma|\phi(x_0, T)u_0^{q_+}(x_0) \right]^{-\frac{1}{n_+-1}} (T - t)^{-\frac{1}{n_+-1}}, \quad n_+ > 0, \quad t \in [0, T). \quad (14)$$

For convenience, define

$$f(t) = \int_{\Omega} u^{m(x)}(x, t)e^{p(x)v(x, t)} dx, \quad g(t) = \int_{\Omega} u^{q(x)}(x, t)e^{n(x)v(x, t)} dx,$$

$$F(t) = \int_0^t f(s) ds, \quad G(t) = \int_0^t g(s) ds.$$

By Lemmas 4.3, 4.4 and Theorem 4.1 of [2], one can obtain that

**Lemma 3.2.** *Let  $(u, v)$  be a simultaneous blowup solution of (1). Then there exists some constant  $C > 0$  such that*

$$\lim_{t \rightarrow T} \frac{u(x, t)}{F(t)} = \lim_{t \rightarrow T} \frac{\|u(\cdot, t)\|_{\infty}}{F(t)} = 1, \quad \lim_{t \rightarrow T} \frac{v(x, t)}{G(t)} = \lim_{t \rightarrow T} \frac{\|v(\cdot, t)\|_{\infty}}{G(t)} = 1,$$

$$\text{and} \quad \lim_{t \rightarrow T} \frac{u_t(x, t)}{f(t)} = \lim_{t \rightarrow T} \frac{v_t(x, t)}{g(t)} = 1$$

hold uniformly on any compact subset of  $\Omega$ .  $\square$

**Remark 3.2.** The corresponding results in Lemma 3.2 hold also for the blowup component if non-simultaneous blowup occurs.

*Proof of Theorem 2.2.* (i) Assume  $m_+ > q_+ + 1$ . Let

$$\Gamma(x, y, t, \tau) = \frac{1}{[4\pi(t - \tau)]^{N/2}} \exp \left\{ -\frac{|x - y|^2}{4(t - \tau)} \right\}$$

be the fundamental solution of the heat equation, and  $(\tilde{u}_0, \tilde{v}_0)$  be a pair of initial data such that the solution of (1) blows up. Take  $v_0 = \tilde{v}_0$  and denote  $M_v = \|v_0\|_{\infty} + 1$ . Let  $u_0 \geq \tilde{u}_0$  be large such that the blowup time  $T$  satisfies

$$M_v \geq \|v_0\|_{\infty} + \frac{(m_+ - 1)|\Omega|}{m_+ - 1 - q_+} \left[ \varepsilon(m_+ - 1)|\sigma|\phi(x_0, T)e^{p_+v_0(x_0)} \right]^{-\frac{q_+}{m_+ - 1}} T^{\frac{m_+ - 1 - q_+}{m_+ - 1}} e^{n_+M_v}.$$

Consider the auxiliary problem

$$\begin{cases} \bar{v}_t = \Delta \bar{v} + |\Omega|C_u^{-\frac{q_+}{m_+ - 1}}(T - t)^{-\frac{q_+}{m_+ - 1}}e^{n_+M_v}, & (x, t) \in \Omega \times (0, T), \\ \bar{v}(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ \bar{v}(x, 0) = v_0(x), & x \in \Omega, \end{cases}$$

with  $C_u = \varepsilon(m_+ - 1)|\sigma|\phi(x_0, T)e^{p_+v_0(x_0)}$ . For  $m_+ > q_+ + 1$ , we have

$$\begin{aligned} \bar{v}(x, t) &\leq \int_{\Omega} \Gamma(x, y, t, 0)v_0(y) dy + \int_0^t \int_{\Omega} \Gamma(x, y, t, \tau)|\Omega|C_u^{-\frac{q_+}{m_+ - 1}}(T - \tau)^{-\frac{q_+}{m_+ - 1}}e^{n_+M_v} dy d\tau \\ &\leq \|v_0\|_{\infty} + \frac{(m_+ - 1)|\Omega|}{m_+ - 1 - q_+} \left[ \varepsilon(m_+ - 1)|\sigma|\phi(x_0, T)e^{p_+v_0(x_0)} \right]^{-\frac{q_+}{m_+ - 1}} T^{\frac{m_+ - 1 - q_+}{m_+ - 1}} e^{n_+M_v} \\ &\leq M_v, \end{aligned}$$

and hence

$$\bar{v}_t \geq \Delta \bar{v} + |\Omega|C_u^{-\frac{q_+}{m_+ - 1}}(T - t)^{-\frac{q_+}{m_+ - 1}}e^{n_+\bar{v}(x_0, t)}, \quad (x, t) \in \Omega \times (0, T).$$

Followed by (14),  $v$  satisfies

$$v_t \leq \Delta v + |\Omega| C_u^{-\frac{q_+}{m_+-1}} (T-t)^{-\frac{q_+}{m_+-1}} e^{n_+v(x_0,t)}, \quad (x,t) \in \Omega \times (0,T).$$

By the comparison principle,  $v$  is bounded for  $v \leq \bar{v} \leq M_v$ .

Now, assume  $u$  blows up alone. It can be checked that  $m_+ > 1$ . By Remark 3.2, we obtain

$$\lim_{t \rightarrow T} \frac{u_t(x,t)}{\int_{\Omega} u^{m(x)}(x,t) e^{p(x)v(x,t)} dx} = 1,$$

uniformly on compact subsets of  $\Omega$ . Then there must exist positive constants  $c$  and  $C$  such that  $u_t(x_0,t) \leq C u^{m_+}(x_0,t)$ , and hence  $u(x_0,t) \geq c(T-t)^{-\frac{1}{m_+-1}}$  by integrating from  $t$  to  $T$ . Using (12), one obtains  $v_t(x_0,t) \geq c(T-t)^{-\frac{q_+}{m_+-1}}$ , and so  $v(x_0,t) \geq \int_0^t c(T-\tau)^{-\frac{q_+}{m_+-1}} d\tau$  after integrating the inequality over  $(0,t)$ . The boundedness of  $v$  requires  $m_+ > q_+ + 1$ .

(ii) Assume  $n_+ > p_+$ . Let  $(\tilde{u}_0, \tilde{v}_0)$  be the initial data such that the solution of (1) blows up. Take  $u_0 = \tilde{u}_0$ , and denote  $M_u = \|u_0\|_{\infty} + 1$ . Let  $v_0 \geq \tilde{v}_0$  be large such that the blowup time  $T$  satisfies

$$M_u \geq \|u_0\|_{\infty} + \frac{n_+}{n_+ - p_+} |\Omega| [\varepsilon n_+ |\sigma| \phi(x_0, T) u_0^{q_+}(x_0)]^{-\frac{p_+}{n_+}} T^{\frac{n_+ - p_+}{n_+}} M_u^{m_+}.$$

Consider the auxiliary problem

$$\begin{cases} \bar{u}_t = \Delta \bar{u} + |\Omega| C_v^{-\frac{p_+}{n_+}} (T-t)^{-\frac{p_+}{n_+}} M_u^{m_+}, & (x,t) \in \Omega \times (0,T), \\ \bar{u}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ \bar{u}(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

with  $C_v = \varepsilon n_+ |\sigma| \phi(x_0, T) u_0^{q_+}(x_0)$ . By Green's identity with  $n_+ > p_+$ , we have  $\bar{u}(x,t) \leq M_u$ , and hence

$$\bar{u}_t \geq \Delta \bar{u} + |\Omega| C_v^{-\frac{p_+}{n_+}} (T-t)^{-\frac{p_+}{n_+}} \bar{u}^{m_+}(x_0,t), \quad (x,t) \in \Omega \times (0,T).$$

It follows from (13) that  $v$  satisfies

$$v_t \leq \Delta v + |\Omega| C_v^{-\frac{p_+}{n_+}} (T-t)^{-\frac{p_+}{n_+}} u^{m_+}(x_0,t), \quad (x,t) \in \Omega \times (0,T),$$

and thus  $u \leq \bar{u} \leq M_u$ .

Now, assume that  $v$  blows up alone. It can be checked that  $n_+ > 0$ , and

$$\lim_{t \rightarrow T} \frac{v_t(x,t)}{\int_{\Omega} u^{q(x)}(x,t) e^{n(x)v(x,t)} dx} = 1,$$

uniformly on compact subsets of  $\Omega$  by Remark 3.2. We have  $v_t(x_0,t) \leq C e^{n_+v(x_0,t)}$ , and hence  $e^{v(x_0,t)} \geq c(T-t)^{-\frac{1}{n_+}}$ . Furthermore,  $u_t(x_0,t) \geq c(T-t)^{-\frac{p_+}{n_+}}$  due

to (11), and consequently,  $u(x_0, t) \geq \int_0^t c(T - \tau)^{-\frac{p_+}{n_+}} d\tau$ . The boundedness of  $u$  implies  $n_+ > p_+$ .  $\square$

*Proof of Theorem 2.4.* (i) Assume  $m_+ > q_+ + 1$  and  $n_+ \leq p_+$ . By Theorem 2.2, there exists initial data such that  $u$  blows up alone if  $m_+ > q_+ + 1$  and  $v$  cannot blow up alone due to  $n_+ \leq p_+$ , hence we only need to exclude the possibility of simultaneous blowup with  $n_+ \leq p_+$ . Otherwise, by Lemma 3.2, we have

$$\lim_{t \rightarrow T} \frac{u_t(x, t)}{\int_{\Omega} u^{m(x)}(x, t)e^{p(x)v(x, t)} dx} = \lim_{t \rightarrow T} \frac{v_t(x, t)}{\int_{\Omega} u^{q(x)}(x, t)e^{n(x)v(x, t)} dx} = 1,$$

uniformly on compact subsets of  $\Omega$ . Hence there exist positive constants  $c$  and  $C$  such that

$$cu^{m_+}(x_0, t)e^{p_+v(x_0, t)} \leq u_t(x_0, t) \leq Cu^{m_+}(x_0, t)e^{p_+v(x_0, t)}, \tag{15}$$

$$cu^{q_+}(x_0, t)e^{n_+v(x_0, t)} \leq v_t(x_0, t) \leq Cu^{q_+}(x_0, t)e^{n_+v(x_0, t)}. \tag{16}$$

One can obtain the contradictions with simultaneous blowup as follows,

$$e^{(p_+ - n_+)v(x_0, t)} \leq C + Cu^{q_+ + 1 - m_+}(x_0, t) \quad \text{for } m_+ > q_+ + 1, n_+ < p_+,$$

$$v(x_0, t) \leq C + Cu^{q_+ + 1 - m_+}(x_0, t) \quad \text{for } m_+ > q_+ + 1, n_+ = p_+.$$

Now assume that any blowup must be  $u$  blowing up alone. Theorem 2.2-(i) requires  $m_+ > q_+ + 1$ . On the other hand, Theorem 2.2-(ii) says  $v$  may blow up alone if and only if  $n_+ > p_+$ . Thus  $n_+ \leq p_+$ . Case (ii) can be treated by using the same techniques as above for (i).  $\square$

In order to prove Theorem 2.5, we introduce a lemma. Denote  $\mathbb{V}_0$  as a set making up of the initial data satisfying (4).

**Lemma 3.3.** *The set of  $(u_0, v_0)$  in  $\mathbb{V}_0$  such that  $u$  ( $v$ ) blows up while  $v$  ( $u$ ) remains bounded is open in  $L^\infty$ -topology.*

*Proof.* Without loss of generality, we only prove the case for  $u$  blowing up with  $v$  remaining bounded. Let  $(u, v)$  be a solution of (1) with initial data  $(u_0, v_0) \in \mathbb{V}_0$  such that  $u$  blows up while  $v$  remains bounded up to blowup time  $T$ , say  $\|v(\cdot, t)\|_\infty \leq M$ . It suffices to find an  $L^\infty$ -neighborhood of  $(u_0, v_0)$  in  $\mathbb{V}_0$  such that any solution  $(\hat{u}, \hat{v})$  of (1) coming from this neighborhood maintains the property that  $\hat{u}$  blows up while  $\hat{v}$  remains bounded.

By Theorem 2.2, we know  $m_+ > q_+ + 1$ . Take  $M_1 = M + 2\xi$ . Let  $(\tilde{u}, \tilde{v})$  be the solution of (1) with the initial data  $(\tilde{u}_0, \tilde{v}_0) \in \mathbb{V}_0$  and the maximal existence time  $T_0$ . Define

$$\mathbb{N}(u_0, v_0) = \left\{ (\tilde{u}_0, \tilde{v}_0) \in \mathbb{V}_0 \mid \|\tilde{u}_0(x) - u_0(x)\|_\infty, \|\tilde{v}_0(x) - v_0(x)\|_\infty < \xi \right\}.$$

Since  $u$  blows up at time  $T$ , there exists some small constant  $\varepsilon_0 > 0$  such that  $(\tilde{u}, \tilde{v})$  blows up and  $T_0$  satisfies

$$M_1 \geq M + \xi + \frac{(m_+ - 1)|\Omega|}{m_+ - 1 - q_+} \left[ \varepsilon(m_+ - 1)|\sigma|\phi(x_0, T_0)e^{p_+v_0(x_0)} \right]^{-\frac{q_+}{m_+ - 1}} T_0^{\frac{m_+ - 1 - q_+}{m_+ - 1}} e^{n_+M_1},$$

provided that  $(\tilde{u}_0, \tilde{v}_0) \in \mathbb{N}(u_0, v_0)$ .

Consider the auxiliary problem

$$\begin{cases} \bar{v}_t = \Delta \bar{v} + |\Omega|C_1^{-\frac{q_+}{m_+ - 1}}(T_0 - t)^{-\frac{q_+}{m_+ - 1}}e^{n_+M_1}, & (x, t) \in \Omega \times (0, T_0), \\ \bar{v}(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T_0), \\ \bar{v}(x, 0) = \tilde{v}_0(x), & x \in \Omega, \end{cases}$$

where  $C_1 = \varepsilon(m_+ - 1)|\sigma|\phi(x_0, T_0)e^{p_+v_0(x_0)}$ . By Green's identity,  $\bar{v} \leq M_1$ . Hence

$$\bar{v}_t \geq \Delta \bar{v} + |\Omega|C_1^{-\frac{q_+}{m_+ - 1}}(T_0 - t)^{-\frac{q_+}{m_+ - 1}}e^{n_+\bar{v}(x_0, t)}, \quad (x, t) \in \Omega \times (0, T_0).$$

On the other hand, by (13), we have

$$\tilde{v}_t \leq \Delta \tilde{v} + |\Omega|C_1^{-\frac{q_+}{m_+ - 1}}(T_0 - t)^{-\frac{q_+}{m_+ - 1}}e^{n_+\tilde{v}(x_0, t)}, \quad (x, t) \in \Omega \times (0, T_0).$$

We have  $\tilde{v} \leq \bar{v} \leq M_1$  by the comparison principle.

According to the continuity with respect to initial data for bounded solutions, there must exist a neighborhood of  $(u_0, v_0)$  in  $\mathbb{V}_0$  such that every solution  $(\hat{u}, \hat{v})$  starting from the neighborhood will enter  $\mathbb{N}(u_0, v_0)$  at time  $T - \varepsilon_0$ , and keeps the property that  $\hat{u}$  blows up while  $\hat{v}$  keep bounded.  $\square$

*Proof of Theorem 2.5.* Firstly, let  $m_+ > q_+ + 1$ ,  $n_+ > p_+$ , and assume that the solution of (1) blows up with initial data  $(u_0, v_0) \in \mathbb{V}_0$ . Then the family of initial data  $(u_0/\lambda, v_0/(1 - \lambda)) \in \mathbb{V}_0$  with  $\lambda \in (0, 1)$  makes the related solutions blow up also. By Theorem 2.2,  $u$  blows up with  $v$  remaining bounded for some  $\lambda = \lambda_1$  near 0, and  $v$  blows up alone with some  $\lambda = \lambda_2$  close to 1. By Lemma 3.3, such sets of initial data are open and connected. Therefore, there must exist some  $\lambda \in (\lambda_1, \lambda_2)$  such that simultaneous blowup happens.

Now assume both simultaneous and non-simultaneous blowup may occur there. Since any blowup must be simultaneous in  $\{m_+ \leq q_+ + 1, n_+ \leq p_+\}$  by Corollary 2.3, and any blowup must be non-simultaneous in  $\{m_+ > q_+ + 1, n_+ \leq p_+\}$  (or  $\{m_+ \leq q_+ + 1, n_+ > p_+\}$ ) by Theorem 2.4, then it has to be satisfied that  $m_+ > q_+ + 1$  and  $n_+ > p_+$ .  $\square$

#### 4. Proof of Theorem 2.6

Followed from (15) and (16), We show a lemma for the relationships between  $u(x_0, t)$  and  $v(x_0, t)$  without proof.

**Lemma 4.1.** *Let  $(u, v)$  be a simultaneous blowup solution of (1) ensured by Corollary 2.3 and Theorem 2.5 with blow-up time  $T$ . For any given  $\delta, \varepsilon \in (0, 1)$*

and  $\sigma > 1$ , there exists  $\tilde{T} < T$  such that the following relationships hold for any  $t \in [\tilde{T}, T)$ .

(i) If  $m_+ < q_+ + 1$  and  $n_+ < p_+$ , then

$$\frac{\varepsilon}{q_+ + 1 - m_+} u^{q_+ + 1 - m_+}(x_0, t) \leq \frac{\sigma}{p_+ - n_+} e^{(p_+ - n_+)v(x_0, t)},$$

$$\frac{\delta\varepsilon}{p_+ - n_+} e^{(p_+ - n_+)v(x_0, t)} \leq \frac{1}{q_+ + 1 - m_+} u^{q_+ + 1 - m_+}(x_0, t).$$

(ii) If  $m_+ < q_+ + 1$  and  $n_+ = p_+$ , then

$$\frac{\varepsilon}{q_+ + 1 - m_+} u^{q_+ + 1 - m_+}(x_0, t) \leq \sigma v(x_0, t),$$

$$\delta\varepsilon v(x_0, t) \leq \frac{1}{q_+ + 1 - m_+} u^{q_+ + 1 - m_+}(x_0, t).$$

(iii) If  $m_+ = q_+ + 1$  and  $n_+ < p_+$ , then

$$\varepsilon \log u(x_0, t) \leq \frac{\sigma}{p_+ - n_+} e^{(p_+ - n_+)v(x_0, t)}, \quad \frac{\delta\varepsilon}{p_+ - n_+} e^{(p_+ - n_+)v(x_0, t)} \leq \log u(x_0, t).$$

(iv) If  $m_+ = q_+ + 1$  and  $n_+ = p_+$ , then

$$\varepsilon \log u(x_0, t) \leq \sigma v(x_0, t), \quad \delta\varepsilon v(x_0, t) \leq \log u(x_0, t).$$

(v) If  $m_+ > q_+ + 1$  and  $n_+ > p_+$ , then

$$\frac{1}{m_+ - 1 - q_+} u^{q_+ + 1 - m_+}(x_0, t) \leq \frac{\sigma}{n_+ - p_+} e^{(p_+ - n_+)v(x_0, t)},$$

$$\frac{\delta}{n_+ - p_+} e^{(p_+ - n_+)v(x_0, t)} \leq \frac{1}{m_+ - 1 - q_+} u^{q_+ + 1 - m_+}(x_0, t).$$

*Proof of Theorem 2.6.* We only prove case (ii):  $m_+ < q_+ + 1$  and  $n_+ = p_+$ , and the other cases can be treated similarly. By Lemma 4.1-(ii),

$$\frac{\varepsilon}{q_+ + 1 - m_+} u^{q_+ + 1 - m_+}(x_0, t) \leq \sigma v(x_0, t), \tag{17}$$

$$\delta\varepsilon v(x_0, t) \leq \frac{1}{q_+ + 1 - m_+} u^{q_+ + 1 - m_+}(x_0, t), \tag{18}$$

we have  $c \leq v_t(x_0, t) v^{-\frac{q_+}{q_+ + 1 - m_+}}(x_0, t) e^{-n_+v(x_0, t)} \leq C$ , and hence, by integrating the above inequalities over  $(t, T)$ ,

$$c(T - t) \leq \int_{v(x_0, t)}^{+\infty} e^{-n_+s} s^{-\frac{q_+}{q_+ + 1 - m_+}} ds \leq C(T - t). \tag{19}$$

It can be checked that

$$\lim_{t \rightarrow T} \frac{\int_{v(x_0, t)}^{+\infty} e^{-n_+s} s^{-\frac{q_+}{q_+ + 1 - m_+}} ds}{e^{-n_+v(x_0, t)} v^{-\frac{q_+}{q_+ + 1 - m_+}}(x_0, t)} = \frac{1}{n_+}. \tag{20}$$

By (19) and (20), we obtain (7).

For  $p_+ = n_+$ , there is

$$cu^{q_+}(x_0, t)e^{n_+v(x_0, t)} \leq u^{q_+ - m_+}(x_0, t)u_t(x_0, t) \leq Cu^{q_+}(x_0, t)e^{n_+v(x_0, t)}. \quad (21)$$

By Lemma 4.1-(ii) with (21),

$$cv^{\frac{q_+}{q_+ + 1 - m_+}}(x_0, t)e^{n_+v(x_0, t)} \leq u^{q_+ - m_+}(x_0, t)u_t(x_0, t) \leq Cv^{\frac{q_+}{q_+ + 1 - m_+}}(x_0, t)e^{n_+v(x_0, t)}. \quad (22)$$

Due to (7) and (22), we have

$$c(T - t)^{-1} \leq u^{q_+ - m_+}(x_0, t)u_t(x_0, t) \leq C(T - t)^{-1}.$$

By integration, we have  $c|\log(T - t)| \leq u^{q_+ + 1 - m_+}(x_0, t) \leq C|\log(T - t)|$ . By Lemma 3.2, there exist positive constants  $c, C$  such that

$$cu(x_0, t) \leq u(x, t) \leq Cu(x_0, t), \quad cv(x_0, t) \leq v(x, t) \leq Cv(x_0, t). \quad (23)$$

Hence

$$c|\log(T - t)| \leq u^{q_+ + 1 - m_+}(x, t) \leq C|\log(T - t)|. \quad (24)$$

Combining the above inequalities with (17), (18), and (23), the estimate for  $v(x, t)$  is obtained as follows,

$$c|\log(T - t)| \leq v(x, t) \leq C|\log(T - t)|. \quad (25)$$

Then (6) is obtained.  $\square$

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