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h-STABILITY OF THE NONLINEAR PERTURBED DIFFERENCE SYSTEMS VIA n_∞-SIMILARITY†

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Abstract. In this paper, we investigate h-stability of the nonlinear perturbed difference system via n_∞-similarity.

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1. Introduction

Discrete Volterra systems arise mainly in the process of modeling of some real phenomena or by applying a numerical method to a Volterra integral equation. when we study the asymptotic stability it is not easy to work with non-exponential types of stability. Medina and Pinto [13-15] extended the study of exponential stability to a variety of reasonable systems called h-systems. They introduced the notion of h-stability for difference systems as well as for differential systems. To study the various stability notions of nonlinear difference systems, the comparison principle [12] and variation of constants formula by Agarwal [1] play a fundamental role.

Media and Pinto [13-15] applied the h-stability to obtain a uniform treatment for the various stability notions in difference systems and gave new insights about stability for weakly stable difference systems. Also, Choi , Koo [3] and Goo, Park [9] obtained results for hS of nonlinear difference systems via n_∞-similarity. The stability problem for Volterra difference systems was studied by Elaydi [10], Elaydi and Murakami [11], Raffoul [16], Zouyousefain and Leela [17], Choi and Koo [2], and others.

In this paper, we investigate h-stability of the nonlinear difference systems via n_∞-similarity.

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2. Preliminaries

We consider the nonlinear Volterra difference system

$$x(n+1) = f(n, x(n)), \tag{1}$$

where $f : N(n_0) \times \mathbb{R}^m \to \mathbb{R}^m$, $N(n_0)=\{n_0, n_0+1, \cdots, n_0+k, \cdots\}$ (n_0 a nonnegative integer), $\mathbb{R}^m$ is the m-dimensional real euclidean space. We assume that $f_x= \partial f/\partial x$ exists and is continuous and invertible on $N(n_0) \times \mathbb{R}^m$, $f(n, 0) = 0$. Let $x(n) = x(n, n_0, x_0)$ be the unique solution of (1) with $x(n_0, n_0, x_0) = x_0$. Also, we consider its associated variational system

$$v(n+1) = f_x(n, 0)v(n) \tag{2}$$

and

$$z(n+1) = f_x(n, x(n, n_0, x_0))z(n) \tag{3}$$

of (1). The fundamental matrix $\Phi(n, n_0, 0)$ of (2) is given by

$$\Phi(n, n_0, 0) = \frac{\partial}{\partial x_0} x(n, n_0, 0)$$

and the fundamental matrix $\Phi(n, n_0, x_0)$ of (3) is given by

$$\Phi(n, n_0, x_0) = \frac{\partial}{\partial x_0} x(n, n_0, x_0)$$

(See [12]).

The symbol $| \cdot |$ will be used to denote any convenient vector norm on $\mathbb{R}^m$. We now recall the main definitions [13] that we need in the sequel.

Definition 2.1. The zero solution of (1), or more briefly system (1), is called (hS) h-stable if there exist $c \geq 1, \delta > 0$ and a positive bounded function $h : N(n_0) \to \mathbb{R}$ such that

$$| x(n, n_0, x_0) | \leq c \mid x_0 \mid h(n)h^{-1}(n_0)$$

for $n \geq n_0$ and $| x_0 | < \delta$ (here $h^{-1}(n) = 1/h(n)$),
(hSV) h-stable in variation if the zero solution of system (3) is hS.

The notion of n_∞-similarity in $\mathcal{M}$ was introduced by Choi and Koo [3]. Let $\mathcal{M}$ denote the set of all $m \times m$ invertible matrices $A(n)$ defined on $N(n_0)$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular bounded matrices $S(n)$ such that $S^{-1}(n)$ is also bounded.

Definition 2.2. A matrix $A(n) \in \mathcal{M}$ is n_∞-similar to a matrix $B(n) \in \mathcal{M}$ if there exists an $m \times m$ matrix $F(n)$ absolutely summable over $N(n_0)$, i.e.,

$$\sum_{l=n_0}^{\infty} | F(l) | < \infty$$

such that

$$S(n+1)B(n) - A(n)S(n) = A(n)F(n)$$

for some $S(n) \in \mathcal{N}$.

For the example of n_∞-similarity, see [3].

Remark 2.3. The notion of t_∞-similarity is an equivalence relation in the set of all $m \times m$ continuous matrices on $\mathbb{R}^+$ but the n_∞-similarity is not an equivalence relation in general.

We consider the nonlinear difference system

$$x(n+1) = f(n, x(n))$$

and its perturbed difference system

$$y(n+1) = f(n, y(n)) + \sum_{l=n_0}^{n} g(l, y(l), Ty(l)), y(n_0) = y_0 \tag{4}$$

where $f : N(n_0) \times \mathbb{R}^m \to \mathbb{R}^m$, and $g : N(n_0) \times \mathbb{R}^m \times F(N(n_0), \mathbb{R}^m) \to \mathbb{R}^m$,and $T : F(N(n_0), \mathbb{R}^m) \to F(N(n_0), \mathbb{R}^m)$ is an operator on $F(N(n_0), \mathbb{R}^m)$={$y|y : N(n_0) \to \mathbb{R}^m$ is a sequence}, and $f(n, 0) = g(n, 0, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (4) satisfying the initial condition $y(n_0, n_0, y_0) = y_0$.

We give some related properties that we need in the sequal.

Theorem 2.4 ([15]). *If the solution $x = 0$ of (1) is hS ,then the solution $v = 0$ of (2) is hS.*

Theorem 2.5 ([3]). *Assume that $f_x(n, 0)$ is n_∞-similar to $f_x(n, x(n, n_0, x_0))$ for $n \geq n_0 \geq 0$ and $| x_0 | \leq \delta$ for some constant $\delta > 0$. Then, the solution $v = 0$ of (2) is hS if and only if the solution $z = 0$ of (3) is hS.*

Lemma 2.6 ([7]). *Let $k(n, r, u)$ be a nondecreasing function in r and u for any fixed $n \in N(n_0)$. Suppose that for $n \geq n_0$,*

$$v(n) - \sum_{l=n_0}^{n-1} k(l, v(l), |T|v(l)) < u(n) - \sum_{n=n_0}^{n-1} k(l, u(l), |T|u(l))$$

If $v(n_0) < u(n_0)$, then $v(n) < u(n)$ for all $n \geq n_0$.

Lemma 2.7 ([4]). *Let $a(n)$,$b(n)$, and $c(n)$ be nonnegative functions definded on $N(n_0)$ and d be a positive number. If for any $n \geq n_0$, the following inequality holds*

$$u(n) \leq d + \sum_{s=n_0}^{n-1} a(s)u(s) + \sum_{s=n_0}^{n-1} b(s) \sum_{l=n_0}^{s-1} c(l)u(l),$$

then

$$u(n) \leq dexp[\sum_{s=n_0}^{n-1} (a(s) + b(s) \sum_{l=n_0}^{s-1} c(l)], n \geq n_0.$$

3. Main Results

In this section, we investigate hS for the nonlinear difference systems via n_∞-similarity using the comparison principle and Bihari-type inequalities. In our subsequent discussion we assume that for any two sequences $y(n)$ and $z(n) \in F(N(n_0), \mathbb{R}^m)$, the operator S satisfies the following property: $|y(n)| \leq |z(n)|$ implies $|Sy(n)| \leq |Sz(n)|$ and $|Sy(n)| \leq |S||y(n)|$ for each finite interval $n_0 \leq n \leq l$ of $N(n_0)$ and $|S| : F(N(n_0), \mathbb{R}^+) \to F(N(n_0), \mathbb{R}^+)$ is a nondecreasing operator.

Theorem 3.1. *Suppose that $f_x(n, 0)$ is n_∞-similar to $f_x(n, x(n, n_0, x_0))$ for $n \geq n_0 \geq 0$ and $| x_0 | \leq \delta$ for some constant $\delta > 0$ and the solution $x = 0$ of* (1) *is hS. Also, suppose that*

$$| \sum_{l=n_0}^{n} g(n, y, Ty) | \leq a(n) \mid y| + b(n) \sum_{l=n_0}^{n-1} c(l)|y(l)|$$

where $a, b, c \in F(N(n_0), \mathbb{R}^+)$ and

$$M(n) = expc_1 \sum_{l=n_0}^{n-1} [\frac{h(l)}{h(l+1)} a(l) + \frac{b(l)}{h(l+1)} \sum_{k=n_0}^{l-1} h(k)c(k)] < \infty.$$

Then the zero solution $y = 0$ of (4) *is hS*

Proof. Using the discrete analogue of Alekseev's formula[13], the solutions of (1) and (4) with the same initial value are related by

$$y(n, n_0, y_0) = x(n, n_0, y_0) + \sum_{l=n_0}^{n-1} \int_0^1 \Phi(n, l+1, \mu(y(l), \tau)) d\tau \cdot \sum_{k=n_0}^{l} g(k, y(k), Ty(k)),$$

where $\mu(y(n), \tau) = f(n, y(n)) + \tau \sum_{l=n_0}^{n} g(l, y(l), Ty(l))$, $\tau \in [0, 1]$ and $\Phi(n, n_0, x_0)$ is the fundamental matrix of (3). In view of the assumptions, Theorem 1.4 and Theorem 1.5, the zero solution $z = 0$ of (3) is hS. Hence, we have

$$\begin{aligned}
&| y(n, n_0, y_0) | \\
&\leq | x(n, n_0, y_0) | + \sum_{l=n_0}^{n-1} \int_0^1 |\Phi(n, l+1, \mu(y(l), \tau)) \mid d\tau \cdot \sum_{k=n_0}^{l} g(k, y(k), Ty(k)) \\
&\leq c \mid y_0 | h(n) h^{-1}(n_0) + c \sum_{l=n_0}^{n-1} h(n) h^{-1}(l+1) [a(l) \mid y(l)| + b(l) \sum_{k=n_0}^{l-1} c(k)|y(k)|].
\end{aligned}$$

Letting $u(n) = \frac{|y(n)|}{h(n)}$, by Lemma 1.7, we obtain

$$\begin{aligned} u(n) &\leq cu(n_0) + c\sum_{l=n_0}^{n-1}[\frac{h(l)}{h(l+1)}a(l)u(l) + \frac{b(l)}{h(l+1)}\sum_{k=n_0}^{l-1}h(k)c(k)u(k) \\ &\leq cu(n_0)expc[\sum_{l=n_0}^{n-1}[\frac{h(l)}{h(l+1)}a(l) + \frac{b(l)}{h(l+1)}\sum_{k=n_0}^{l-1}h(k)c(k)] \\ &\leq c_1u(n_0)M(\infty). \end{aligned}$$

Hence, we obtain

$$| y(n) | \leq M | y_0 | h(n)h^{-1}(n_0), M = c_1M(\infty) \geq 1,$$

for all $n \geq n_0$. This completes the proof. □

Corollary 3.2. *Suppose that $f_x(n,0)$ is n_∞-similar to $f_x(n,x(n,n_0,x_0))$ for $n \geq n_0 \geq 0$ and $| x_0 | \leq \delta$ for some constant $\delta > 0$ and the solution $x = 0$ of* (1) *is hS with the positive increasing function $h(n)$ and for any $n \geq n_0$.*

$$| \sum_{l=n_0}^{n} g(n,y,Ty) | \leq a(n) | y| + b(n)\sum_{l=n_0}^{n-1}c(l)|y(l)|,$$

where $a,b,c \in l_1(N(n_0))$. Then the zero solution $y = 0$ of (4) *is also hS.*

Theorem 3.3. *Suppose that $f_x(n,0)$ is n_∞-similar to $f_x(n,x(n,n_0,x_0))$ for $n \geq n_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$ and the solution $x = 0$ of* (1) *is hS with the nonincreasing function $h(n)$. Also, suppose that*

$$| \sum_{l=n_0}^{n} g(l,z,Tz) | \leq r(n,| z |,|Tz|) \quad for \ n \geq n_0, \quad | z | < \infty,$$

where $r : N(n_0) \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing in u and v for each fixed $n \in N(n_0)$ with $r(n,0,0) = 0$. Consider the scalar difference equation

$$u(n+1) = u(n) + cr(n,u(n),|T|u(n)), \quad u(n_0) = u_0, \quad c > 1. \tag{5}$$

If the zero solution $u = 0$ of (5) *is hS, then the zero solution $y = 0$ of* (4) *is also hS whenever $u_0 = c | y_0 |$.*

Proof. Using the discrete analogue of Alekseev's formula [13], we have

$$\begin{aligned} &y(n,n_0,y_0) \\ &= x(n,n_0,y_0) + \sum_{l=n_0}^{n-1}\int_0^1 \Phi(n,l+1,\mu(y(l),\tau))d\tau \cdot \sum_{k=n_0}^{l} g(k,y(k),T(k)), \end{aligned}$$

where $\mu(y(n),\tau) = f(n,y(n)) + \tau\sum_{k=n_0}^{l} g(k,y(k),Ty(k)), \tau \in [0,1]$ and $\Phi(n,n_0,x_0)$ is the fundamental matrix of (3). By the assumptions, Theorem 1.4 and Theorem

1.5, the zero solution $z = 0$ of (3) is hS. Hence, we obtain

$$\begin{aligned} &| y(n, n_0, y_0) | \\ &\leq | x(n, n_0, y_0) | + \sum_{l=n_0}^{n-1} \int_0^1 | \Phi(n, l+1, \mu(y(l), \tau)) | d\tau | \sum_{k=n_0}^{l} g(k, y(k), Ty(k)) |, \\ &\leq c | y_0 | h(n)h^{-1}(n_0) + c \sum_{l=n_0}^{n-1} h(n)h^{-1}(l+1) r(l, | y(l) |, |Ty(l)|) \\ &\leq c | y_0 | + c \sum_{l=n_0}^{n-1} r(l, | y(l) |, |Ty(l)|), \end{aligned}$$

since $h(n)$ is nonincreasing. Thus, we have

$$| y(n) | - c \sum_{l=n_0}^{n-1} r(l, | y(l) |, |Ty(l)|) \leq c | y_0 | = u_0 = u(n) - c \sum_{l=n_0}^{n-1} r(l, u(l), |T|u(l)).$$

By Lemma 1.6, we get $y(n) < u(n)$ for all $n \geq n_0$. In view of the assumption, since $u = 0$ of (5) is hS, we obtain

$$\begin{aligned} | y(n) | < u(n) &\leq c_1 u_0 h(n) h(n_0)^{-1} \\ &= c_1 c | y_0 | h(n) h(n_0)^{-1} \\ &= d | y_0 | h(n) h(n_0)^{-1}, \qquad d = c_1 c > 1 \end{aligned}$$

Hence, the proof is complete. □

Remark 3.4. Letting $g(n, y, Ty) = g(n, y)$ and $r(n, u, w) = r(n, u)$ in Theorem 2.3, we obtain the same result as that of Theorem 3.5 in [9] .

Remark 3.5. If we consider the linear difference system

$$x(n+1) = f(n, x(n)) = A(n)x(n) \tag{6}$$

and its perturbation

$$y(n+1) = A(n)y(n) + \sum_{l=n_0}^{n} g(l, y(l), Ty(l)), \tag{7}$$

where $A(n)$ is an $m \times m$ matrix defined on $N(n_0)$, then the zero solution $y = 0$ of (7) is hS under the same conditions in Theorem 2.3.

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