

**LIE SYMMETRY ANALYSIS AND INVARIANT SOLUTIONS
OF THE GENERALIZED FIFTH-ORDER KDV EQUATION
WITH VARIABLE COEFFICIENTS[†]**

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ABSTRACT. This paper studies the generalized fifth-order KdV equation with variable coefficients using Lie symmetry methods. Lie group classification with respect to the time dependent coefficients is performed. Then we get the similarity reductions using the symmetry and give some exact solutions.

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Key words and phrases : Fifth-order KdV equation, Variable coefficients, Lie symmetries, Group-invariant solutions.

1. Introduction

It is well known that the celebrated KdV types of equations have been around for a very long time. Lot of studies have been conducted with these types of equations [1-8]. In this paper, the generalized fifth-order KdV equation

$$u_t + u^n u_x + \alpha(t)u + \beta(t)u_{xxxx} = 0, \quad (1.1)$$

of time dependent variable coefficients of the linear damping and dispersion is studied. Here in (1.1) the first term represents the evolution term while the second term represents the nonlinear term. The third term represents the linear damping while the fourth term is the dispersion term. The time dependent coefficients of damping and dispersion are, respectively, $\alpha(t)$ and $\beta(t)$ are arbitrary smooth functions of the variable t .

These fifth-order KdV types of equations have been derived to model many physical phenomena, such as gravity-capillary waves on a shallow layer and magneto-sound propagation in plasmas, and so on. In [9] similarity solutions for some classes of the Eq.(1.1) were considered. The paper [10] is mainly concerned

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with the local well-posedness of the initial-value problems for the Kawahara and the modified Kawahara equations in Sobolev spaces.

Lie's method of infinitesimal transformation groups which essentially reduces the number of independent variables in partial differential equation (PDE) and reduces the order of ODE has been widely used in equations of mathematical physics. Lie's method [11-19] is an effective and simplest method among group theoretic techniques and a large number of equations are solved with the aid of this method.

Our aim in the present work is to perform the variable coefficients version of the generalized fifth-order KdV equation with the help of Lie's method. Then we get symmetry reductions and group-invariant solutions.

2. Lie group classification

2.1. Lie symmetry analysis of (1.1)

If (1.1) is invariant under a one parameter Lie group of point transformations $t^* = t + \epsilon\tau(x, t, u) + O(\epsilon^2)$, $x^* = x + \epsilon\xi(x, t, u) + O(\epsilon^2)$, $u^* = u + \epsilon\eta(x, t, u) + O(\epsilon^2)$, (2.1)

with infinitesimal generator

$$V = \tau(x, t, u) \frac{\partial}{\partial t} + \xi(x, t, u) \frac{\partial}{\partial x} + \eta(x, t, u) \frac{\partial}{\partial u}, \quad (2.2)$$

then the invariant condition reads as

$$\eta^t + u^n \eta^x + n\eta u_x u^{n-1} + \alpha'(t)\tau u + \beta'(t)\tau u_{xxxxx} + \alpha(t)\eta + \beta(t)\eta^{xxxxx} = 0, \quad (2.3)$$

where

$$\begin{aligned} \eta^t &= D_t(\eta) - u_x D_t(\xi) - u_t D_t(\tau), \\ \eta^x &= D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau), \\ \eta^{xx} &= D_x(\eta^x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi), \\ \eta^{xxx} &= D_x(\eta^{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi), \\ \eta^{xxxx} &= D_x(\eta^{xxx}) - u_{xxxt} D_x(\tau) - u_{xxxx} D_x(\xi), \\ \eta^{xxxxx} &= D_x(\eta^{xxxx}) - u_{xxxxt} D_x(\tau) - u_{xxxxx} D_x(\xi). \end{aligned}$$

Here, D_i denotes the total derivative operator and is defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \dots \quad i = 1, 2,$$

and $(x^1, x^2) = (t, x)$.

The infinitesimals are determined from invariance condition (2.3), by setting the coefficients of different differentials equal to zero. We obtain an over determined system of linear partial differential equations (PDEs). Therefore, the determining equation for symmetries after some tedious calculations yield

$$\tau = \tau(t), \quad \xi_u = 0, \quad \xi_t = 0, \quad \xi_{xx} = 0, \quad \eta_{uu} = 0, \quad \eta_x = 0, \quad (2.4)$$

$$\beta(t)\tau_t + \beta'(t)\tau - 5\beta(t)\xi_x = 0, \quad (2.5)$$

$$\tau_t u - \xi_x u + n\eta = 0, \tag{2.6}$$

$$\eta_t + \alpha(t)\eta + \alpha(t)u\tau_t - \alpha(t)u\eta_u + u\alpha'(t)\tau = 0. \tag{2.7}$$

Solving the determining Eqs.(2.4)-(2.7) we get three sets of infinitesimals
Set.1

$$\eta = \frac{(c_1 + c_3 n e^{nt\alpha})}{n} u, \tau = -\frac{c_3 e^{nt\alpha}}{\alpha} + c_4, \xi = c_1 x + c_2, \tag{2.8}$$

$$\frac{\beta'}{\beta} = \frac{5c_1\alpha + c_3 n\alpha e^{nt\alpha}}{c_4\alpha - c_3 e^{nt\alpha}},$$

where α is a constant.

For this case, the symmetry Lie algebra is four-dimensional and is spanned by generators of symmetry

$$V_1 = \frac{\partial}{\partial x}, V_2 = x \frac{\partial}{\partial x} + \frac{1}{n} u \frac{\partial}{\partial u}, V_3 = -\frac{1}{\alpha} e^{n\alpha t} \frac{\partial}{\partial t} + e^{n\alpha t} u \frac{\partial}{\partial u}, V_4 = \frac{\partial}{\partial t}. \tag{2.9}$$

Set.2

$$\xi = c_1 x + c_2, \tau = (c_1 - nc_3)t + c_4, \eta = c_3 u, \tag{2.10}$$

$$\frac{\beta'}{\beta} = \frac{4c_1 + nc_3}{(c_1 - nc_3)t + c_4}, \alpha = \frac{c_5}{(c_1 - nc_3)t + c_4}.$$

The Lie algebra extends in this case by symmetry generator

$$V_1 = \frac{\partial}{\partial x}, V_2 = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t}, V_3 = -nt \frac{\partial}{\partial t} + u \frac{\partial}{\partial u}, V_4 = \frac{\partial}{\partial t}. \tag{2.11}$$

Set.3

$$\xi = (c_1 + nc_2)x + c_3, \tau = c_1 t + c_4, \eta = c_2 u, \tag{2.12}$$

$$\frac{\beta'}{\beta} = \frac{4c_1 + 5nc_2}{c_1 t + c_4}, \alpha = 0.$$

For this case, the symmetry Lie algebra is extended by the generators of symmetry

$$V_1 = \frac{\partial}{\partial x}, V_2 = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t}, V_3 = nx \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}, V_4 = \frac{\partial}{\partial t}. \tag{2.13}$$

It easy to check that the symmetry generators found in (2.9), (2.11) and (2.13) form a closed Lie algebra whose commutation relations are given in Tables 1, 2 and 3, respectively. The entry in row i and column j representing $[V_i, V_j]$. Here $[V_i, V_j]$ is the commutator for the Lie algebra[11] given by

$$[V_i, V_j] = V_i V_j - V_j V_i.$$

Table 1. Commutator table of the Lie algebra of (2.9).

	V_1	V_2	V_3	V_4
V_1	0	V_1	0	0
V_2	$-V_1$	0	0	0
V_3	0	0	0	$-n\alpha V_3$
V_4	0	0	$n\alpha V_3$	0

Table 2. Commutator table of the Lie algebra of (2.11).

	V_1	V_2	V_3	V_4
V_1	0	V_1	0	0
V_2	$-V_1$	0	0	$-V_4$
V_3	0	0	0	nV_4
V_4	0	V_4	$-nV_4$	0

Table 3. Commutator table of the Lie algebra of (2.13).

	V_1	V_2	V_3	V_4
V_1	0	V_1	nV_1	0
V_2	$-V_1$	0	0	$-V_4$
V_3	$-nV_1$	0	0	0
V_4	0	V_4	0	0

3. Symmetry reductions and exact group-invariant solutions

In order to obtain similarity reductions and solutions of Eq(1.1), one first solves the characteristic equation, to obtain invariant transformations and then substitutes these results into Eq(1.1) to determine the corresponding reduced equations. Finally, similarity solutions can be obtained. We write the characteristic equations in the form

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}.$$

Here we discuss the following cases:

3.1. Symmetry reductions to case Set.1

3.1.1. $c_1 = c_2 = c_4 = 0(V_3)$.

In this case, one can get the following form similarity solution

$$u(x, t) = e^{-\alpha t} f(h), \tag{3.1}$$

where $h = x$ is the group-invariant and $\beta = b_0 e^{-n\alpha t}$. Substitution of (3.1) into the (1.1), we reduce it to the following ODE

$$b_0 f^{(5)} + f^n f' = 0. \tag{3.2}$$

Integration of (3.2), yields

$$b_0 f^{(4)} + \frac{1}{n+1} f^{n+1} - C_1 = 0, \tag{3.3}$$

where b_0, C_1 are constants of integration.

3.1.2. $c_1 = c_4 = 0$.

we derive the following expression of u

$$u(x, t) = e^{-\alpha t} f(h), \tag{3.4}$$

where $h = x - \frac{c_2}{c_3 n} e^{-n\alpha t}$ is the group-invariant and $\beta = b_1 e^{-n\alpha t}$. Substitution of (3.4) into the (1.1), one can get

$$c_3 b_1 f^{(5)} + c_3 f^n f' + c_2 \alpha f' = 0. \tag{3.5}$$

It is also possible to integrate once the ODE (3.5) to get

$$c_3 b_1 f^{(4)} + \frac{c_3}{n+1} f^{n+1} + c_2 \alpha f - C_2 = 0, \tag{3.6}$$

where b_1, C_2 are constants of integration.

3.1.3. $c_4 = 0$.

In this case, we obtain the similarity solution is given by

$$u = \exp\left[-\frac{c_1}{c_3 n^2 \alpha} e^{-nt\alpha} - \alpha t\right] f(h),$$

where $h = (c_1 x + c_2) \exp\left[\frac{c_1}{c_3 n} e^{-nt\alpha}\right]$ is the group-invariant and

$$\beta = b_2 \exp\left[-\frac{5c_1}{c_3 n} e^{-nt\alpha} - n\alpha t\right].$$

We have

$$c_3 b_2 n c_1^4 f^{(5)} + n c_3 f^n f' - \alpha n h f' + \alpha f = 0, \tag{3.7}$$

where b_2 is a constant of integration. Here $f' = \frac{df}{dh}$.

3.2. Symmetry reductions to case Set.2

3.2.1. $c_1 = c_2 = 0$.

In this case, one can get the following form similarity solution

$$u(x, t) = (-nc_3 t + c_4)^{-\frac{1}{n}} f(h), \tag{3.8}$$

where $h = x$ is the group-invariant and $\beta = b_3 (-nc_3 t + c_4)^{-1}$, $\alpha = \frac{c_5}{(c_1 - nc_3)t + c_4}$. Substitution of (3.8) into the (1.1), we reduce it to the following ODE

$$b_3 f^{(5)} + f^n f' + (c_3 + c_5) f = 0. \tag{3.9}$$

Integration of (3.9), with $c_3 = -c_5$, yields

$$b_3 f^{(4)} + \frac{1}{n+1} f^{n+1} - C_3 = 0, \tag{3.10}$$

where b_3, C_3 are constants of integration. Which is similar to (3.3).

3.2.2. $c_1 = 0$.

we derive the following expression

$$u(x, t) = (-nc_3 t + c_4)^{-\frac{1}{n}} f(h), h(x, t) = x + \frac{c_2}{nc_3} \lg(c_4 - nc_3 t), \tag{3.11}$$

and

$$\beta = \frac{b_4}{-nc_3 t + c_4}, \alpha = \frac{c_5}{(c_1 - nc_3)t + c_4}. \tag{3.12}$$

Inserting (3.11), (3.12) in (1.1), we have

$$b_4 f^{(5)} + f^n f' - c_2 f' + (c_3 + c_5) f = 0. \quad (3.13)$$

When $c_3 = -c_5$, it is also possible to integrate once the ODE (3.13) to get

$$b_4 f^{(4)} + \frac{1}{n+1} f^{n+1} - c_2 f - C_4 = 0, \quad (3.14)$$

where b_4, C_4 are constants of integration.

3.2.3. The general case.

In this case, we obtain the similarity solution is given by

$$u = [(c_1 - nc_3)t + c_4]^{\frac{c_3}{c_1 - nc_3}} f(h), \quad (3.15)$$

where

$$h = (nc_1 x + c_2) [(c_1 - nc_3)t + c_4]^{\frac{-c_1}{c_1 - nc_3}}, \quad (3.16)$$

is the group-invariant and

$$\beta = b_5 [(c_1 - nc_3)t + c_4]^{\frac{4c_1 + nc_3}{c_1 - nc_3}}, \alpha = \frac{c_5}{(c_1 - nc_3)t + c_4}. \quad (3.17)$$

We have

$$b_5 c_1^5 f^{(5)} + c_1 f^n f' - c_1 h f' + (c_3 + c_5) f = 0. \quad (3.18)$$

If $-c_1 = c_3 + c_5$, then we obtain

$$b_5 c_1^4 f^{(5)} + f^n f' - h f' - f = 0. \quad (3.19)$$

Integrate once the ODE (3.19), we get

$$b_5 c_1^4 f^{(4)} + \frac{1}{n+1} f^{n+1} - h f = 0. \quad (3.20)$$

where b_5 is a constant of integration. Here $f' = \frac{df}{dh}$.

3.3. Symmetry reductions to case Set.3

3.3.1. $c_1 = 0$.

In this case, one can get the following form similarity solution

$$u(x, t) = e^{\frac{c_2}{c_4} t} f(h), \quad (3.21)$$

where

$$h = (nc_2 x + c_3) e^{-\frac{nc_2}{c_4} t}$$

is the group-invariant and $\beta = b_6 e^{\frac{5nc_2}{c_4} t}$. Substitution of (3.21) into the (1.1), we reduce it to the following ODE

$$b_6 c_4 n^4 c_2^4 f^{(5)} + c_4 n f^n f' - n h f' + f = 0, \quad (3.22)$$

where b_6 is a constant of integration.

3.3.2. $c_1 = c_2 = 0$.

In this case, we have $u = f(h), h = x - \frac{c_3}{c_2}t, \beta = b_7$. Thus (1.1) can become

$$b_7 c_2 f^{(5)} + c_2 f^n f' - c_3 f' = 0, \tag{3.23}$$

where b_7 is a constant. It is similar to (3.5).

3.3.3. The general case.

In this case, we obtain the similarity solution is given by

$$u = (c_1 t + c_4)^{\frac{c_2}{c_1}} f(h), \tag{3.24}$$

where

$$h = [(c_1 + n c_2)x + c_3](c_1 t + c_4)^{-\frac{c_1 + n c_2}{c_1}}, \tag{3.25}$$

is the group-invariant and

$$\beta = b_8 [c_1 t + c_4]^{\frac{5 n c_2 + 4 c_1}{c_1}}. \tag{3.26}$$

We have

$$b_8 (c_1 + n c_2)^5 f^{(5)} + (c_1 + n c_2) f^n f' - (c_1 + n c_2) h f' + c_2 f = 0. \tag{3.27}$$

If $-c_2 = c_1 + n c_2$, then we obtain

$$b_8 (c_1 + n c_2)^5 f^{(5)} + (c_1 + n c_2) f^n f' + c_2 h f' + c_2 f = 0. \tag{3.28}$$

Integrate once the ODE (3.28), one can get

$$b_8 (c_1 + n c_2)^5 f^{(4)} + \frac{c_1 + n c_2}{n + 1} f^{n+1} + c_2 h f = 0. \tag{3.29}$$

where b_8 is a constant of integration. Here $f' = \frac{df}{dh}$.

It is not difficult to find that the reduced ODEs may be classified into four classes [20]

$$f^{(5)} + P f^n f' + Q h f' + R f = 0, \tag{3.30}$$

$$f^{(5)} + P f^n f' + Q f' = 0, \tag{3.31}$$

$$f^{(5)} + P f^n f' + Q f = 0, \tag{3.32}$$

$$f^{(5)} + P f^n f' = 0, \tag{3.33}$$

where P, Q and R are constants.

4. The exact power series solutions

In this section, we will consider the exact analytic solutions to the reduced equations by using the power series method.

Now, we seek a solution of Eq.(3.31) in a power series of the form

$$f(h) = \sum_{n=0}^{\infty} a_n h^n. \tag{4.1}$$

Integrate once the ODE (3.31), we have

$$f^{(4)} + \frac{P}{n + 1} f^{(n+1)} + Q f - C = 0, \tag{4.2}$$

Substituting (4.1) into (4.2), we can get

$$\begin{aligned}
 & 24a_4 + \sum_{n=1}^{\infty} (n+1)(n+2)(n+3)(n+4)a_{n+4}h^n + \frac{P}{n+1}a_0^{n+1} \\
 & + \frac{P}{n+1} \sum_{n=1}^{\infty} \underbrace{\left(\sum_{k_1=0}^n \sum_{k_2=0}^{k_1} \cdots \sum_{k_{n-1}=0}^{k_{n-2}} \sum_{k_n=0}^{k_{n-1}} a_{k_n} a_{k_{n-1}-k_n} \cdots a_{n-k_1} \right)}_{n-1} h^n \quad (4.3) \\
 & + Qa_0 + Q \sum_{n=1}^{\infty} a_n h^n - C = 0,
 \end{aligned}$$

From (4.3), comparing coefficients, for $n = 0$, we obtain

$$a_4 = \frac{1}{24} \left(C - Qa_0 - \frac{P}{n+1} a_0^{n+1} \right). \quad (4.4)$$

Generally, for $n \geq 1$, we have

$$\begin{aligned}
 a_{n+4} = & \frac{-1}{(n+1)(n+2)(n+3)(n+4)} \left\{ Qa_n \right. \\
 & \left. + \frac{P}{n+1} \underbrace{\left(\sum_{k_1=0}^n \sum_{k_2=0}^{k_1} \cdots \sum_{k_{n-1}=0}^{k_{n-2}} \sum_{k_n=0}^{k_{n-1}} a_{k_n} a_{k_{n-1}-k_n} \cdots a_{n-k_1} \right)}_{n-1} \right\}. \quad (4.5)
 \end{aligned}$$

From (4.4) and (4.5), we can get all the coefficients $a_n (n \geq 4)$ of the power series (4.1). For arbitrary chosen constant numbers a_0, a_1, a_2 , and a_3 , the other terms can be determined successively from (4.4) and (4.5) in a unique way. In addition, it is easy to prove that the convergence of the power series (4.1) with the coefficients give by (4.4) and (4.5)[21]. The details are omitted here. For this reason, this power series solution is an exact analytic solution.

For example, the power series solution of Eq.(3.6) can be written as following

$$\begin{aligned}
 f(\xi) = & a_0 + a_1h + a_2h^2 + a_3h^3 + a_4h^4 + \sum_{n=1}^{\infty} a_{n+4}h^{n+4} \\
 = & a_0 + a_1h + a_2h^2 + a_3h^3 + \frac{1}{24c_3b_1} (C_2 - c_2\alpha a_0 - \frac{c_3}{n+1} a_0^{n+1}) h^4 \\
 - & \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)} \left[\frac{c_2\alpha}{c_3b_1} a_n \right. \quad (4.6) \\
 & \left. + \frac{1}{b_1(n+1)} \underbrace{\left(\sum_{k_1=0}^n \sum_{k_2=0}^{k_1} \cdots \sum_{k_{n-1}=0}^{k_{n-2}} \sum_{k_n=0}^{k_{n-1}} a_{k_n} a_{k_{n-1}-k_n} \cdots a_{n-k_1} \right)}_{n-1} \right] h^{n+4}.
 \end{aligned}$$

Thus, the exact power series solution of Eq.(1.3) is

$$\begin{aligned}
 u(x, t) &= \left[a_0 + a_1 \left(x - \frac{c_2}{c_3 n} e^{-n\alpha t} \right) + a_2 \left(x - \frac{c_2}{c_3 n} e^{-n\alpha t} \right)^2 + a_3 \left(x - \frac{c_2}{c_3 n} e^{-n\alpha t} \right)^3 \right. \\
 &\quad + \frac{1}{24c_4 b_1} \left(C_2 - c_2 \alpha a_0 - \frac{c_3}{n+1} a_0^{n+1} \right) \left(x - \frac{c_2}{c_3 n} e^{-n\alpha t} \right)^4 \\
 &\quad \left. + \sum_{n=1}^{\infty} c_{n+4} \left(x - \frac{c_2}{c_3 n} e^{-n\alpha t} \right)^{n+4} \right] e^{-t\alpha} \\
 &= \left\{ a_0 + a_1 \left(x - \frac{c_2}{c_3 n} e^{-n\alpha t} \right) + a_2 \left(x - \frac{c_2}{c_3 n} e^{-n\alpha t} \right)^2 + a_3 \left(x - \frac{c_2}{c_3 n} e^{-n\alpha t} \right)^3 \right. \\
 &\quad + \frac{1}{24c_4 b_1} \left(C_2 - c_2 \alpha a_0 - \frac{c_3}{n+1} a_0^{n+1} \right) \left(x - \frac{c_2}{c_3 n} e^{-n\alpha t} \right)^4 \tag{4.7} \\
 &\quad - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)} \left[\frac{c_2 \alpha}{c_3 b_1} a_n \right. \\
 &\quad \left. + \frac{1}{b_1(n+1)} \underbrace{\left(\sum_{k_1=0}^n \sum_{k_2=0}^{k_1} \dots \sum_{k_{n-1}=0}^{k_{n-2}} \sum_{k_n=0}^{k_{n-1}} a_{k_n} a_{k_{n-1}-k_n} \dots a_{n-k_1} \right)}_{n-1} \right] \\
 &\quad \left. \left(x - \frac{c_2}{c_3 n} e^{-n\alpha t} \right)^{n+4} \right\} e^{-t\alpha},
 \end{aligned}$$

where $a_i (i = 0, 1, 2, 3)$ are arbitrary constants, the other coefficients $a_n (n \geq 4)$ can be determined successively from (4.4) and (4.5).

Of course, in physical applications, it will be convenient to write the solution of Eq.(1.3) in the approximate form

$$\begin{aligned}
 u(x, t) &= \left[a_0 + a_1 \left(x - \frac{c_2}{c_3 n} e^{-n\alpha t} \right) + a_2 \left(x - \frac{c_2}{c_3 n} e^{-n\alpha t} \right)^2 + a_3 \left(x - \frac{c_2}{c_3 n} e^{-n\alpha t} \right)^3 \right. \\
 &\quad \left. + \frac{1}{24c_4 b_1} \left(C_2 - c_2 \alpha a_0 - \frac{c_3}{n+1} a_0^{n+1} \right) \left(x - \frac{c_2}{c_3 n} e^{-n\alpha t} \right)^4 + \dots \right] e^{-t\alpha}, \tag{4.8}
 \end{aligned}$$

in terms of the above computation.

Remark 1. The exact solution of the rest of Eqs and the solution in the approximate form can be written in terms of the above computation. but for brevity we have omitted them here.

5. Conclusion

In this paper we have studied the generalized fifth-order KdV equation with variable coefficients using the Lie symmetry group methods. A Lie group classification of the symmetries with respect to the special forms of the time dependent variable coefficients was presented. At the same time, we generalize the corresponding results in [4]. This study contains a number of new and important

insights. Then some exact exact analytic solutions are obtained by using the power series method. It is important that the reduced ODEs may be classified into four classes. Furthermore, how to get the other forms of exact solutions to these reduced ODEs? We hope to investigate this in the future.

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