J. Appl. Math. & Informatics Vol. **31**(2013), No. 1 - 2, pp. 199 - 209
Website: http://www.kcam.biz

OSCILLATION OF HIGHER-ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS AND MIXED ARGUMENTS[†]

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Abstract. In this paper, we study the oscillation problem of the following higher-order neutral differential equation with positive and negative coefficients and mixed arguments

$$z^{(n)}(t)+q_1(t)|x(t-\sigma_1)|^{\alpha-1}x(t-\sigma_1)+q_2(t)|x(t-\sigma_2)|^{\beta-1}x(t-\sigma_2)=e(t),$$

where $t \geq t_0$, $z(t)=x(t)-p(t)x(t-\tau)$ with $p(t)>0$, $\beta>1>\alpha>0$, τ, σ_1 and σ_2 are real numbers. Without imposing any restriction on τ, we establish several oscillation criteria for the above equation in two cases: (i) $q_1(t)\leq 0$, $q_2(t)>0$, $\sigma_1\geq 0$ and $\sigma_2\leq\tau$; (ii) $q_1(t)\geq 0$, $q_2(t)<0$, $\sigma_1\geq\tau$ and $\sigma_2\leq 0$. As an interesting application, our results can also be applied to the following higher-order differential equation with positive and negative coefficients and mixed arguments

$$x^{(n)}(t)+q_1(t)|x(t-\sigma_1)|^{\alpha-1}x(t-\sigma_1)+q_2(t)|x(t-\sigma_2)|^{\beta-1}x(t-\sigma_2)=e(t).$$

Two numerical examples are also given to illustrate the main results.

AMS Mathematics Subject Classification : 34C10.
Key words and phrases : Oscillation, higher-order, neutral differential equation, positive and negative coefficients.

1. Introduction

Consider the following nth-order neutral differential equation of the form

$$z^{(n)}(t)+q_1(t)|x(t-\sigma_1)|^{\alpha-1}x(t-\sigma_1)+q_2(t)|x(t-\sigma_2)|^{\beta-1}x(t-\sigma_2)=e(t), \quad (1.1)$$

where $t\geq t_0$, $z(t)=x(t)-p(t)x(t-\tau)$, $n\geq 1$ is an integer. Throughout this paper, we assume that

(A_1) $p\in C^n[t_0,\infty)$ and $p(t)>0$;

Received December 31, 2011. Revised July 2, 2012. Accepted July 3, 2012. *Corresponding author. †This work was supported by the Natural Science Foundation of Shandong Province (grant nos. ZR2010AL002 and JQ201119) and the National Natural Science Foundation of China (grant no. 61174217).

(A_2) $q_1, q_2 \in C[t_0, \infty)$ are real-valued functions with different sign;
(A_3) $\beta > 1 > \alpha > 0$, τ, σ_1 and σ_2 are real numbers;
(A_4) $e \in C[t_0, \infty)$ is an oscillatory real-valued function.

We are here only concerned with the nontrivial solutions of Eq. (1.1) that are defined for all large t. The oscillatory behavior is considered in the usual sense, i.e., a solution of Eq. (1.1) is said to be oscillatory if it has arbitrarily large zeros, and nonoscillatory otherwise. Eq. (1.1) is said to be oscillatory if all of its nontrivial solutions are oscillatory.

In this paper, we will focus on oscillation of Eq. (1.1) in the following two cases which do not impose any restriction on τ:

(i) $q_1(t) \leq 0$, $q_2(t) > 0$, $\sigma_1 \geq 0$ and $\sigma_2 \leq \tau$;
(ii) $q_1(t) \geq 0$, $q_2(t) < 0$, $\sigma_1 \geq \tau$ and $\sigma_2 \leq 0$.

In the literature, most of oscillation results for neutral differential equations with positive and negative coefficients and mixed arguments are restricted to the lower-order case, i.e., $n = 1$ or $n = 2$ (see the monograph [1] and references therein). For other oscillation results of neutral differential equations of the form of (1.1), we refer the readers to [2-6, 8-12, 16-18, 20].

To the best of our knowledge, nothing has been known about oscillation of the higher-order forced neutral differential equation (1.1) in both cases (i) and (ii). Generally speaking, the investigation on oscillation of Eq. (1.1) becomes difficult due to the existence of positive and negative coefficients, mixed (delayed and advanced) arguments and mixed nonlinearities.

Following the method used in Agarwal and Grace [1], Ou and Wang [13], Sun and Wong [14], Sun and Mingarelli [15], and Yang [20], we will establish oscillation criteria for Eq. (1.1) in both cases (i) and (ii) by using a nonnegative kernel function $H(t, s)$ defined on $D = \{(t, s) : t \geq s \geq t_0\}$ which is sufficiently smooth in the variable s and satisfy:

(a) $H(t, s) > 0$ for $t > s \geq t_0$ and $H(t, t) = 0$;
(b) $h_i(t, s) = (-1)^i (\partial^i H/\partial s^i)$, $h_i(t, s) > 0$ for $t > s \geq t_0$, $h_i(t, t) = 0$, $H^{-1}(t, t_0) h_i(t, t_0) = O(1)$ as $t \to \infty$, $i = 0, 1, 2, \cdots, n$, where $h_0(t, s) = H(t, s)$.

Our results will include some existing results as special cases, and can be used to reveal the oscillatory behavior of solutions of Eq. (1.1) that are not covered by the existing results in the literature.

2. Main results

Theorem 2.1. *Assume that $(A_1) - (A_4)$ and (i) hold. If there exists a kernel function $H(t, s)$ satisfying (a) and (b) such that*

$$\limsup_{t\to\infty} \frac{1}{H(t, r)} \int_r^t [H(t, s)e(s) + P_1(t, s) + P_2(t, s)]ds = +\infty \tag{2.1}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t, r)} \int_r^t [H(t, s)e(s) - P_1(t, s) - P_2(t, s)]ds = -\infty \tag{2.2}$$

hold for some $r \geq t_0$, where

$$P_1(t,s) = (\alpha-1)\alpha^{\frac{\alpha}{1-\alpha}}[h_n(t,s-\sigma_1)]^{\frac{\alpha}{\alpha-1}}[H(t,s)|q_1(s)|]^{\frac{1}{1-\alpha}},$$

$$P_2(t,s) = (1-\beta)\beta^{\frac{\beta}{1-\beta}}[H(t,s+\sigma_2-\tau)q_2(s+\sigma_2-\tau)]^{\frac{1}{1-\beta}}[h_n(t,s)p(s)]^{\frac{\beta}{\beta-1}},$$

then Eq. (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Without loss of generality, we may assume $x(t-m) > 0$ for $t \geq t_1 \geq t_0$, where $m = \max\{\tau, \sigma_1, \sigma_2\}$. When $x(t)$ is eventually negative, the proof follows the same argument. Multiplying Eq. (1.1) by $H(t,s)$ and integrating from t_1 to t gives

$$\begin{aligned}
\int_{t_1}^{t} H(t,s)e(s)ds &= \int_{t_1}^{t} H(t,s)z^{(n)}(s)ds \\
&\quad - \int_{t_1}^{t} H(t,s)|q_1(s)|x^{\alpha}(s-\sigma_1)ds \\
&\quad + \int_{t_1}^{t} H(t,s)q_2(s)x^{\beta}(s-\sigma_2)ds \\
&= -\sum_{i=0}^{n-1} h_i(t,t_1)z^{(n-i-1)}(t_1) + \int_{t_1}^{t} h_n(t,s)z(s)ds \\
&\quad - \int_{t_1}^{t} H(t,s)|q_1(s)|x^{\alpha}(s-\sigma_1)ds \\
&\quad + \int_{t_1}^{t} H(t,s)q_2(s)x^{\beta}(s-\sigma_2)ds.
\end{aligned} \tag{2.3}$$

Note that

$$\int_{t_1}^{t} h_n(t,s)z(s)ds = \int_{t_1}^{t} h_n(t,s)[x(s)-p(s)x(s-\tau)]ds.$$

From (2.3) we get

$$\begin{aligned}
&\int_{t_1}^{t} H(t,s)e(s)ds \\
= &-\sum_{i=0}^{n-1} h_i(t,t_1)x^{(n-i-1)}(t_1) \\
&+ \int_{t_1}^{t} [h_n(t,s)x(s) - H(t,s)|q_1(s)|x^{\alpha}(s-\sigma_1)]ds \\
&+ \int_{t_1}^{t} [H(t,s)q_2(s)x^{\beta}(s-\sigma_2) - h_n(t,s)p(s)x(s-\tau)]ds.
\end{aligned} \tag{2.4}$$

On the other hand, since $\sigma_1 \geq 0$ and $\sigma_2 \leq \tau$, we have

$$\begin{aligned}
&\int_{t_1}^{t} [h_n(t,s)x(s) - H(t,s)|q_1(s)|x^{\alpha}(s-\sigma_1)]ds \\
= &\int_{t_1+\sigma_1}^{t+\sigma_1} h_n(t,s-\sigma_1)x(s-\sigma_1)ds - \int_{t_1}^{t} H(t,s)|q_1(s)|x^{\alpha}(s-\sigma_1)ds \\
\geq &\int_{t_1}^{t} [h_n(t,s-\sigma_1)x(s-\sigma_1) - H(t,s)|q_1(s)|x^{\alpha}(s-\sigma_1)]ds \\
&- \int_{t_1}^{t_1+\sigma_1} h_n(t,s-\sigma_1)x(s-\sigma_1)ds,
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
&\int_{t_1}^{t} [H(t,s)q_2(s)x^{\beta}(s-\sigma_2) - h_n(t,s)p(s)x(s-\tau)]ds \\
= &\int_{t_1+\tau-\sigma_2}^{t+\tau-\sigma_2} H(t,s+\sigma_2-\tau)q_2(s+\sigma_2-\tau)x^{\beta}(s-\tau)ds \\
&- \int_{t_1}^{t} h_n(t,s)p(s)x(s-\tau)ds \\
\geq &\int_{t_1}^{t} H(t,s+\sigma_2-\tau)q_2(s+\sigma_2-\tau)x^{\beta}(s-\tau)ds \\
&- \int_{t_1}^{t} h_n(t,s)p(s)x(s-\tau)ds \\
&- \int_{t_1}^{t_1+\tau-\sigma_2} H(t,s+\sigma_2-\tau)q_2(s+\sigma_2-\tau)x^{\beta}(s-\tau)ds.
\end{aligned} \tag{2.6}$$

For $x > 0$, set

$$F_1(x) = h_n(t,s-\sigma_1)x - H(t,s)|q_1(s)|x^{\alpha},$$

and

$$F_2(x) = H(t,s+\sigma_2-\tau)q_2(s+\sigma_2-\tau)x^{\beta} - h_n(t,s)p(s)x.$$

It is not difficult to verify that $F_1(x)$ and $F_2(x)$ obtain their minimums at

$$x = \left[\frac{\alpha H(t,s)|q_1(s)|}{h_n(t,s-\sigma_1)}\right]^{1/(1-\alpha)}$$

and

$$x = \left[\frac{h_n(t,s)p(s)}{\beta H(t,s+\sigma_2-\tau)q_2(s+\sigma_2-\tau)}\right]^{1/(\beta-1)},$$

respectively. Moreover,

$$F_1^{\min} = P_1(t,s), \quad F_2^{\min} = P_2(t,s). \tag{2.7}$$

Substituting (2.5)-(2.7) into (2.4) and dividing $H(t,t_1)$ on both sides of (2.4), by assumptions (a) and (b) we have that there exists a constant $M \in R$ such that

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}[H(t,s)e(s) - P_1(t,s) - P_2(t,s)]ds \ge M,$$

which contradicts (2.2). This completes the proof of Theorem 2.1. $\square$

Theorem 2.2. *Assume that $(A_1) - (A_4)$ and (ii) hold. If there exists a kernel function $H(t,s)$ satisfying (a) and (b) such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,r)}\int_{r}^{t}[H(t,s)e(s) - Q_1(t,s) - Q_2(t,s)]ds = +\infty \tag{2.8}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,r)}\int_{r}^{t}[H(t,s)e(s) + Q_1(t,s) + Q_2(t,s)]ds = -\infty \tag{2.9}$$

hold for some $r \ge t_0$, where

$$Q_1(t,s) = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}[H(t,s)q_1(s)]^{\frac{1}{1-\alpha}}[h_n(t,s+\tau-\sigma_1)p(s+\tau-\sigma_1)]^{\frac{\alpha}{\alpha-1}},$$

$$Q_2(t,s) = (\beta-1)\beta^{\frac{\beta}{1-\beta}}[h_n(t,s)]^{\frac{\beta}{\beta-1}}[H(t,s+\sigma_2)|q_2(s+\sigma_2)|]^{\frac{1}{1-\beta}},$$

then Eq. (1.1) is oscillatory.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of Eq. (1.2). Say $x(t-m) > 0$ for $t \ge t_1 \ge t_0$, where $m = \max\{\tau, \sigma_1, \sigma_2\}$. When $x(t)$ is eventually negative, the proof follows the same argument. Proceeding as in the proof of Theorem 1, we can get

$$\begin{aligned} &\int_{t_1}^{t} H(t,s)e(s)ds \\ = \;& -\sum_{i=0}^{n-1} h_i(t,t_1)z^{(n-i-1)}(t_1) \\ &+ \int_{t_1}^{t}[H(t,s)q_1(s)x^{\alpha}(s-\sigma_1) - h_n(t,s)p(s)x(s-\tau)]ds \\ &+ \int_{t_1}^{t}[h_n(t,s)x(s) - H(t,s)|q_2(s)|x^{\beta}(s-\sigma_2)]ds. \end{aligned} \tag{2.10}$$

Note that $\sigma_1 \geq \tau$ and $\sigma_2 \leq 0$, we have

$$\begin{aligned}
&\int_{t_1}^{t}[H(t,s)q_1(s)x^{\alpha}(s-\sigma_1) - h_n(t,s)p(s)x(s-\tau)]ds \\
= &\int_{t_1}^{t} H(t,s)q_1(s)x^{\alpha}(s-\sigma_1)ds \\
&-\int_{t_1+\sigma_1-\tau}^{t+\sigma_1-\tau} h_n(t,s+\tau-\sigma_1)p(s+\tau-\sigma_1)x(s-\sigma_1)ds \\
\leq &\int_{t_1}^{t} H(t,s)q_1(s)x^{\alpha}(s-\sigma_1)ds \\
&-\textstyle\int_{t_1}^{t} h_n(t,s+\tau-\sigma_1)p(s+\tau-\sigma_1)x(s-\sigma_1)ds \\
&+\int_{t_1}^{t_1+\sigma_1-\tau} h_n(t,s+\tau-\sigma_1)p(s+\tau-\sigma_1)x(s-\sigma_1)ds,
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
&\int_{t_1}^{t}[h_n(t,s)x(s) - H(t,s)|q_2(s)|x^{\beta}(s-\sigma_2)]ds \\
= &\int_{t_1}^{t} h_n(t,s)x(s)ds - \int_{t_1-\sigma_2}^{t-\sigma_2} H(t,s+\sigma_2)|q_2(s+\sigma_2)|x^{\beta}(s)]ds \\
\leq &\int_{t_1}^{t}[h_n(t,s)x(s) - H(t,s+\sigma_2)|q_2(s+\sigma_2)|x^{\beta}(s)]ds \\
&+\int_{t_1}^{t_1-\sigma_2} H(t,s+\sigma_2)|q_2(s+\sigma_2)|x^{\beta}(s)ds.
\end{aligned} \tag{2.12}$$

For $x > 0$, set

$$G_1(x) = H(t,s)q_1(s)x^{\alpha} - h_n(t,s+\tau-\sigma_1)p(s+\tau-\sigma_1)x,$$

and

$$G_2(x) = h_n(t,s)x - H(t,s+\sigma_2)|q_2(s+\sigma_2)|x^{\beta}.$$

Then, we can easily see that G_1 and G_2 obtain their maximums at

$$x = \left[\frac{\alpha H(t,s)q_1(s)}{h_n(t,s+\tau-\sigma_1)p(s+\tau-\sigma_1)}\right]^{1/(1-\alpha)}$$

and

$$x = \left[\frac{h_n(t,s)}{\beta H(t,s+\sigma_2)|q_2(s+\sigma_2)|}\right]^{1/(\beta-1)},$$

respectively. We also have

$$G_1^{\max} = Q_1(t,s), \quad G_2^{\max} = Q_2(t,s). \tag{2.13}$$

Substituting (2.11)-(2.13) into (2.10) and dividing $H(t,t_1)$ on both sides of (2.10), we get a contradiction with (2.8). The proof is complete. $\square$

Following the similar steps as in the proofs of Theorem 2.1 and Theorem 2.2, we can extend Theorem 2.1 and Theorem 2.2 to the case of mixed time-varying arguments. We assume here that $\tau, \sigma_1, \sigma_2 \in C^1[t_0, \infty)$ are time-varying delayed or advanced arguments satisfying $\lim_{t\to\infty} \tau(t) = \infty$, $\lim_{t\to\infty} \sigma_1(t) = \infty$, $\lim_{t\to\infty} \sigma_2(t) = \infty$ and $\tau'(t), \sigma_1'(t), \sigma_2'(t) > 0$ on $[t_0, \infty)$. Set f^{-1} to be the inverse of f and $(f \circ g)(t) = f(g(t))$. By repeating the procedure in the proof of Theorem 2.1 and Theorem 2.2, we have the following oscillation results for Eq. (1.1) with time-varying arguments.

Theorem 2.3. *Assume that $(A_1) - (A_4)$ hold, $q_1(t) \le 0$, $q_2(t) > 0$, $\sigma_1(t) \le t$ and $\sigma_2(t) \ge \tau(t)$. If there exists a kernel function $H(t,s)$ satisfying (a) and (b) such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,r)} \int_r^t [H(t,s)e(s) + \tilde{P}_1(t,s) + \tilde{P}_2(t,s)]ds = +\infty \tag{2.14}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,r)} \int_r^t [H(t,s)e(s) - \tilde{P}_1(t,s) - \tilde{P}_2(t,s)]ds = -\infty \tag{2.15}$$

hold for some $r \ge t_0$, where

$$\tilde{P}_1(t,s) = (\alpha-1)\alpha^{\frac{\alpha}{1-\alpha}}[h_n(t,\sigma_1(s))\sigma_1'(s)]^{\frac{\alpha}{\alpha-1}}[H(t,s)|q_1(s)|]^{\frac{1}{1-\alpha}},$$

$$\begin{aligned}\tilde{P}_2(t,s) &= (1-\beta)\beta^{\frac{\beta}{1-\beta}}[h_n(t,s)p(s)]^{\frac{\beta}{\beta-1}} \\ &\quad \times[H(t,(\sigma_2^{-1}\circ\tau)(s))q_2((\sigma_2^{-1}\circ\tau)(s))((\sigma_2^{-1}\circ\tau)(s))']^{\frac{1}{1-\beta}},\end{aligned}$$

then Eq. (1.1) is oscillatory.

Theorem 2.4. *Assume that $(A_1) - (A_4)$ hold, $q_1(t) \ge 0$, $q_2(t) < 0$, $\sigma_1(t) \le \tau(t)$ and $\sigma_2(t) \ge t$. If there exists a kernel function $H(t,s)$ satisfying (a) and (b) such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,r)} \int_r^t [H(t,s)e(s) - \tilde{Q}_1(t,s) - \tilde{Q}_2(t,s)]ds = +\infty \tag{2.16}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,r)} \int_r^t [H(t,s)e(s) + \tilde{Q}_1(t,s) + \tilde{Q}_2(t,s)]ds = -\infty \tag{2.17}$$

hold for some $r \ge t_0$, where

$$\begin{aligned}\tilde{Q}_1(t,s) &= (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}[H(t,s)q_1(s)]^{\frac{1}{1-\alpha}} \\ &\quad \times[h_n(t,(\tau^{-1}\circ\sigma_1)(s))p((\tau^{-1}\circ\sigma_1)(s))((\tau^{-1}\circ\sigma_1)(s))']^{\frac{\alpha}{\alpha-1}},\end{aligned}$$

$$\tilde{Q}_2(t,s) = (\beta-1)\beta^{\frac{\beta}{1-\beta}}[h_n(t,s)]^{\frac{\beta}{\beta-1}}[H(t,\sigma_2^{-1}(s))|q_2(\sigma_2^{-1}(s))|(\sigma_2^{-1}(s))']^{\frac{1}{1-\beta}},$$

then Eq. (1.1) is oscillatory.

As an interesting application, we apply the method used in this paper to the following higher-order differential equation with delayed and advanced arguments and mixed nonlinearities:

$$x^{(n)}(t)+q_1(t)|x(t-\sigma_1)|^{\alpha-1}x(t-\sigma_1)+q_2(t)|x(t-\sigma_2)|^{\beta-1}x(t-\sigma_2)=e(t), \quad (2.18)$$

where $t \geq t_0$. We have the following oscillation results for Eq. (2.18).

Corollary 2.5. *Assume that $(A_1)-(A_4)$ hold, $q_1(t) \leq 0$, $q_2(t) > 0$, $\sigma_1 \geq 0$ and $\sigma_2 \leq 0$. If there exist a constant $\gamma > 0$ and a kernel function $H(t,s)$ satisfying (a) and (b) such that (2.1) and (2.2) hold for some $r \geq t_0$, where*

$$P_1(t,s)=(\alpha-1)\alpha^{\frac{\alpha}{1-\alpha}}[(1+\gamma)h_n(t,s-\sigma_1)]^{\frac{\alpha}{\alpha-1}}[H(t,s)|q_1(s)|]^{\frac{1}{1-\alpha}},$$

$$P_2(t,s)=(1-\beta)\beta^{\frac{\beta}{1-\beta}}[H(t,s+\sigma_2-\tau)q_2(s+\sigma_2-\tau)]^{\frac{1}{1-\beta}}[\gamma h_n(t,s)p(s)]^{\frac{\beta}{\beta-1}},$$

then Eq. (2.18) is oscillatory.

Proof. In fact, if we rewrite Eq. (2.16) into the following form

$$z^{(n)}(t)+q_1(t)|x(t-\sigma_1)|^{\alpha-1}x(t-\sigma_1)+q_2(t)|x(t-\sigma_2)|^{\beta-1}x(t-\sigma_2)=e(t),$$

where $z^{(n)}(t)=(1+\gamma)x^{(n)}(t)-\gamma x^{(n)}(t)$. Then, proceeding as in the proof of Theorem 2.1 yields the desired result. □

Based on proofs of Theorem 2.2-Theorem 2.4 and the analysis of Corollary 2.1, we have the following corollaries.

Corollary 2.6. *Assume that $(A_1)-(A_4)$ hold, $q_1(t) \geq 0$, $q_2(t) < 0$, $\sigma_1 \geq 0$ and $\sigma_2 \leq 0$. If there exist a constant $\gamma > 0$ and a kernel function $H(t,s)$ satisfying (a) and (b) such that (2.8) and (2.9) hold for some $r \geq t_0$, where*

$$Q_1(t,s)=(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}[H(t,s)q_1(s)]^{\frac{1}{1-\alpha}}[\gamma h_n(t,s+\tau-\sigma_1)]^{\frac{\alpha}{\alpha-1}},$$

$$Q_2(t,s)=(\beta-1)\beta^{\frac{\beta}{1-\beta}}[(1+\gamma)h_n(t,s)]^{\frac{\beta}{\beta-1}}[H(t,s+\sigma_2)|q_2(s+\sigma_2)|]^{\frac{1}{1-\beta}},$$

then Eq. (2.18) is oscillatory.

Corollary 2.7. *Assume that $(A_1)-(A_4)$ hold, $q_1(t) \leq 0$, $q_2(t) > 0$, $\sigma_1(t) \leq t$ and $\sigma_2(t) \geq t$. If there exist a constant $\gamma > 0$ and a kernel function $H(t,s)$ satisfying (a) and (b) such that (2.14) and (2.15) hold for some $r \geq t_0$, where*

$$\tilde{P}_1(t,s)=(\alpha-1)\alpha^{\frac{\alpha}{1-\alpha}}[(1+\gamma)h_n(t,\sigma_1(s))\sigma_1'(s)]^{\frac{\alpha}{\alpha-1}}[H(t,s)|q_1(s)|]^{\frac{1}{1-\alpha}},$$

$$\begin{aligned}\tilde{P}_2(t,s) &= (1-\beta)\beta^{\frac{\beta}{1-\beta}}[\gamma h_n(t,s)p(s)]^{\frac{\beta}{\beta-1}} \\ &\quad \times[H(t,(\sigma_2^{-1}\circ\tau)(s))q_2((\sigma_2^{-1}\circ\tau)(s))((\sigma_2^{-1}\circ\tau)(s))']^{\frac{1}{1-\beta}},\end{aligned}$$

then Eq. (2.18) is oscillatory.

Corollary 2.8. *Assume that $(A_1)-(A_4)$ hold, $q_1(t) \geq 0$, $q_2(t) < 0$, $\sigma_1(t) \leq t$ and $\sigma_2(t) \geq t$. If there exist a constant $\gamma > 0$ and a kernel function $H(t,s)$ satisfying (a) and (b) such that (2.16) and (2.17) hold for some $r \geq t_0$, where*

$$\begin{aligned}\tilde{Q}_1(t,s) &= (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}[H(t,s)q_1(s)]^{\frac{1}{1-\alpha}} \\ &\quad \times[\gamma h_n(t,(\tau^{-1}\circ\sigma_1)(s))((\tau^{-1}\circ\sigma_1)(s))']^{\frac{\alpha}{\alpha-1}},\end{aligned}$$

$$\tilde{Q}_2(t,s) = (\beta-1)\beta^{\frac{\beta}{1-\beta}}[(1+\gamma)h_n(t,s)]^{\frac{\beta}{\beta-1}} \times [H(t,\sigma_2^{-1}(s))|q_2(\sigma_2^{-1}(s))|(\sigma_2^{-1}(s))']^{\frac{1}{1-\beta}},$$

then Eq. (2.18) is oscillatory.

3. Examples

The following two examples are given to illustrate the main results.

Example 3.1. Consider the following second-order neutral differential equation

$$z''(t) - t^{\lambda_1}|x(t-\sigma_1)|^{-1/2}x(t-\sigma_1) + (t+2)^{\lambda_2}|x(t+1)|x(t+1) = t^{\gamma}\cos t, \quad (3.1)$$

where $t>0$, $z(t) = x(t) - t^{\rho}x(t-1)$, $\sigma_1 > 0$, $\lambda_1 > -1/2$, $\lambda_2 - 2\rho < 1$ and $\gamma > 0$ are constants. Let $H(t,s) = (t-s)^2$. A straightforward computation yields

$$P_1(t,s) = -\frac{1}{8}(t-s)^4 s^{2\lambda_1}, \quad P_2(t,s) = -(t+2-s)^{-2}s^{2\rho-\lambda_2}.$$

Therefore,

$$\int_0^t P_1(t,s)ds = -\frac{1}{8}t^{5+2\lambda_1}B(2\lambda_1+1,5),$$

where B is the Beta function, $B(2\lambda_1+1,5) > 0$ due to $\lambda_1 > -1/2$, and

$$\int_0^t P_2(t,s)ds \geq -\frac{1}{4}\int_0^t s^{2\lambda-\lambda_2}ds = -\frac{t^{1+2\rho-\lambda_2}}{4(1+2\rho-\lambda_2)}.$$

On the other hand,

$$\int_0^t (t-s)^2 s^{\gamma}\cos s ds = t^{3+\gamma}\int_0^1 (1-u)^2 u^{\gamma}\cos(tu)du = t^{3+\gamma}I_{\gamma}(t), \quad (3.2)$$

where $I_{\gamma}(t)$ has the asymptotic formula

$$I_{\gamma}(t) = -\Gamma(3)t^{-3}\sin t + o(t^{-3}) \quad (3.3)$$

as $t \to \infty$ (see [7] pp. 49-50). By Theorem 2.1, Eq. (3.1) is oscillatory if $\gamma > \max\{5+2\lambda_1, 1+2\lambda-\lambda_2\}$.

Example 3.2. Consider the following second-order neutral differential equation

$$z''(t) + t^{\lambda_1}|x(t-1)|^{-1/2}x(t-1) - (t+1)^{\lambda_2}|x(t+1)|x(t+1) = t^{\gamma}\cos t, \quad (3.4)$$

where $t>0$, $z(t) = x(t) - t^{\rho}x(t-1)$, $\lambda_1 > (\rho-1)/2$ and $\lambda_2 < 1$ are constants. If we choose $H(t,s) = (t-s)^2$, then we have

$$Q_1(t,s) = \frac{1}{2}(t-s)^4 s^{2\lambda_1-\rho}, \quad Q_2(t,s) = (t+1-s)^{-1}s^{-\lambda_2}.$$

Thus,

$$\int_0^t Q_1(t,s)ds = \frac{1}{2}t^{5+2\lambda_1-\rho}B(2\lambda_1-\rho+1,5),$$

where $B(2\lambda_1-\rho+1,5) > 0$ due to $\lambda_1 > (\rho-1)/2$, and

$$\int_0^t Q_2(t,s)ds \leq \int_0^t s^{-\lambda_2} = \frac{t^{1-\lambda_2}}{1-\lambda_2}.$$

By (3.2), (3.3) and Theorem 2.2, we have that Eq. (3.4) is oscillatory if $\gamma > \max\{5 + 2\lambda_1 - \lambda, 1 - \lambda_2\}$.

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