J. Appl. Math. & Informatics Vol. 31(2013), No. 1 - 2, pp. 189 - 197
Website: http://www.kcam.biz

SUFFICIENCY IN NONSMOOTH MULTIOBJECTIVE FRACTIONAL PROGRAMMING

SARITA SHARMA* AND I. AHMAD

ABSTRACT. In this paper, Karush-Kuhn-Tucker type sufficient optimality conditions are obtained for a feasible point of a nonsmooth multiobjective fractional programming problem to be an efficient or properly efficient by using generalized (F, ρ, σ)-type I functions.

AMS Mathematics Subject Classification : 90C29, 90C30, 90C32, 90C46.
Key words and phrases : Nonsmooth multiobjective optimization, generalized type I functions, sufficient optimality conditions.

1. Introduction

Consider the following nonsmooth multiobjective programming problem:

$$(NP) \text{ Minimize } f(x) = [f_1(x), f_2(x), \ldots, f_k(x)]$$
$$\text{subject to } x \in X = \{x \in S : g(x) \leqq 0\},$$

where $S \subseteq R^n$, the functions $f = (f_1, f_2, \ldots, f_k) : S \to R^k$ and $g = (g_1, g_2, \ldots, g_m) : S \to R^m$ are locally Lipschitz functions.

Zhao [16] obtained Karush-Kuhn-Tucker type sufficient conditions and duality results for a nonsmooth scalar optimization assuming Clarke [4] generalized subgradients under type I functions. Kuk and Tanino [8] considered a nonsmooth multiobjective program (NP) and established sufficient optimality conditions and duality theorems involving generalized type I vector-valued functions. Gulati and Agarwal [7] defined generalized (F, α, ρ, d)-type I functions for (NP) and obtained sufficiency and duality results. In [12], Nobakhtian used the concept of infine functions to establish optimality conditions and duality results for (NP). Ahmad and Sharma [1] introduced a new class of (F, ρ, σ)-type I functions for a nonsmooth multiobjective program and derived optimality conditions and duality theorems. Recently, Nobakhtian [15] introduced generalized (F, ρ)-convexity for (NP) and proved duality results for a mixed type dual.

Received August 26, 2011. Revised February 23, 2012. Accepted April 9, 2012. *Corresponding author.

In this paper, we consider the following multiobjective fractional programming problem:

$$\text{(FP) Minimize} \left[\frac{f_1(x)}{h_1(x)}, \frac{f_2(x)}{h_2(x)}, \dots, \frac{f_k(x)}{h_k(x)}\right]$$
$$\text{subject to } x \in X = \{x \in S : g(x) \leqq 0\},$$

where the functions $f = (f_1, f_2, \dots, f_k) : S \to R^k, h = (h_1, h_2, \dots, h_k) : S \to R^k$ and $g = (g_1, g_2, \dots, g_m) : S \to R^m$ are locally Lipschitz on S. Let $f_i(x) \geqq 0$ and $h_i(x) > 0$ for each $i = 1, 2, \dots, k$ and $x \in S$.
We derive sufficient optimality conditions for (FP) by using the concept of generalized (F, ρ, σ)-type I functions. Our results improve the results appeared in [9, 10, 11, 13, 14, 15].

2. Definitions and preliminaries

The following conventions of vectors in R^n will be followed throughout this paper: $x \geqq y \Leftrightarrow x_p \geqq y_p$, $p = 1, 2, \dots, n$; $x \geq y \Leftrightarrow x \geqq y$, $x \neq y$; $x > y \Leftrightarrow x_p > y_p$, $p = 1, 2, \dots, n$. Let $K = \{1, 2, \dots, k\}$, $M = \{1, 2, \dots, m\}$ be the index sets.
A function $f : R^n \longrightarrow R$ is said to be locally Lipschitz at $\bar{x} \in R^n$, if there exist scalars $\delta > 0$ and $\epsilon > 0$ such that

$$| f(x^1) - f(x^2) | \leqq \delta \| x^1 - x^2 \|, \text{ for all } x^1, x^2 \in \bar{x} + \epsilon B,$$

where $\bar{x} + \epsilon B$ is the open ball of radius ϵ about $\bar{x}$.
The generalized directional derivative [4] of a locally Lipschitz function f at x in the direction v, denoted by $f^o(x; v)$, is as follows:

$$f^\circ(x; v) = \lim_{y \to x} \sup_{t \downarrow 0} \frac{f(y + tv) - f(y)}{t}.$$

The generalized gradient [4] of f at x is denoted by

$$\partial f(x) = \left\{\xi \in R^n : f^\circ(x; v) \geqq \xi^t v, \text{ for all } v \in R^n\right\}.$$

Now consider the following multiobjective optimization problem:

$$\text{(MP) Minimize } f(x) = [f_1(x), f_2(x), \dots, f_k(x)]$$
$$\text{subject to } x \in X.$$

The following definitions are from Geoffrion [6]:

Definition 1. A point $\bar{x} \in X$ is said to be an efficient solution of (MP), if there exists no $x \in X$ such that $f(x) \leq f(\bar{x})$.

Definition 2. A point $\bar{x} \in X$ is said to be a weakly efficient solution of (MP), if there exists no $x \in X$ such that $f(x) < f(\bar{x})$.

Definition 3. An efficient solution $\bar{x}$ of (MP) is said to be properly efficient, if there exists a scalar $N > 0$ such that for each $i, f_i(x) < f_i(\bar{x})$ and $x \in X$ imply that

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leqq N$$

for at least one j satisfying $f_j(\bar{x}) < f_j(x)$.

Definition 4. A functional $F : S \times S \times R^n \longrightarrow R$ is sublinear in its third argument, if for all $x, \bar{x} \in S$,

(i) $F(x, \bar{x}; a + b) \leqq F(x, \bar{x}; a) + F(x, \bar{x}; b),$ for all $a, b \in R^n$,

(ii) $F(x, \bar{x}; \alpha a) = \alpha F(x, \bar{x}; a),$ for all $\alpha \in R, \alpha \geqq 0$ and $a \in R^n$.

We recall the following generalized $(F, \rho, \sigma)-$type I functions [1]. Let $f : S \to R^k$ and $g : S \to R^m$ be locally Lipschitz at a given point $\bar{x} \in S$, $\rho = (\rho_1, \rho_2, \ldots, \rho_k) \in R^k, \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in R^m$, and $d(\cdot, \cdot) : S \times S \to R$. Also, for $\bar{x} \in X, J(\bar{x}) = \{j \in M : g_j(\bar{x}) = 0\}$ and g_J will denote the vector of active constraints at $\bar{x}$.

Definition 5. For each $i \in K$ and $j \in M$, (f_i, g_j) is said to be (F, ρ, σ)-type I at $\bar{x} \in S$, if for all $x \in X$, we have

$$f_i(x) - f_i(\bar{x}) \geqq F(x, \bar{x}; \xi_i) + \rho_i d^2(x, \bar{x}), \text{ for all } \xi_i \in \partial f_i(\bar{x}), \tag{2.1}$$

$$-g_j(\bar{x}) \geqq F(x, \bar{x}; \eta_j) + \sigma_j d^2(x, \bar{x}), \text{ for all } \eta_j \in \partial g_j(\bar{x}).$$

If (2.1) is a strict inequality, then we say that (f_i, g_j) is (F, ρ, σ)-semistrictly-type I at $\bar{x}$.

Definition 6. For each $i \in K$ and $j \in M$, (f_i, g_j) is said to be (F, ρ, σ)-prestrictquasi-strictlypseudo-type I at $\bar{x} \in S$, if for all $x \in X$, we have

$$f_i(x) < f_i(\bar{x}) \Longrightarrow F(x, \bar{x}; \xi_i) \leqq -\rho_i d^2(x, \bar{x}), \text{ for all } \xi_i \in \partial f_i(\bar{x}),$$

$$F(x, \bar{x}; \eta_j) \geqq -\sigma_j d^2(x, \bar{x}) \Longrightarrow -g_j(\bar{x}) > 0, \text{ for all } \eta_j \in \partial g_j(\bar{x}).$$

Definition 7. For each $i \in K$ and $j \in M$, (f_i, g_j) is said to be (F, ρ, σ)-pseudoquasi-type I at $\bar{x} \in S$, if for all $x \in X$, we have

$$F(x, \bar{x}; \xi_i) \geqq -\rho_i d^2(x, \bar{x}) \Longrightarrow f_i(x) \geqq f_i(\bar{x}), \text{ for all } \xi_i \in \partial f_i(\bar{x}), \tag{2.2}$$

$$-g_j(\bar{x}) \leqq 0 \Longrightarrow F(x, \bar{x}; \eta_j) \leqq -\sigma_j d^2(x, \bar{x}), \text{ for all } \eta_j \in \partial g_j(\bar{x}).$$

If (2.2) is satisfied as

$$F(x, \bar{x}; \xi_i) \geqq -\rho_i d^2(x, \bar{x}) \Longrightarrow f_i(x) > f_i(\bar{x}), \text{ for all } \xi_i \in \partial f_i(\bar{x}),$$

then we say that (f_i, g_j) is (F, ρ, σ)-strictly-pseudoquasi-type I at $\bar{x}$.

In order to derive sufficient optimality conditions, we will invoke the following results. We use Dinkelbach-type [5] approach to get the following auxiliary parametric problem:

$$(\text{FP})^{\lambda} \text{ Minimize } f(x) = [f_1(x) - \bar{\lambda}_1 h_1(x), f_2(x) - \bar{\lambda}_2 h_2(x), \ldots, f_k(x) - \bar{\lambda}_k h_k(x)]$$

$$\text{subject to } x \in X.$$

where $\bar{\lambda}_i$, $i \in K$, are parameters. The problem is equivalent to (FP) in the sense that for particular choices of $\bar{\lambda}_i$, $i \in K$, the two problems have the same set of efficient solutions.

In relation to $(\mathrm{FP})^{\lambda}$, we consider the following scalar minimization problem on the lines of Geoffrion [6]:

$$(\mathrm{FP})^{\frac{\mu}{\lambda}} \text{ Minimize } \sum_{i\in K} \mu_i(f_i(x) - \bar{\lambda}_i h_i(x))$$
$$\text{subject to } x \in X.$$

Lemma 1 ([6]). *If $\bar{x}$ is an optimal solution of $(\mathrm{FP})^{\frac{\mu}{\lambda}}$, for some $\mu \in R^k$, with strictly positive components, where $\bar{\lambda}_i = \dfrac{f_i(\bar{x})}{h_i(\bar{x})}$, $i \in K$, then $\bar{x}$ is a properly efficient solution of (FP).*

Lemma 2 ([3]). *$\bar{x}$ is an efficient solution of $(\mathrm{FP})^{\lambda}$, if and only if $\bar{x}$ solves $(FP)_r$, $r \in K$:*

$$(\mathrm{FP})_{\mathrm{r}} \; \textit{Minimize } f_r(x) - \bar{\lambda}_r h_r(x)$$
$$\textit{subject to } f_i(x) - \bar{\lambda}_i h_i(x) \leqq f_i(\bar{x}) - \bar{\lambda}_i h_i(\bar{x}), \text{ for all } i \neq r,$$
$$g(x) \leqq 0, \; x \in S.$$

Proposition 1 (*Karush-Kuhn-Tucker type necessary conditions*) ([9]). *If $\bar{x}$ is an efficient solution of (FP), then there exist $\bar{\mu} \in R^k$ and $\bar{\nu} \in R^m$ such that*

$$0 \in \sum_{i\in K} \bar{\mu}_i[\partial f_i(\bar{x}) - \bar{\lambda}_i \partial h_i(\bar{x})] + \sum_{j\in J(\bar{x})} \bar{\nu}_j \partial g_j(\bar{x}),$$
$$\bar{\nu}_j g_j(\bar{x}) = 0, \; j \in M,$$
$$\bar{\mu} > 0, \; \bar{\nu} \geqq 0,$$

where $\bar{\lambda}_i = \dfrac{f_i(\bar{x})}{h_i(\bar{x})}$, $i \in K$.

3. Sufficiency

In this section, we obtain sufficient conditions for a feasible point of (FP) to be efficient and properly efficient.

Theorem 1. *Let $\bar{x} \in X$, and let there exist scalars $\bar{\mu}_i > 0, i \in K$ and $\bar{\nu}_j \geqq 0$, $j \in J(\bar{x})$ such that*

$$0 \in \sum_{i\in K} \bar{\mu}_i[\partial f_i(\bar{x}) - \bar{\lambda}_i \partial h_i(\bar{x})] + \sum_{j\in J(\bar{x})} \bar{\nu}_j \partial g_j(\bar{x}), \tag{3.1}$$
$$\bar{\nu}_j g_j(\bar{x}) = 0, \; j \in M, \tag{3.2}$$

where $\bar{\lambda}_i = \dfrac{f_i(\bar{x})}{h_i(\bar{x})}$, $i \in K$.

If

(*i*) $[f_i - \bar{\lambda}_i h_i, \, g_j]$, $i \in K$, $j \in J(\bar{x})$ *is (F, ρ, σ)-type I at $\bar{x}$; and*

(*ii*) $\sum_{i\in K} \bar{\mu}_i \rho_i + \sum_{j\in J(\bar{x})} \bar{\nu}_j \sigma_j \geqq 0$,

then $\bar{x}$ is a properly efficient solution of (FP).

Proof. By (3.1) we obtain that there exist $\xi_i \in \partial f_i(\bar{x}), \zeta_i \in \partial h_i(\bar{x})$, $i \in K$ and $\eta_j \in \partial g_j(\bar{x}), j \in J(\bar{x})$ satisfying

$$\sum_{i\in K} \bar{\mu}_i[\xi_i - \bar{\lambda}_i\zeta_i] + \sum_{j\in J(\bar{x})} \bar{\nu}_j\eta_j = 0. \tag{3.3}$$

From hypothesis (i), we get

$$\begin{aligned} &[f_i(x) - \bar{\lambda}_i h_i(x)] - [f_i(\bar{x}) - \bar{\lambda}_i h_i(\bar{x})] \\ &\geq F(x, \bar{x}; \xi_i - \bar{\lambda}_i\zeta_i) + \rho_i d^2(x, \bar{x}) \text{ for all } \xi_i \in \partial f_i(\bar{x}),\, \zeta_i \in \partial h_i(\bar{x}), \end{aligned} \tag{3.4}$$

$$-g_j(\bar{x}) \geqq F(x, \bar{x}; \eta_j) + \sigma_j d^2(x, \bar{x}) \text{ for all } \eta_j \in \partial g_j(\bar{x}). \tag{3.5}$$

On multiplying (3.4) by $\bar{\mu}_i > 0$, $i \in K$, and (3.5) by $\bar{\nu}_j \geqq 0$, $j \in J(\bar{x})$, using the sublinearity of F; and taking summation over i and j, respectively, we get

$$\begin{aligned} &\sum_{i\in K} \bar{\mu}_i[f_i(x) - \bar{\lambda}_i h_i(x)] - \sum_{i\in K} \bar{\mu}_i[f_i(\bar{x}) - \bar{\lambda}_i h_i(\bar{x})] \\ &\geq F(x, \bar{x}; \sum_{i\in K} \bar{\mu}_i(\xi_i - \bar{\lambda}_i\zeta_i)) + \sum_{i\in K} \bar{\mu}_i\rho_i d^2(x, \bar{x}) \text{ for all } \xi_i \in \partial f_i(\bar{x}),\, \zeta_i \in \partial h_i(\bar{x}), \end{aligned}$$

$$0 \geqq -\sum_{j\in J(\bar{x})} \bar{\nu}_j g_j(\bar{x}) \geqq F(x, \bar{x}; \sum_{j\in J(\bar{x})} \bar{\nu}_j\eta_j) + \sum_{j\in J(\bar{x})} \bar{\nu}_j\sigma_j d^2(x, \bar{x}) \text{ for all } \eta_j \in \partial g_j(\bar{x}).$$

Combining these inequalities, and using the sublinearity of F, we obtain

$$\begin{aligned} &\sum_{i\in K} \bar{\mu}_i[f_i(x) - \bar{\lambda}_i h_i(x)] - \sum_{i\in K} \bar{\mu}_i[f_i(\bar{x}) - \bar{\lambda}_i h_i(\bar{x})] \\ &\geq F(x, \bar{x}; \sum_{i\in K} \bar{\mu}_i(\xi_i - \bar{\lambda}_i\zeta_i) + \sum_{j\in J(\bar{x})} \bar{\nu}_j\eta_j) + (\sum_{i\in K} \bar{\mu}_i\rho_i + \sum_{j\in J(\bar{x})} \bar{\nu}_j\sigma_j) d^2(x, \bar{x}) \\ &\geq F(x, \bar{x}; \sum_{i\in K} \bar{\mu}_i(\xi_i - \bar{\lambda}_i\zeta_i) + \sum_{j\in J(\bar{x})} \bar{\nu}_j\eta_j), \text{ (by hyp. (ii))}, \end{aligned}$$

which on using (3.3) with the sublinearity of F, yields

$$\sum_{i\in K} \bar{\mu}_i[f_i(x) - \bar{\lambda}_i h_i(x)] - \sum_{i\in K} \bar{\mu}_i[f_i(\bar{x}) - \bar{\lambda}_i h_i(\bar{x})] \geqq 0,$$

or

$$\sum_{i\in K} \bar{\mu}_i[f_i(x) - \bar{\lambda}_i h_i(x)] \geqq \sum_{i\in K} \bar{\mu}_i[f_i(\bar{x}) - \bar{\lambda}_i h_i(\bar{x})]. \tag{3.6}$$

The inequality (3.6) shows that $\bar{x}$ is an optimal solution of $(\mathrm{FP})^{\frac{\mu}{\lambda}}$. Hence by Lemma 1, we can conclude that $\bar{x}$ is a properly efficient solution of (FP). □

Theorem 2. *Let $\bar{x} \in X$, and let there exist scalars $\bar{\mu}_i > 0, i \in K$ and $\bar{\nu}_j \geqq 0$, $j \in J(\bar{x})$ satisfying (3.1) and (3.2). If*

(i) $[\sum_{i\in K} \bar{\mu}_i(f_i - \bar{\lambda}_i h_i),\, \bar{\nu}_J g_J]$ is (F, ρ_1, σ_1)-pseudoquasi-type I at $\bar{x}$; and

(ii) $\rho_1 + \sigma_1 \geqq 0$,

then $\bar{x}$ is a properly efficient solution of (FP).

Proof. From (3.2), we get

$$\bar{\nu}_j g_j(\bar{x}) = 0,\ j \in J(\bar{x}), \quad \text{or}\ -\sum_{j\in J(\bar{x})} \bar{\nu}_j g_j(\bar{x}) \leqq 0.$$

Then the second part of hypothesis (i) implies

$$F(x,\bar{x}; \sum_{j\in J(\bar{x})} \bar{\nu}_j \eta_j) + \sigma_1 d^2(x,\bar{x}) \leqq 0,$$

which in view of (3.3), hypothesis (ii), and the sublinearity of F, gives

$$F(x,\bar{x}; \sum_{i\in K} \bar{\mu}_i(\xi_i - \bar{\lambda}_i \zeta_i) + \rho_1 d^2(x,\bar{x}) \geqq 0.$$

The above inequality along with the first part of assumption (i) yields

$$\sum_{i\in K} \bar{\mu}_i[f_i(x) - \bar{\lambda}_i h_i(x)] \geqq \sum_{i\in K} \bar{\mu}_i[f_i(\bar{x}) - \bar{\lambda}_i h_i(\bar{x})],$$

which is precisely (3.6). Hence, $\bar{x}$ is a properly efficient solution of (FP). □

Theorem 3. *Let $\bar{x} \in X$, and let there exist scalars $\bar{\mu}_i > 0, i \in K$ and $\bar{\nu}_j \geqq 0$, $j \in J(\bar{x})$ satisfying (3.1) and (3.2). If*

(i) $[\sum_{i\in K} \bar{\mu}_i(f_i - \bar{\lambda}_i h_i),\ \bar{\nu}_J g_J]$ *is (F, ρ_2, σ_2)-prestrictquasi-strictlypseudo-type I at $\bar{x}$; and*

(ii) $\rho_2 + \sigma_2 \geqq 0$,

then $\bar{x}$ is a properly efficient solution of (FP).

Proof. The proof follows on the similar lines of Theorem 2. □

Remark 1. If we replace $\bar{\mu}_i > 0, i \in K$ by $\bar{\mu}_i \geqq 0, i \in K$, $\sum_{i\in K} \bar{\mu}_i = 1$ in the above theorems and other conditions are imposed on $[\sum_{i\in K} \bar{\mu}_i(f_i - \bar{\lambda}_i h_i),\ \bar{\nu}_J g_J]$, we get stronger conclusion that $\bar{x}$ is an efficient solution of (FP). The results are shown below:

Theorem 4. *Let $\bar{x} \in X$, and let there exist scalars $\bar{\mu}_i \geqq 0, i \in K$, $\sum_{i\in K} \bar{\mu}_i = 1$ and $\bar{\nu}_j \geqq 0, j \in J(\bar{x})$ satisfying (3.1) and (3.2). If*

(i) $[\sum_{i\in K} \bar{\mu}_i(f_i - \bar{\lambda}_i h_i),\ \bar{\nu}_J g_J]$ *is (F, ρ_3, σ_3)-semistrictly-type I at $\bar{x}$; and*

(ii) $\rho_3 + \sigma_3 \geqq 0$,

then $\bar{x}$ is an efficient solution of (FP).

Proof. Suppose to the contrary that $\bar{x}$ is not an efficient solution of (FP), then there exists $x \in X$ such that

$$\left[\frac{f_1(x)}{h_1(x)}, \frac{f_2(x)}{h_2(x)}, \ldots, \frac{f_k(x)}{h_k(x)}\right] \leq \left[\frac{f_1(\bar{x})}{h_1(\bar{x})}, \frac{f_2(\bar{x})}{h_2(\bar{x})}, \ldots, \frac{f_k(\bar{x})}{h_k(\bar{x})}\right]. \tag{3.7}$$

By hypothesis (i), we have

$$\sum_{i\in K}\bar{\mu}_i[f_i(x)-\bar{\lambda}_ih_i(x)]-\sum_{i\in K}\bar{\mu}_i[f_i(\bar{x})-\bar{\lambda}_ih_i(\bar{x})]$$
$$> F(x,\bar{x};\sum_{i\in K}\bar{\mu}_i(\xi_i-\bar{\lambda}_i\zeta_i))+\rho_3d^2(x,\bar{x}) \text{ for all } \xi_i\in\partial f_i(\bar{x}),\ \zeta_i\in\partial h_i(\bar{x}),$$

$$0\geqq-\sum_{j\in J(\bar{x})}\bar{\nu}_jg_j(\bar{x})\geqq F(x,\bar{x};\sum_{j\in J(\bar{x})}\bar{\nu}_j\eta_j)+\sigma_3d^2(x,\bar{x}) \text{ for all } \eta_j\in\partial g_j(\bar{x}).$$

Now following the proof of Theorem 1, we reach at

$$\sum_{i\in K}\bar{\mu}_i[f_i(x)-\bar{\lambda}_ih_i(x)]>\sum_{i\in K}\bar{\mu}_i[f_i(\bar{x})-\bar{\lambda}_ih_i(\bar{x})].$$

As $\bar{\lambda}_i=\dfrac{f_i(\bar{x})}{h_i(\bar{x})}$, $i\in K$, it follows that

$$\sum_{i\in K}\bar{\mu}_i[f_i(x)-\bar{\lambda}_ih_i(x)]>0.$$

Since $\bar{\mu}_i\geqq 0$, $i\in K$, $\sum_{i\in K}\bar{\mu}_i=1$, we get

$$(f_1(x)-\bar{\lambda}_1h_1(x),f_2(x)-\bar{\lambda}_2h_2(x),\ldots,f_k(x)-\bar{\lambda}_kh_k(x))>0,$$

which in turn yields

$$\left[\frac{f_1(x)}{h_1(x)},\frac{f_2(x)}{h_2(x)},\ldots,\frac{f_k(x)}{h_k(x)}\right]>(\bar{\lambda}_1,\bar{\lambda}_2,\ldots,\bar{\lambda}_k),$$

or

$$\left[\frac{f_1(x)}{h_1(x)},\frac{f_2(x)}{h_2(x)},\ldots,\frac{f_k(x)}{h_k(x)}\right]>\left[\frac{f_1(\bar{x})}{h_1(\bar{x})},\frac{f_2(\bar{x})}{h_2(\bar{x})},\ldots,\frac{f_k(\bar{x})}{h_k(\bar{x})}\right],$$

which is a contradiction to (3.7). Hence $\bar{x}$ is an efficient solution of (FP). □

The following theorem can be proved along the lines of Theorem 4.

Theorem 5. *Let $\bar{x}\in X$, and let there exist scalars $\bar{\mu}_i\geqq 0, i\in K$, $\sum_{i\in K}\bar{\mu}_i=1$ and $\bar{\nu}_j\geqq 0, j\in J(\bar{x})$ satisfying (1) and (2). If*

(i) $[\sum_{i\in K}\bar{\mu}_i(f_i-\bar{\lambda}_ih_i),\ \bar{\nu}_Jg_J]$ is (F,ρ_4,σ_4)-strictlypseudoquasi-type I at $\bar{x}$; and

(ii) $\rho_4+\sigma_4\geqq 0$,

then $\bar{x}$ is an efficient solution of (FP).

Remark 2. It may be noted that Theorems 4 and 5 also hold for weakly efficient solution of (FP).

References

1. I. Ahmad and S Sharma, *Optimality conditions and duality in nonsmooth multiobjective optimization*, J. Nonlinear Convex Anal. **8** (2007), 417-430 .
2. C.R. Bector, S Chandra and I. Husain, *Optimality conditions and subdifferentiable multiobjective fractional programming*, J. Optim. Theory Appl. **79** (1993), 105-125 .
3. V. Chankong and Haimes, *Multiobjective Decision Making: Theory and Methods*, North Holland, New York (1983).
4. F.H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley and Sons, New York (1983).
5. W. Dinkelbach, *On nonlinear fractional programming*, Manag. Sci. **13** (1967), 492-498 .
6. A.M. Geoffrion, *Proper efficiency and theory of vector maximization*, J. Math. Anal. Appl. **22**, 618-630 (1968).
7. T.R. Gulati and D. Agarwal, *Sufficiency and duality in nonsmooth multiobjective optimization involving generalized* (F, α, ρ, d)*-type I functions*, Comput. Math. Appl. **52** (2006), 81-94 .
8. H. Kuk and T. Tanino, *Optimality and duality in nonsmooth multiobjective optimization involving generalized type I functions*, Comput. Math. Appl. **45** (2003), 1497-1506 .
9. H. Kuk, G.M. Lee and T. Tanino, *Optimality and duality for nonsmooth multiobjective fractional with generalized invexity*, J. Math. Anal. Appl. **262** (2001), 365-375 .
10. J.C. Liu, *Optimality and duality for multiobjective fractional programming involving nonsmooth* (F, ρ)*-convex functions*, Optimization **36**, 333-346 .
11. J.C. Liu, *Optimality and duality for multiobjective fractional programming involving nonsmooth pseudoinvex functions*, Optimization **37** (1996), 27-39.
12. S. Nobakhtian, *Infine functions and Nonsmooth multiobjective optimization problems*, Comp. Math. Appl. **51** (2006), 1385-1394 .
13. S. Nobakhtian, *Sufficiency in nonsmooth multiobjective programming involving generalized* (F, ρ) *-convexity*, J. Optim. Theory Appl. **130** (2006), 361-367 .
14. S. Nobakhtian, *Optimality and duality for nonsmooth multiobjective fractional programming with mixed constraints*, J. Global Optim. **41** (2008), 103-115 .
15. S. Nobakhtian, *Generalized* (F, ρ) *-convexity and duality in nonsmooth problems of multiobjective optimization*, J. Optim. Theory Appl. **136** (2008), 61-68 .
16. F. Zhao, *On sufficiency of the Kuhn-Tucker conditions in nondifferentiable programming*, Bull. Austral. Math. Soc. **46** (1992), 385-389 .

Sarita Sharma received M.Sc. from Meerut University, India and Ph.D at Aligarh Muslim University, India. She is a Post Doctoral fellow in Department of Mathematics, Indian Institute of Technology, Roorkee, India. She has published a number of papers in the area of mathematical programming.

Department of Mathematics, Indian Institute of Technology, Roorkee-247 667, India.
e-mail: ssharma05@hotmail.com

I. Ahmad received M.Sc., M.Phil and Ph. D from Indian Institute of Technology, Roorkee, India. He is an Associate Professor at Aligarh Muslim University, India. Presently, he is on leave from Aligarh Muslim University and working as an Associate Professor at King Fahd University of Petroleum and Minerals, Saudi Arabia. His major areas of research interests are Mathematical Programming and Manifolds inclding generalizations of convexity, optimality crieteria, duality theory, geodesic convexity and continous-time programming. He is author and co-author of seventy research papers on previous mentioned field, published in journals of international repute.

Department of Mathematics, King Fahd University of Petroleum and Minerals, Saudi Arabia.

Permanent Address: Department of Mathematics, Aligarh Muslim University, Aligarh-202 002, India.
e-mails: drizhar@kfupm.edu.sa, izharmaths@hotmail.com