J. Appl. Math. & Informatics Vol. 31(2013), No. 1 - 2, pp. 131 - 145
Website: http://www.kcam.biz

AN INITIAL VALUE TECHNIQUE FOR SINGULARLY PERTURBED DIFFERENTIAL-DIFFERENCE EQUATIONS WITH A SMALL NEGATIVE SHIFT†

R. NAGESHWAR RAO AND P. PRAMOD CHAKRAVARTHY*

Abstract. In this paper, we present an initial value technique for solving singularly perturbed differential difference equations with a boundary layer at one end point. Taylor's series is used to tackle the terms containing shift provided the shift is of small order of singular perturbation parameter and obtained a singularly perturbed boundary value problem. This singularly perturbed boundary value problem is replaced by a pair of initial value problems. Classical fourth order Runge–Kutta method is used to solve these initial value problems. The effect of small shift on the boundary layer solution in both the cases, i.e., the boundary layer on the left side as well as the right side is discussed by considering numerical experiments. Several numerical examples are solved to demonstate the applicability of the method.

AMS Mathematics Subject Classification : 65L11.
Key words and phrases : Initial value technique, singular perturbation, differential–difference equation, boundary layer.

1. Introduction

The numerical treatment of singular perturbations is far from the trivial because of the boundary layer behavior. In recent decades this is a field of increasing interest to applied mathematicians and numerical analysts in view of the challenges the problems there in pose to the researchers. For a detailed theory and analytical discussion on singular perturbation problems one may refer to the books and high level monographs: O' Malley [13], Nayfeh [12], Kevorkian and Cole [11], Bender and Orszag [10]. A Singularly perturbed delay differential equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving atleast one delay term. In 1979, the Japanese physicist Kensuke Ikeda considered a nonlinear absorbing medium

Received August 26, 2011. Revised February 28, 2012. Accepted April 4, 2012. *Corresponding author. †The authors wish to thank the Department of Science & Technology, Government of India, for their financial support under the project No. SR/S4/MS: 598/09.

containing two-level atoms placed in a ring cavity and subject to a constant input of light. If the total length of the cavity is sufficiently large, the optical system undergoes a time delayed feedback that destabilizes its steady state output. The phenomena has been formulated as a differential difference equation which was known as Ikeda DDE. In Besancon (France), work has been done on a delayed optical system where the dynamical variable is the wavelength. An improved device using a tunable DBR laser was then realized. This experience then led to the development of a system based on coherence modulation. The dynamical variable is the optical-path difference in a coherent modulator driven electrically by a nonlinear delayed feedback loop. The system is realized from an MZ coherence modulator powered by a short coherence source and driven by a nonlinear feedback loop that contains a second MZ interferometer and a delay line. In dimensionless variables, the response of the system is described by a differential difference equation.

Delay differential equations appear in fluid mechanics when some memory effects need to be taken into account. For example, in the case of vertical water fountains exhibiting oscillatory motion of their tips, the pressure at the orifice of the jet is related to its instantaneous height and its variation oscillates with a dominant period. If the fountain is oriented slightly away from the vertical, the back flow is no longer possible and the jet describes a parabola with a fixed maximum elevation. The gravity-induced back flow is thus essential for the onset of the oscillatory behavior. The oscillations were found to be the result of the interplay between linear growth and a delayed nonlinear saturation and in turn the mathematical model describing the phenomena is a differential difference equation where the delay is the transit time through the recirculation loop. For detailed discussion one may refer Thomas Erneux [14].

In recent years, there has been a growing interest in the numerical treatment of differential difference equations. This is also due to the occurrence of these types of differential equations in various fields such as epidemics, population dynamics [9], where the small delay play an important role in the modeling of the real life phenomena. For example, in depolarization in the Stein's model [7] which is a continuous time, continuous state space Markov process whose sample paths have discontinuities of the first kind [8]. The time between nerve impulses is the time of first passage to a level at or above a threshold value, determining the moments of this random variable involves differential difference equations. In [5], Lange and Miura considered the problem of determining the expected time for the generation of action potentials in nerve cells by random synaptic inputs in the dendrites which was formulated as a general boundary value problem for the linear second order differential difference equation. This biological model motivated them towards the study of boundary value problems for singularly perturbed differential difference equations with small shifts. They gave an asymptotic approach to the solution of boundary value problems for two classes of singularly perturbed differential difference equations with small shifts. Lange and Miura[6] have shown the effect of very small shifts on the solution and pointed out that they drastically affect the solution and therefore cannot be neglected. Kadalbajoo and Sharma[1] presented a numerical approach to solve the boundary value problem for singularly perturbed differential difference equation with a negative shift in the differentiated term. The shifted term was approximated by Taylor series as the shifts were of $o(\epsilon)$ and hence the effects of small shifts on the boundary layer solution of the problem were illustrated. In [4] they presented a parameter uniform numerical difference scheme based on finite differences on piecewise uniform mesh to solve boundary value problem for singularly perturbed differential difference equations of convection-diffusion type with small delay. The concept of replacing singularly perturbed two-point boundary value problem by an initial-value problem is presented by Kadalbajoo and Reddy [2] and pair of initial value problems by Reddy and Pramod Chakravarthy [3]. Kadalbajoo and Devendra Kumar [15] discussed a computational method for a singularly perturbed nonlinear differential equation with a small shift. They used quasilinearization method to convert the nonlinear problem into a sequence of linear problems. To tackle the delay term they used Taylor's series expansion on sequence of linear problems.

In this paper we present an Initial value technique to solve the boundary value problem for singularly perturbed differential difference equation which contains only negative shift in the differentiated term, provided the shifts are of $o(\epsilon)$. We first approximate the shifted term by Taylor series so that the boundary value problem of singularly perturbed differential difference equation is converted to a singularly perturbed ordinary differential equation. This boundary value problem is then replaced by a pair of initial value problems. Classical fourth order Runge-Kutta method is used to solve these initial value problems. In fact any standard analytical or numerical method can be used. The effect of small shift on the boundary layer solution in both the cases, i.e., the boundary layer on the left side as well as the right side is discussed by considering numerical experiments. Several numerical examples are solved using the presented method, compared the computed result with exact solution and plotted the graphs of the solution of the problems.

2. Initial value technique

To describe the method, we consider a linear singularly perturbed differential-difference equation, which contains only negative shift in the convection term [5, 6]

$$\epsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x) \tag{2.1}$$

on $0 < x < 1,\ 0 < \epsilon \ll 1$, subject to the interval and boundary conditions

$$\begin{aligned} &y(x) = \phi(x),\ -\delta \le x \le 0 \\ &y(1) = \beta \end{aligned} \tag{2.2}$$

where $a(x)$, $b(x)$, $f(x)$ and $\phi(x)$ are smooth functions, β is a constant and $\delta(\epsilon)$ is a small shifting parameter. For $\delta = 0$, the problem (2.1,2.2) is a singularly perturbed differential equation. We further assume that $a(x) \geq M > 0$ throughout the interval [0,1], where M is some positive constant. This assumption implies that the boundary layer will be in the neighborhood of $x = 0$. We set $\delta = \tau\epsilon$ with $\tau = O(1)$. If τ is not too large, the layer structure is modified but maintained at the same end (C.G. Lange and R.M. Miura [5]).

We expand the retarded term $y'(x-\delta)$ by Taylor series which converts the singularly perturbed differential difference equation to a singularly perturbed ordinary differential equation. By Taylor series expansion, we have

$$y'(x-\delta) \approx y'(x) - \delta y''(x) \tag{2.3}$$

By substituting (2.23) in (2.21) and considering $\delta = \tau\epsilon$ we have

$$\epsilon y''(x) + \frac{a(x)}{(1-\tau a(x))} y'(x) + \frac{b(x)}{(1-\tau a(x))} y(x) = \frac{f(x)}{(1-\tau a(x))} \tag{2.4}$$

Hence, we have the boundary value problem of the form

$$\epsilon y''(x) + \tilde{a}(x) y'(x) + \tilde{b}(x) y(x) = \tilde{f}(x) \tag{2.5}$$

on $0 < x < 1,\ 0 < \epsilon \ll 1$, subject to the boundary conditions

$$y(0) = \phi(0) \quad \text{and} \quad y(1) = \beta \tag{2.6}$$

where $\tilde{a}(x) = \frac{a(x)}{(1-\tau a(x))}$, $\tilde{b}(x) = \frac{b(x)}{(1-\tau a(x))}$, and $\tilde{f}(x) = \frac{f(x)}{(1-\tau a(x))}$.

The initial value technique consists of the following steps:

Step1. Obtain the reduced problem by setting $\epsilon = 0$ in equation (2.5) and solve it for the solution with the appropriate boundary condition. Let $y_0(x)$ be the solution of the reduced problem of (2.5)-(2.6), i.e.;

$$\tilde{a}(x) y_0(x) + \tilde{b}(x) y_0(x) = \widetilde{f}(x) \tag{2.7}$$

$$y_0(1) = \beta \tag{2.8}$$

Step 2. Setup the two first order equations equivalent to the equation (2.5) as follows:

$$z\prime(x) + [\tilde{b}(x) - \tilde{a}'(x)] y(x) = \tilde{f}(x), \tag{2.9}$$

$$\epsilon y'(x) + \tilde{a}(x) y(x) = z(x) \tag{2.10}$$

Step 3. Setup the initial conditions as follows:

Using $y_0(x)$, the solution of the reduced problem, in equation (2.10) we have

$$z(1) = \epsilon y_0'(1) + \tilde{a}(1) y_0(1) \tag{2.11}$$

This will be the initial condition for equation (2.9) and $y(0) = \phi(0)$ will be the initial condition for equation (2.10).

Step 4. Get the pair of initial value problems as follows: Replacing $y(x)$ by $y_0(x)$ in (2.9), we get

$$z'(x) + [\tilde{b}(x) - \tilde{a}'(x)] y_0(x) = \tilde{f}(x) \tag{2.12}$$

Now the differential equation (2.12) with the condition (2.11) constitute an initial value problem and the differential equation (2.10) with the condition $y(0) = \phi(0)$ constitute another initial value problem.

Therefore the pair of initial value problems corresponding to (2.5)-(2.6) are given by

$$\textbf{(IVP 1)}\ z'(x) + [\tilde{b}(x) - \tilde{a}'(x)]y_0(x) = \tilde{f}(x) \text{ with } z(1) = \epsilon y_0'(1) + \tilde{a}(1)y_0(1) \tag{2.13}$$

$$\textbf{(IVP 2)}\ \epsilon y'(x) + \tilde{a}(x)y(x) = z(x) \text{ with } y(0) = \phi(0) \tag{2.14}$$

Thus in a manner of speaking, we have replaced the original boundary value problem (2.5)-(2.6) by a pair of initial value problems. The integration of these initial value problems goes in opposite direction, and the second problem is solved only if the solution of the first one is known. We solve these initial value problems (2.13)-(2.14) to obtain the solution over the interval [0, 1]. There now exist a number of efficient methods for the solution of initial value problems. In order to solve the initial value problems in our numerical experimentation, we make use of classical fourth order Runge-Kutta method. In fact, any standard analytical or numerical method can be used.

2.1. Linear examples with left layer. To demonstrate the applicability of Initial value technique, we have applied it to two linear singularly perturbed differential difference equations with a small shift and the boundary layer being to the left of the domain. These examples have been chosen because they have been widely discussed in literature and for some examples approximate solutions are available for comparison.

Example 1. We consider the following singularly perturbed linear differential-difference equation

$$\epsilon y''(x) + y'(x-\delta) - y(x) = 0$$

under the interval and boundary conditions

$$y(x) = 1;\ -\delta \le x \le 0\,; \quad y(1) = 1$$

For $\delta = o(\epsilon)$, the approximate solution available for comparison is

$$y(x) = \frac{(1-e^{m_2})e^{m_1 x} + (e^{m_1}-1)e^{m_2 x}}{e^{m_1} - e^{m_2}},$$

where $m_1 \frac{-1-\sqrt{1+4(\epsilon-\delta)}}{2(\epsilon-\delta)}$ and $m_2 = \frac{-1+\sqrt{1+4(\epsilon-\delta)}}{2(\epsilon-\delta)}$

Expanding the retarded term $y'(x-\delta)$ by Taylor series,

$$y'(x-\delta) \approx y'(x) - \delta y''(x)$$

and considering $\delta = \tau\epsilon$ we have the modified problem

$$\epsilon y''(x) + \frac{1}{1-\tau} y'(x) - \frac{1}{1-\tau} y(x) = 0 \tag{2.15}$$

subject to the conditions

$$y(0) = 1; \quad y(1) = 1 \tag{2.16}$$

The reduced problem corresponding to (2.15)-(2.16) is

$$y_0' - y_0 = 0, \quad y_0(1) = 1$$

whose solution is $y_0 = e^{x-1}$.

Hence, the set of initial value problems corresponding to (2.15)-(2.16) are

(IVP 1) $z'(x) - \frac{e^{x-1}}{1-\tau} = 0; \quad z(1) = \epsilon + \frac{1}{1-\tau}$

(IVP 2) $\epsilon y'(x) + \frac{1}{1-\tau} y(x) = z(x); \quad y(0) = 1$

The numerical results are given in table 1 and coresponding graphs in figures (2.1)-(2.2)

Example 2. We consider the following singularly perturbed linear differential-difference equation

$$\epsilon y''(x) + e^{-x/2} y'(x-\delta) - y(x) = 0$$

under the interval and boundary conditions

$$y(x) = 1;\ -\delta \le x \le 0\,; \quad y(1) = 1$$

The exact solution is not known for this example.

Expanding the retarded term $y'(x-\delta)$ by Taylor series,

$$y'(x-\delta) \approx y'(x) - \delta y''(x)$$

and considering $\delta = \tau\epsilon$ we have the modified problem

$$\epsilon y''(x) + \frac{e^{-x/2}}{\left(1 - \tau e^{-x/2}\right)} y'(x) - \frac{1}{\left(1 - \tau e^{-x/2}\right)} y(x) = 0 \tag{2.17}$$

subject to the conditions

$$y(0) = 1; \quad y(1) = 1 \tag{2.18}$$

The reduced problem corresponding to (2.17)-(2.18) is

$$y_0' - e^{x/2} y_0 = 0, \quad y_0(1) = 1$$

whose solution is $y_0 = \exp\left[2\left(e^{x/2} - e^{1/2}\right)\right]$

Hence, the set of initial value problems corresponding to (2.17)-(2.18) are

(IVP 1) $$z'(x) + \left[-\frac{1}{1-\tau e^{-x/2}} + \frac{e^{-x/2}}{2\left(1-\tau e^{-x/2}\right)^2}\right] \exp\left[2\left(e^{x/2} - e^{1/2}\right)\right] = 0\,;$$

$$z(1) = \epsilon e^{1/2} + \frac{e^{-1/2}}{1-\tau e^{-1/2}}$$

(IVP 2) $$\epsilon y'(x) + \frac{e^{-x/2}}{1-\tau e^{-x/2}} y(x) = z(x); \quad y(0) = 1$$

The numerical results are given in tables 2 and corresponding graph in figure (2.3)

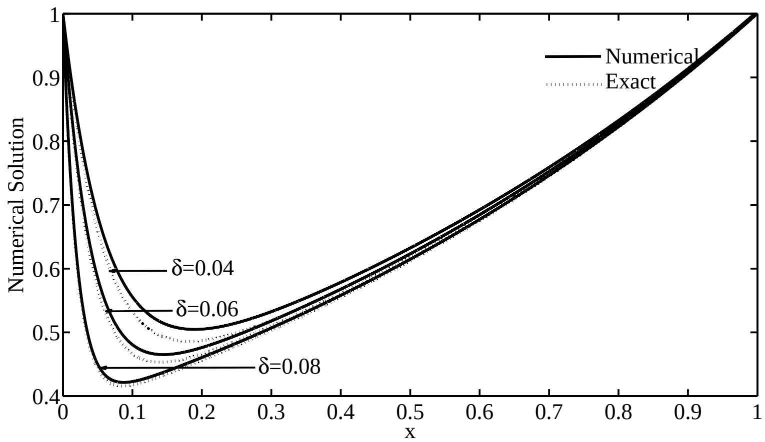

FIGURE 2.1. Graph of the solution of the example 1 for $\epsilon = 0.1$ and different values of δ

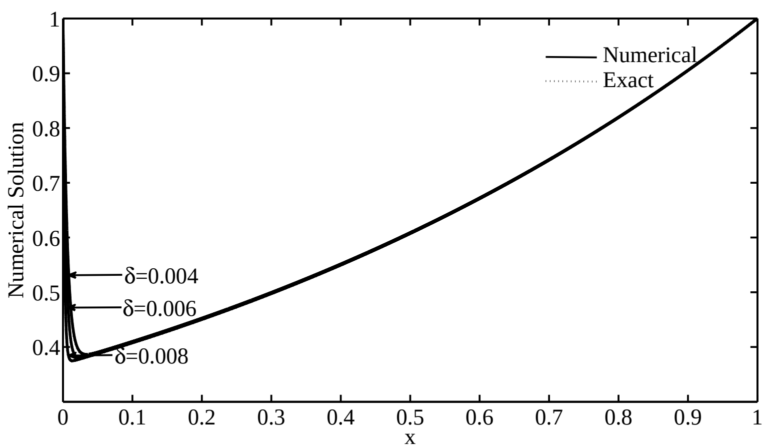

FIGURE 2.2. Graph of the solution of the example 1 for $\epsilon = 0.01$ and different values of δ

2.2. Right End Boundary Layer Problems. Now we assume that $a(x) \leq M < 0$ throughout the interval [0, 1], where M is a constant. This assumption implies that the boundary layer will be in the neighborhood of x=1 for the singularly perturbed differential difference equation (2.1)-(2.2). We set $\delta = \tau\epsilon$ with $\tau = O(1)$.

The initial value technique consists of the following steps:

Step1. Obtain the reduced problem by setting $\epsilon = 0$ in equation (2.5) and solve it for the solution with the appropriate boundary condition. Let $y_0(x)$ be

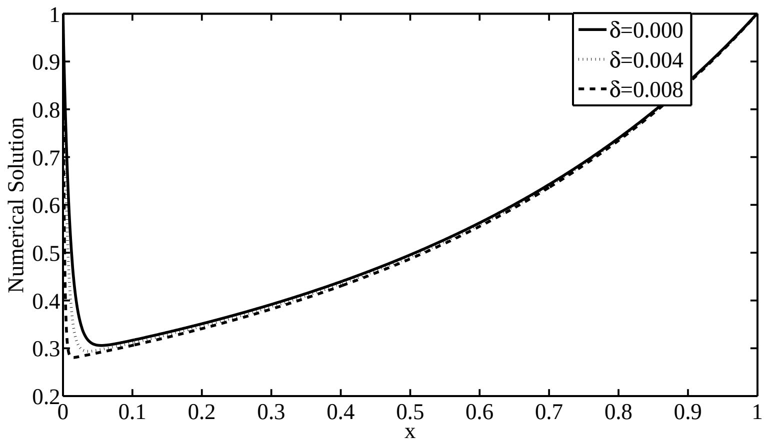

FIGURE 2.3. Graph of the solution of the example 2 for $\epsilon = 0.01$ and different values of δ

TABLE 1. Numerical Results of Example 1 with $\epsilon = 10^{-3}$, $\delta = 0.0004$, $h = 10^{-4}$

x	Numerical Solution	Exact Solution
0.0000	1.0000000	1.0000000
0.0001	0.9030536	0.9029751
0.0002	0.8209958	0.8208592
0.0003	0.751541	0.7513622
0.0004	0.6927543	0.6925458
0.0005	0.642998	0.6427694
0.0010	0.4879487	0.4876993
0.0050	0.3702536	0.3700951
0.0100	0.3719539	0.3717972
0.0500	0.3871091	0.3869613
0.1000	0.4069259	0.4067890
0.2000	0.4496596	0.4495444
0.3000	0.4968876	0.4967937
0.4000	0.5490826	0.5490090
0.5000	0.606767	0.6067124
0.6000	0.6705181	0.6704807
0.7000	0.7409740	0.7409514
0.8000	0.8188398	0.8188289
0.9000	0.9048948	0.9048916
1.0000	1.0000000	1.0000000

TABLE 2. Numerical Results of Example 2 with $\epsilon = 10^{-3}$, $h = 10^{-3}$ for different values of δ

x	Numerical Solution			
	$\delta = 0$	$\delta = 0.0002$	$\delta = 0.0004$	$\delta = 0.0006$
0.000	1.0000000	1.0000000	1.0000000	1.0000000
0.001	0.5469622	0.4977724	0.4718476	0.7438532
0.002	0.3772521	0.3435459	0.3283316	0.5768190
0.003	0.3137571	0.2962924	0.2894893	0.4684561
0.004	0.2900983	0.2819389	0.2791255	0.3985296
0.005	0.2813869	0.2777096	0.2765081	0.3536586
0.020	0.2801579	0.2798844	0.2796111	0.2793874
0.040	0.2858806	0.2856089	0.2853372	0.2850657
0.060	0.2917795	0.2915096	0.2912397	0.2909699
0.080	0.2978612	0.2975931	0.2973251	0.2970572
0.100	0.3041326	0.3038665	0.3036004	0.3033344
0.200	0.3386029	0.3383476	0.3380924	0.3378372
0.300	0.3790610	0.3788189	0.3785768	0.3783348
0.400	0.4268172	0.4265911	0.4263651	0.4261392
0.500	0.4835252	0.4833189	0.4831127	0.4829066
0.600	0.5512866	0.5511048	0.5509231	0.5507416
0.700	0.6327912	0.6326400	0.6324889	0.6323379
0.800	0.7315078	0.7313950	0.7312823	0.7311698
0.900	0.8519448	0.8518805	0.8518165	0.8517528
1.000	1.0000096	1.0000074	1.0000056	1.0000040

the solution of the reduced problem of (2.5)-(2.6), i.e.;

$$\tilde{a}(x)y_0'(x) + \tilde{b}(x)y_0(x) = \tilde{f}(x) \tag{2.19}$$

$$\text{with } y_0(0) = \phi(0) \tag{2.20}$$

Step 2. Setup the two first order equations equivalent to the equation (2.5) as follows:

$$z\prime(x) + [\tilde{b}(x) - \tilde{a}'(x)]y(x) = \tilde{f}(x) \tag{2.21}$$

$$\text{and } \epsilon y'(x) + \tilde{a}(x)y(x) = z(x) \tag{2.22}$$

Step 3. Setup the initial conditions as follows:
Using $y_0(x)$, the solution of the reduced problem, in equation (2.22) we have

$$z(0) = \epsilon y_0'(0) + \tilde{a}(0)y_0(0) \tag{2.23}$$

This will be the initial condition for equation (2.21) and $y(1) = \beta$ will be the initial condition for equation (2.22).

Step 4. Get the pair of initial value problems as follows:
Replacing $y(x)$ by $y_0(x)$ in (2.21), we get

$$z'(x) + [\tilde{b}(x) - \tilde{a}'(x)]y_0(x) = \tilde{f}(x) \tag{2.24}$$

Now the differential equation (2.24) with the condition (2.23) constitute an initial value problem and the differential equation (2.22) with the condition $y(1) = \beta$ constitute another initial value problem.

Therefore the pair of initial value problems corresponding to (2.5)-(2.6) are given by

$$\textbf{(IVP 1)}\ z'(x)+[\tilde{b}(x)-\tilde{a}'(x)]y_0(x) = \tilde{f}(x) \text{ with } z(0)=\epsilon y_0'(0)+\tilde{a}(0)y_0(0) \quad (2.25)$$

$$\textbf{(IVP 2)}\ \epsilon y'(x)+\tilde{a}(x)y(x)=z(x) \text{ with } y(1)=\beta \quad (2.26)$$

Thus in a manner of speaking, we have replaced the original boundary value problem (2.5)-(2.6) by a pair of initial value problems. The integration of these initial value problems goes in opposite direction, and the second problem is solved only if the solution of the first one is known. We solve these initial value problems (2.25)-(2.26) to obtain the solution over the interval [0, 1]. There now exist a number of efficient methods for the solution of initial value problems. In order to solve the initial value problems in our numerical experimentation, we make use of classical fourth order Runge-Kutta method. Infact, any standard analytical or numerical method can be used.

2.3. Linear examples with right layer. To demonstrate the applicability of the method, we have applied it to two linear singularly perturbed differential difference equations with a small shift and the boundary layer being to the right of the domain. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison.

Example 3. We consider the following singularly perturbed linear differential-difference equation

$$\epsilon y''(x) - y'(x-\delta) - y(x) = 0$$

under the interval and boundary conditions

$$y(x) = 1;\ -\delta \le x \le 0\,; \quad y(1) = -1$$

For $\delta = o(\epsilon)$, the approximate solution available for comparison is

$$y(x) = \frac{-(1+e^{m_2})e^{m_1 x} + (1+e^{m_1})e^{m_2 x}}{e^{m_1} - e^{m_2}},$$

where $m_1 \frac{-1-\sqrt{1+4(\epsilon+\delta)}}{2(\epsilon+\delta)}$ and $m_2 = \frac{-1+\sqrt{1+4(\epsilon+\delta)}}{2(\epsilon+\delta)}$

Expanding the retarded term $y'(x-\delta)$ by Taylor series,

$$y'(x-\delta) \approx y'(x) - \delta y''(x)$$

and considering $\delta = \tau\epsilon$ we have the modified problem

$$\epsilon y''(x) - \frac{1}{1+\tau} y'(x) - \frac{1}{1+\tau} y(x) = 0 \quad (2.27)$$

subject to the conditions

$$y(0) = 1; \quad y(1) = -1 \quad (2.28)$$

The reduced problem corresponding to (2.27)-(2.28) is

$$y_0' + y_0 = 0, \quad y_0(0) = 1$$

whose solution is $y_0 = e^{-x}$.

Hence, the set of initial value problems corresponding to (2.27)-(2.28) are

(IVP 1) $z'(x) - \frac{e^{-x}}{1+\tau} = 0; \quad z(0) = -\epsilon - \frac{1}{1+\tau}$

(IVP 2) $\epsilon y'(x) - \frac{1}{1+\tau} y(x) = z(x); \quad y(1) = -1$

The numerical results are given in table 3 and corresponding graphs in figures (2.4)-(2.5).

Example 4. We consider the following singularly perturbed linear differential-difference equation

$$\epsilon y''(x) - e^{-x} y'(x-\delta) - xy(x) = 0$$

under the interval and boundary conditions

$$y(x) = 1;\ -\delta \le x \le 0\,; \quad y(1) = 1$$

The exact solution is not known.

Expanding the retarded term $y'(x-\delta)$ by Taylor series,

$$y'(x-\delta) \approx y'(x) - \delta y''(x)$$

and considering $\delta = \tau\epsilon$ we have the modified problem

$$\epsilon y''(x) - \frac{e^{-x}}{(1+\tau e^{-x})} y'(x) - \frac{x}{(1+\tau e^{-x})} y(x) = 0 \tag{2.29}$$

subject to the conditions

$$y(0) = 1; \quad y(1) = 1 \tag{2.30}$$

The reduced problem corresponding to (2.29)-(2.30) is

$$y_0' + xe^{x} y_0 = 0, \quad y_0(0) = 1$$

whose solution is $y_0 = exp\{-1-(x-1)e^{x}\}$

Hence, the set of initial value problems corresponding to (2.29)-(2.30) are

(IVP 1) $z'(x) - \left[\frac{x}{1+\tau e^{-x}} + \frac{e^{-x}}{(1+\tau e^{-x})^2}\right] \exp\left[-1-(x-1)e^{x}\right] = 0;\ z(0) = -\frac{1}{1+\tau}$

(IVP 2) $\epsilon y'(x) - \frac{e^{-x}}{(1+\tau e^{-x})} y(x) = z(x);\ y(1) = 1$

The numerical results are given in table 4 and corresponding graph in figure (2.6).

3. Discussion and conclusions

We have presented an initial value technique for solving singularly perturbed differential difference equations with a boundary layer at one end point. The shift term is approximated by Taylor series which converts the singularly perturbed differential difference equation to a singularly perturbed ordinary differential equation. The solution of this singularly perturbed boundary value problem is computed numerically by solving a pair of initial value problems, which are

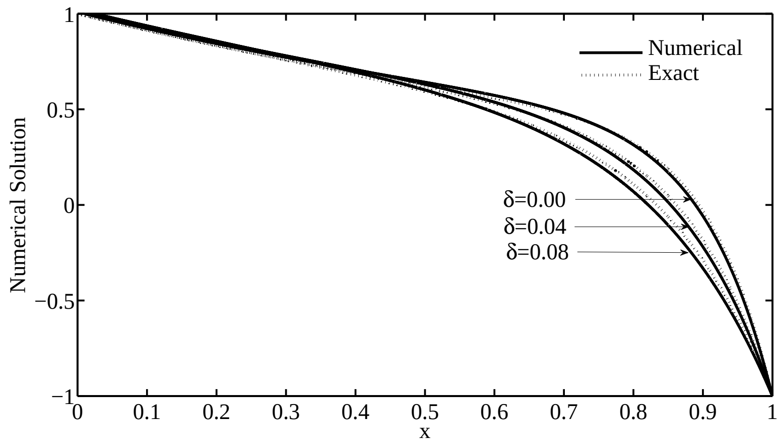

FIGURE 2.4. Graph of the solution of the example 3 for $\epsilon = 0.1$ and different values of δ

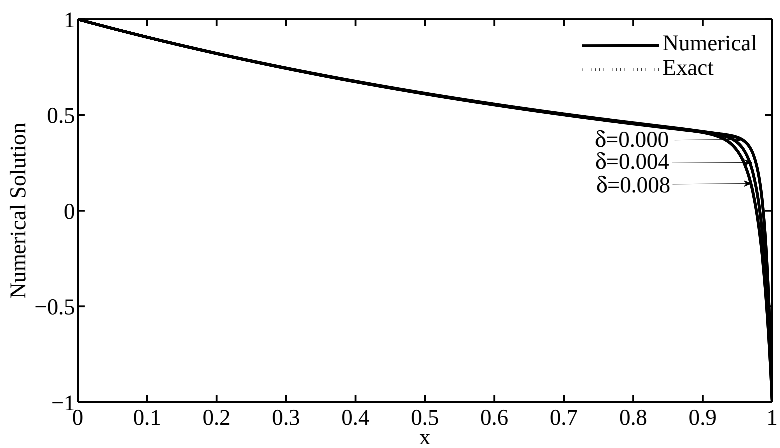

FIGURE 2.5. Graph of the solution of the example 3 for $\epsilon = 0.01$ and different values of δ

deduced from the original problem. This method is very easy to implement on any computer with minimum problem preparation. We have implemented the present method on two linear examples with left-end boundary layer and two examples with right-end boundary layer by taking different values of ϵ.To solve the initial value problems we used the classical fourth order Runge-Kutta method. Infact any standard analytical or numerical method can be used. Computational results are presented in tables. The graphs are plotted to illustrate the behavior of the solution. The approximate solution is compared with exact

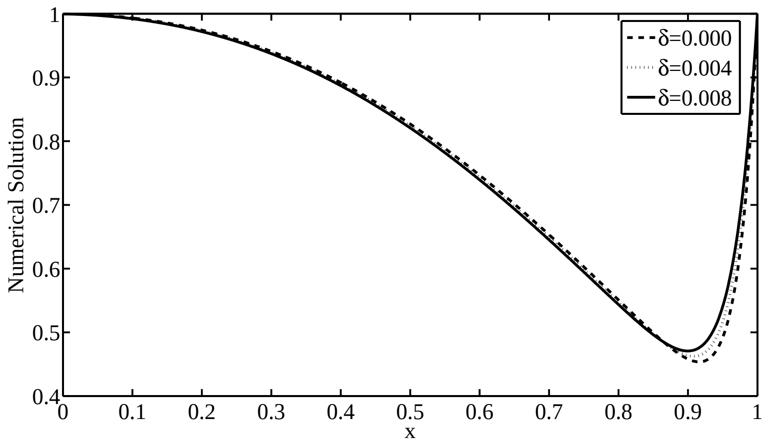

FIGURE 2.6. Graph of the solution of the example 4 for $\epsilon = 0.01$ and different values of δ

TABLE 3. Numerical Results of Example 3 with $\epsilon = 0.001$, $\delta = 0.0006$, $h = 10^{-4}$

x	Numerical Solution	Exact Solution
0.0000	1.0000000	1.0000000
0.1000	0.904992	0.9049817
0.2000	0.8190229	0.8189920
0.3000	0.7412348	0.7411728
0.4000	0.6708493	0.6707478
0.5000	0.6071618	0.6070145
0.6000	0.5495349	0.5493371
0.7000	0.497392	0.4971400
0.8000	0.4502112	0.4499026
0.9000	0.4075202	0.4071537
0.9500	0.3877232	0.3873274
0.9900	0.3699405	0.3695484
0.9950	0.3105879	0.3104847
0.9990	-0.3634558	-0.3629102
0.9995	-0.6324272	-0.6320255
0.9996	-0.6970553	-0.6967067
0.9997	-0.7658492	-0.7655640
0.9998	-0.8390775	-0.8388671
0.9999	-0.9170263	-0.9169033
1.0000	-1.0000000	-1.0000000

Table 4. Numerical Results of Example 4 with $\epsilon = 10^{-3}$, $h = 10^{-3}$ for different values of δ

x	Numerical Solution			
	$\delta = 0$	$\delta = 0.0002$	$\delta = 0.0004$	$\delta = 0.0006$
0.000	1.0000000	1.0000000	1.0000000	1.0000000
0.100	0.9945450	0.9945224	0.9944997	0.9944769
0.200	0.9770879	0.9770393	0.9769907	0.9769419
0.300	0.9458708	0.9457932	0.9457154	0.9456376
0.400	0.8996037	0.8994950	0.8993863	0.8992774
0.500	0.8377743	0.8376347	0.8374950	0.8373551
0.600	0.7609742	0.7608061	0.7606380	0.7604697
0.700	0.6711730	0.6709820	0.6707910	0.6705999
0.800	0.5718520	0.5716467	0.5714413	0.5712359
0.900	0.4678951	0.4676865	0.4674778	0.4672692
0.920	0.4470533	0.4468454	0.4466375	0.4464297
0.940	0.4263091	0.4261025	0.4258960	0.4256895
0.960	0.4057126	0.4055082	0.4053043	0.4051013
0.980	0.3856894	0.3857399	0.3858928	0.3861650
0.995	0.4706318	0.4840198	0.4971599	0.5099797
0.996	0.5145287	0.5298502	0.5445338	0.5585636
0.997	0.5784173	0.5948414	0.6102125	0.6245972
0.998	0.6711637	0.6868021	0.7010957	0.7141981
0.999	0.8055503	0.8167111	0.8266744	0.8356206
1.000	1.0000000	1.0000000	1.0000000	1.0000000

solution wherever available. It can be observed from the results that the present method approximates with exact solution very well, which shows the efficiency of the method. From the graphs it can be observed that in the case of left end boundary value problems, as δ increases, the thickness of the boundary layer decreases while for the right end boundary layer problems as δ increases, the thickness of the boundary layer increases.

References

1. M.K. Kadalbajoo and K.K. Sharma, *Numerical analysis of singularly perturbed delay differential equations with layer behavior*, Applied Mathematics and Computation **157** (2004), 11-28.
2. M.K. Kadalbajoo and Y.N. Reddy, *Initial value technique for a class of non-linear singular perturbation problems*, Journal of Optimization Theory and Applications **53** (1987) 395-406.
3. Y.N. Reddy and P. Pramod Chakravarthy, *Method of Reduction of Order for Solving Singularly Perturbed Two Point Boundary Value Problems*, Applied Mathematics and Computation **136** (2003) 27-45.

4. M.K. Kadalbajoo and K.K. Sharma, *Parameter-Uniform fitted mesh method for singularly perturbed delay differential equations with layer behavior*, Electronic Transactions on Numerical Analysis **23** (2006) 180-201.
5. C.G. Lange and R.M. Miura, *Singular perturbation analysis of boundary-value problems for differential difference equations. V. Small shifts with layer behavior*, SIAM Journal on Applied Mathematics **54** (1994) 249-272.
6. C.G. Lange and R.M. Miura, *Singular perturbation analysis of boundary-value problems for differential difference equations. VI. Small shifts with rapid oscillations*, SIAM Journal on Applied Mathematics **54** (1994) 273-283.
7. R. B. Stein, *Some Models of Neuronal Variability*, Biophysical Journal **7** (1967) 37-68.
8. H.C. Tuckwell and D.K. Cope, *Accuracy of Neuronal Inter spike Times Calculated from a Diffusion Approximation*, Journal of Theoretical Biology **83** (1980) 377-387.
9. Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, 1993.
10. C.M. Bender and S.A. Orszag, Advanced mathematical methods for Scientists and Engineers, Mc Graw-Hill, New York, 1978.
11. J. Kevorkian and J.D. Cole, Perturbation Methods in Applied Mathematics, Springer-Verlag, New York, 1981.
12. A.H. Nayfeh, Perturbation Methods, Wiley, New York, 1973.
13. R.E. O' Malley, Introduction to Singular Perturbations, Academic Press, New York, 1974.
14. Thomas Erneux, Applied Delay Differential Equations, Springer, New York , 2009.
15. M.K.Kadalbajoo and Devendra Kumar, *A computational method for singularly perturbed nonlinear differential-difference equations with small shift*, Applied Mathematical Modelling **34** (2010) 2584-2596.

R. Nageshwar Rao received M.Sc. from National Institute of Technology, Warangal, India and is pursuing Ph.D at Visvesvaraya National Institute of Technology, Nagpur, India. His research interests include Numerical Methods for Singularly Perturbed Differential–Difference Equations.

Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur - 440010, India.
`e-mail: nrao_ragi@yahoo.co.in`

P. Pramod Chakravarthy received M.Sc. and Ph.D. from National Institute of Technology, Warangal, India. He is currently working as an Associate Professor at Visvesvaraya National Institute of Technology, Nagpur, India. Since 2006 he has been at Visvesvaraya National Institute of Technology, Nagpur. His research interests include Numerical Methods for Singular Perturbation Problems in Differential Equations and Differential–Difference Equations.

Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur - 440010, India.
`e-mail: pramodpodila@yahoo.co.in`