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SERIES SOLUTIONS TO INITIAL-NEUMANN BOUNDARY VALUE PROBLEMS FOR PARABOLIC AND HYPERBOLIC EQUATIONS

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Abstract. The purpose of this paper is to employ a new useful technique to solve the initial-Neumann boundary value problems for parabolic, hyperbolic and parabolic-hyperbolic equations and obtain a solution in form of infinite series. The results obtained indicate that this approach is indeed practical and efficient.

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1. Introduction

The method of separation of variable for the solution of the partial differential equations often leads to ordinary differential equations with variable coefficients whose solutions are obtained either in the form of infinite series in term of special functions. In particular, this method requires that the boundary conditions be homogeneous. For inhomogeneous boundary conditions, a transformation formula should be employed to transform inhomogeneous boundary conditions to homogeneous boundary conditions. However, Adomian decomposition method [1]-[4] has been proved to be powerful, effective, and can easily handle a wide class of initial-boundary value problems for linear and nonlinear partial differential, where the components of the solutions are elegantly computed by a recursive relation. Therefore, the solution is obtained in a series form. An important point can be made here in that the method attacks any homogeneous or inhomogeneous problem without any need for transformation formula. Further there is no need to change the inhomogeneous boundary conditions to homogeneous conditions as required by the method of separation of variables.

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Also, The finite difference method is one of several techniques for obtaining numerical solutions to these types of equations and is discussed in many textbooks. Ames [5], Cooper [6] and Morton and Mayers [7] provide a more mathematical development of finite difference methods. See Cooper [6] for modern introduction to the theory of partial differential equations along with a brief coverage of numerical methods. Also, the variational iteration method, Padé approximants and other numerical methods are discussed in [7].

Here, we will prove results on the existence and uniqueness of the solutions to initial-Neumann boundary value problems for linear parabolic, hyperbolic and parabolic-hyperbolic equations, where the solution is searched in form of infinite series $u(t,x) = \sum_{n=0}^{\infty} a_n(x)t^n$. In such cases, a recursion formula is obtained to calculate the unknown coefficients a_n. The most important feature of this method is that it reduces the initial-Neumann value problems into ordinary differential equations that can be easily handled.

2. Initial-Neumann value problem for linear parabolic equations

In the rectangular domain $D = \{(t,x) : \ 0 \le t \le T, \ 0 \le x \le 1\}$, consider the initial-Neumann value problem for parabolic equation

$$u_t - u_{xx} + \lambda u = f(t,x), \ \lambda \ge 0, \tag{1}$$

subject to

$$u(0,x) = \phi(x) \tag{2}$$

and the Neumann boundary conditions

$$u_x(t,0) = \alpha(t), \ u_x(t,1) = \beta(t), \tag{3}$$

where $f \in L_2(D)$, $\phi \in L_2(0,1)$, $\alpha \in L_2(0,T)$ and $\beta \in L_2(0,T)$.

2.1. Existence. To find the solution of problem (1)-(3), integrating both sides of Eq. (1) with respect to x from 0 to 1, we obtain

$$\int_0^1 u_t(t,x)dx - \int_0^1 u_{xx}dx + \lambda \int_0^1 u(t,x)dx = \int_0^1 f(t,x)dx. \tag{4}$$

Since

$$\int_0^1 u_{xx}(t,x)dx = u_x(t,1) - u_x(t,0). \tag{5}$$

Substituting (5) into (4) we obtain

$$\int_0^1 u_t(t,x)dx + \lambda \int_0^1 u(t,x)dx = u_x(t,1) - u_x(t,0) + \int_0^1 f(t,x)dx. \tag{6}$$

Introducing a new function $v(t)$ such that

$$v(t) = \int_0^1 u(t,x)dx, \ \text{where } v(0) = \int_0^1 \phi(x)dx. \tag{7}$$

This transformation will carry out Eq. (6) to

$$\frac{dv}{dt} + \lambda v = \int_0^1 f(t,x)dx + \beta(t) - \alpha(t). \tag{8}$$

Thus problem (1)-(3) can be reduced to an ordinary linear differential equation satisfied by the new dependent variable v, and its solution can be found as

$$v(t) = \frac{\int e^{\lambda t} p(t)dt + C}{e^{\lambda t}}, \tag{9}$$

where $p(t) = \int_0^1 f(t,x)dx + \beta(t) - \alpha(t)$ and C is a constant of integration, which can be found by using the initial condition $v(0) = \int_0^1 u(0,x)dx = \int_0^1 \varphi(x)dx$. Once Eq. (8) is solved, go back to the original dependent variable $u(t,x)$ via the Eq. (7).
We now can seek the solution $u(t,x)$ as $u(t,x) = \sum_{n=0}^{\infty} a_n(x)t^n$ and $v(t)$ be equated to an infinite series of polynomials of the form $v(t) = \sum_{n=0}^{\infty} b_n t^n$.
The substitution yields

$$\int_0^1 \sum_{n=0}^{\infty} a_n(x)t^n dx = \sum_{n=0}^{\infty} b_n t^n. \tag{10}$$

Equating coefficients of like powers of t, we derive the recursion formula for the coefficients $a_n(x)$

$$\int_0^1 a_n(x)dx = b_n, \ n \geq 0. \tag{11}$$

This equation has the solution

$$a_n(x) = \begin{cases} b_n \frac{\Theta(x)}{K}, \text{ if } K \neq 0, \\ b_n \left(1 + C\Theta(x)\right), \text{ if } K = 0, \end{cases} \tag{12}$$

where $K = \int_0^1 \Theta(x)dx$ and $\Theta(x)$ is an arbitrary function.
In view of the boundary conditions (3), we get $a_n'(0) = \alpha_n$ and $a_n'(1) = \beta_n, \ n \geq 0$, where α_n and β_n are the coefficients of the power series $\alpha(t) = \sum_{n=0}^{\infty} \alpha_n t^n$ and $\beta(t) = \sum_{n=0}^{\infty} \beta_n t^n$, respectively.
The final solution is now given by

Theorem 2.1. *There exists a solution $u(t,x)$ of problem (1)-(3), which has the form of a sum as*

$$u(t,x) = \sum_{n=0}^{\infty} a_n(x)t^n$$

such that the coefficients a_n are given by (12) and satisfy

$$a_n'(0) = \alpha_n, \ a_n'(1) = \beta_n, \ n \geq 0,$$

where $\alpha(t) = \sum_{n=0}^{\infty} \alpha_n t^n$ and $\beta(t) = \sum_{n=0}^{\infty} \beta_n t^n$.

2.2. Uniqueness. Here, we establish the uniqueness of the solution of problem (1)-(3).

Theorem 2.2. *There exists at most one solution to problem (1)-(3) in* E_1*, where* E_1 *is a Hilbert space*

$$E_1 = \{u : u, u_t, u_x \in L_2(D^\tau), u, u_x \in L_2(0,1)\}$$

with respect to the norm

$$\|u\|_{E_1} = \sup_{0\le\tau\le T} \int_{D^\tau} u_t^2(t,x)dtdx + \sup_{0\le\tau\le T} \int_0^1 \left[u^2 + u_x^2\right](\tau,x)dx,$$

where $D^\tau = (0,\tau) \times (0,1)$,

Proof. Let $u_1(t,x)$ and $u_2(t,x)$ be two different solutions of problem (1)-(3). Then $u(t,x) = u_1(t,x) - u_2(t,x)$ is a nontrivial solution to the homogeneous problem

$$u_t - u_{xx} + \lambda u = 0, \tag{13}$$

$$u(0,x) = 0 \tag{14}$$

and

$$u_x(t,0) = 0,\ u_x(t,1) = 0. \tag{15}$$

Multiplying both sides of Eq. (13) by u_t, employing integration by parts over D^τ, and taking into account the initial-Neumann conditions (14) and (15), we obtain

$$\int_{D^\tau} u_t^2(t,x)dtdx + \frac{1}{2}\int_0^1 u_x^2(\tau,x)dx + \frac{\lambda}{2}\int_0^1 u^2(\tau,x)dx = 0. \tag{16}$$

Now, taking the upper bound with respect to τ in the interval $(0,T)$, we obtain

$$\|u\|_{E_1} = 0. \tag{17}$$

Thus, $u(t,x) = 0$ in E_1. Hence $u_1(t,x) = u_2(t,x)$. □

Example 1. Consider the initial-boundary value problem

$$\begin{cases} u_t - u_{xx} = x, \\ u(0,x) = 0, \\ u_x(t,0) = t,\ u_x(t,1) = t. \end{cases} \tag{18}$$

A simple computation yields $v(t) = \frac{t}{2}$. Thus

$$b_0 = 0,\ b_1 = \frac{1}{2},\ b_n = 0,\ n \ge 2,$$

$$a_0(x) = 0,\ a_n(x) = 0,\ n \ge 2$$

and

$$a_1(x) = \begin{cases} b_1 \dfrac{\Theta(x)}{K}, \text{ if } K \ne 0, \\ b_1\left(1 + C\Theta(x)\right), \text{ if } K = 0. \end{cases}$$

If we choose $\Theta(x) = 2x - 1$, then $a_1(x) = \frac{1}{2}[1 + C(2x - 1)]$, and C can be easily determined by using the boundary condition $a_1'(0) = \alpha_1 = 1$ or $a_1'(1) = \beta_1 = 1$, hence $C = 1$. Then the solution is given by

$$u(t, x) = \sum_{n=0}^{\infty} a_n(x)t^n = tx.$$

Example 2. Consider the Neumann heat conduction problem

$$\begin{cases} u_t - u_{xx} = 0, \\ u(0, x) = e^x, \\ u_x(t, 0) = e^t, \ u_x(t, 1) = e^{t+1}. \end{cases} \tag{19}$$

Following the analysis introduced before leads to

$$v(t) = ke^t, \ k = e - 1, \quad b_n = \frac{k}{n!}, \ n \geq 0$$

and

$$a_n(x) = \begin{cases} b_n \dfrac{\Theta(x)}{K}, \text{ if } K \neq 0, \\ b_n \left(1 + C\Theta(x)\right), \text{ if } K = 0, \end{cases}$$

where

$$a_n'(0) = \frac{1}{n!}, \ a_n'(1) = \frac{e}{n!}, \ n \geq 0.$$

If we choose $\Theta(x) = e^x$, then $a_n(x) = \frac{e^x}{n!}$. Thus the solution is given by

$$u(t, x) = \sum_{n=0}^{\infty} a_n(x)t^n = \sum_{n=0}^{\infty} \frac{e^x}{n!}t^n = e^{x+t}.$$

3. Initial-Neumann value problems for linear hyperbolic equations

The same analysis can also be applied to the initial- Neumann boundary value problem for linear hyperbolic equations

$$u_{tt} - u_{xx} + \lambda u = g(t, x), \tag{20}$$

$$u(0, x) = \phi(x), \ u_t(0, x) = \psi(x) \tag{21}$$

$$u_x(t, 0) = \alpha(t), \ u_x(t, 1) = \beta(t), \tag{22}$$

where $g \in L_2(D)$, $\phi \in L_2(0, 1)$ and $\psi \in L_2(0, 1)$.

3.1. Existence. Proceeding as before, we obtain a linear equation satisfied by the new dependent variable w

$$\frac{d^2w}{dt^2} + \lambda w = \int_0^1 g(t, x)dx + \beta(t) - \alpha(t), \tag{23}$$

where $w(t) = \int_0^1 u(t, x)dx$.

If $\lambda < 0$, then the general solution is given by

$$w(t) = C_1 e^{\sqrt{-\lambda}t} + C_2 e^{-\sqrt{-\lambda}} + w_p(t), \tag{24}$$

where $w_p(t)$ is a particular solution of Eq. (23).
If $\lambda > 0$, then the general solution is given by

$$w(t) = C_1 \cos\sqrt{\lambda}t + C_2 \sin\sqrt{-\lambda}t + w_p(t). \tag{25}$$

If $\lambda = 0$, then the general solution is given by

$$w(t) = \int\int p(t)dtdt,\ p(t) = \int_0^1 f(t,x)dx + \beta(t) - \alpha(t). \tag{26}$$

To find the constants $C_i,\ i = 1, 2$, we use the initial conditions $w(0) = \int_0^1 u(0,x)dx = \int_0^1 \phi(x)dx$ and $w'(0) = \int_0^1 u_t(0,x)dx = \int_0^1 \psi(x)dx$. Once Eq. (23) is solved, go back to the original dependent variable $u(t,x)$ via the equation $w(t) = \int_0^1 u(t,x)dx$, and assuming that $u(t,x) = \sum_{n=0}^{\infty} a_n(x)t^n$ and $w(t)$ be equated to an infinite series of polynomials of the form $w(t) = \sum_{n=0}^{\infty} c_n t^n$. Thus we have

Theorem 3.1. *There exists a solution $u(t,x) = \sum_{n=0}^{\infty} a_n(x)t^n$, of problem (20)-(22), where the coefficients are given by*

$$a_n(x) = \begin{cases} c_n \dfrac{\Theta(x)}{K},\ \text{if } K \neq 0, \\ c_n\left(1 + C\Theta(x)\right),\ \text{if } K = 0 \end{cases} \tag{27}$$

such that

$$a'_n(0) = \alpha_n,\ a'_n(1) = \beta_n,\ n \geq 0.$$

3.2. Uniqueness.

Theorem 3.2. *There exists at most one solution to problem (20)-(22) in E_2, where E_2 is a Hilbert space*

$$E_2 = \{u : u, u_t, u_x \in L_2(0,1)\}$$

with respect to the norm

$$\|u\|_{E_2} = \sup_{0\leq\tau\leq T} \int_0^1 \left[u^2 + u_x^2 + u_t^2\right](\tau,x)dtdx.$$

Proof. Let $u_1(t,x)$ and $u_2(t,x)$ be two different solutions of problem (20)-(22). Then $u(t,x) = u_1(t,x) - u_2(t,x)$ is a nontrivial solution to the homogeneous problem

$$u_{tt} - u_{xx} + \lambda u = 0, \tag{28}$$

$$u(0,x) = 0, \tag{29}$$

$$u_x(t,0) = 0,\ u_x(t,1) = 0. \tag{30}$$

Multiplying both sides of Eq. (28) by u_t, employing integration by parts over D^τ, and taking into account the initial-Neumann conditions (29)-(30), we obtain

$$\frac{\lambda}{2}\int_0^1 u^2(\tau,x)dx + \int_0^1 u_x^2(\tau,x)dx + \frac{1}{2}\int_0^1 u_t^2(\tau,x)dx = 0. \tag{31}$$

Proceeding as before, we obtain

$$\|u\|_{E_2} = 0. \tag{32}$$

Thus, $u(t, x) = 0$ in E_2. So that $u_1(t, x) = u_2(t, x)$. □

Example 3. Let

$$\begin{cases} u_t - u_{xx} = 0, \\ u(0, x) = 0, \\ u_t(0, x) = e^x, \\ u_x(t, 0) = \sin t, \;\; u_x(t, 1) = e \sin t, \end{cases} \tag{33}$$

where the exact solution to this problem is given by $u(t, x) = e^x \sin t$. A simple computation yields $w(t) = (1-e)\sin t e^t + C_1 t + C_2$. Using $w(0) = \int_0^1 u(0, x)dx = 0$ and $w'(0) = \int_0^1 u_t(0, x)dx = e - 1$, thus

$$w(t) = k \sin t = \frac{(-1)^n k t^{(2n+1)}}{(2n+1)!}, \;\; n \geq 0, \;\; k = 1 - e,$$

$$c_{2n+1} = \frac{(-1)^n k}{(2n+1)!}, \;\; c_{2n} = 0, \;\; a_{2n} = 0, \;\; n \geq 0$$

and

$$a_{2n+1}(x) = \begin{cases} c_n \dfrac{\Theta(x)}{K}, \text{ if } K \neq 0, \\ c_n \left(1 + C\Theta(x)\right), \text{ if } K = 0, \end{cases}$$

where

$$a'_{2n}(0) = 0, \;\; a'_{2n+1}(0) = \frac{(-1)^n}{(2n+1)!}, \;\; n \geq 0, \;\; a'_{2n}(1) = 0, \;\; a'_{2n+1}(1) = \frac{(-1)^n e}{(2n+1)!}, \;\; n \geq 0.$$

If we choose $\Theta(x) = e^x$, then $a_{2n+1}(x) = \frac{(-1)^n e^x}{(2n+1)!}$, and the solution is given by

$$u(t, x) = \sum_{n=0}^{\infty} a_n(x) t^n = \sum_{n=0}^{\infty} \frac{(-1)^n e^x}{(2n+1)!} t^{2n+1} = e^x \sin t.$$

4. Initial-Neumann value problems for parabolic-hyperbolic equations

Consider the Neumann boundary value problem for linear parabolic- hyperbolic equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right) = h(t, x), \;\; 0 < x < 1, \;\; 0 < t \leq T, \tag{34}$$

$$u(0, x) = \phi(x), \;\; u_t(0, x) = \psi(x), \;\; u_{tt}(0, x) = \omega(x), \tag{35}$$

$$u_x(t, 0) = \alpha_1(t), \;\; u_x(t, 1) = \beta_1(t), \tag{36}$$

$$u_{xxx}(t, 0) = \alpha_2(t), \;\; u_{xxx}(t, 1) = \beta_2(t), \tag{37}$$

where $g \in L_2(D)$, $\phi \in L_2(0, 1)$, $\psi \in L_2(0, 1)$, $\omega \in L_2(0, 1)$, $\alpha_i \in L_2(0, T)$ and $\beta_i \in L_2(0, T)$, $i = 1, 2$.

This type of problem has been considered in [7], where the author has proved the existence and uniqueness of the generalized solutions, using energy inequality and the density of the range of the operator generated by the problem.

4.1. Existence. Proceeding as before, integrating both sides of Eq. (34) with respect to x from 0 to 1, we obtain

$$\int_0^1 \frac{\partial^3 u}{\partial t^3}(t,x)dx = \int_0^1 h(t,x)dx + \left[\beta_1'(t) - \alpha_1'(t)\right] + \left[\beta_1''(t) - \alpha_1''(t)\right] - \left[\beta_2(t) - \alpha_2(t)\right]. \quad (38)$$

Introducing a new function $\chi(t)$

$$\chi(t) = \int_0^1 u(t,x)dx, \quad (39)$$

where

$$\chi(0) = \int_0^1 \phi(x)dx,\ \chi'(0) = \int_0^1 \psi(x)dx \text{ and } \chi''(0) = \int_0^1 \omega(x)dx. \quad (40)$$

This transformation will carry out Eq. (39) to

$$\frac{d^3\chi}{dt^3} = H(t), \quad (41)$$

where

$$H(t) = \int_0^1 h(t,x)dx + \left[\beta_1'(t) - \alpha_1'(t)\right] + \left[\beta_1''(t) - \alpha_1''(t)\right] - \left[\beta_2(t) - \alpha_2(t)\right].$$

Thus problem (34)-(37) can be reduced to the ordinary linear differential equations (41) satisfied by the new variable χ and its solution can be found as

$$\chi(t) = \int_0^t \int_0^t \int_0^t H(t)dtdtdt + \frac{t^2}{2}\int_0^1 \omega(x)dx + t\int_0^1 \psi(x)dx + \int_0^1 \phi(x)dx. \quad (42)$$

Once Eq. (41) is solved, go back to the original dependent variable $u(t,x)$ via the Eq. (39). Proceeding as before, we can seek the solution $u(t,x)$ as $u(t,x) = \sum_{n=0}^{\infty} a_n(x)t^n$ and $\chi(t)$ be equated to an infinite series of polynomials $\chi(t) = \sum_{n=0}^{\infty} d_n t^n$. The substitution yields

$$\int_0^1 a_n(x)dx = d_n,\ n \geq 0. \quad (43)$$

Thus, we have

Theorem 4.1. *There exists a solution $u(t,x)$ of problem (34)-(37), which has the form of a sum as $u(t,x) = \sum_{n=0}^{\infty} a_n(x)t^n$, where the coefficients are given by*

$$a_n(x) = \begin{cases} d_n \dfrac{\Theta(x)}{K},\ \text{if } K \neq 0, \\ d_n\left(1 + C\Theta(x)\right),\ \text{if } K = 0 \end{cases} \quad (44)$$

such that

$$a_n'(0) = \alpha_{1,n},\ a_n'(1) = \beta_{1,n},\ n \geq 0 \quad \text{and} \quad a_n'''(0) = \alpha_{2,n},\ a_n'''(1) = \beta_{2,n},\ n \geq 0.$$

4.2. Uniqueness. In proving the uniqueness result we shall make use of

Lemma 4.2. *For* $v,\ v_t \in L_2(D^\tau)$, *we have*

$$\int_0^1 v^2(\tau,x)dx \leq k\left(\int_{D^\tau}(v_t)^2(\tau,x)dtdx + \int_{D^\tau} v^2(\tau,x)dtdx + \int_0^1 v^2(0,x)dx\right), \tag{45}$$

where $k > 0$.

Proof. We have

$$\int_0^\tau \frac{\partial}{\partial t}(v^2)dt = 2\int_0^\tau v_t v dt. \tag{46}$$

Thus

$$v^2(\tau,x) = 2\int_0^\tau v_t v dt + v^2(0,x). \tag{47}$$

So that

$$\int_0^1 v^2(\tau,x)dx = 2\int_{D^\tau} v_t v dt + \int_0^1 v^2(0,x)dx. \tag{48}$$

Applying the ϵ_1- inequality, we obtain

$$\int_0^1 v^2(\tau,x)dx \leq \frac{1}{\epsilon_1}\int_{D^\tau}(v_t)^2dtdx + \epsilon_1\int_{D^\tau} v^2dtdx + \int_0^1 v^2(0,x)dx,\ \epsilon_1 > 0, \tag{49}$$

or

$$\int_0^1 v^2(\tau,x)dx \leq k\left(\int_{D^\tau}(v_t)^2dtdx + \int_{D^\tau} v^2dtdx + \int_0^1 v^2(0,x)dx\right), \tag{50}$$

where $k = \max(\frac{1}{\epsilon_1}, \epsilon_1, 1)$. □

Theorem 4.3. *There exists at most one solution to problem (34)-(37) in* E_2.

Proof. The uniqueness of the solution to this problem can be proved via a substitution. Indeed, consider the new function $v(t,x)$ defined by

$$v(t,x) = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1,\ 0 < t \leq T. \tag{51}$$

Then problem (34)-(37) becomes

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = h(t,x),\ 0 < x < 1,\ 0 < t \leq T, \tag{52}$$

$$v(0,x) = \omega(x) - \phi''(x), \tag{53}$$

$$v_x(t,0) = \alpha_1''(t) - \alpha_2(t),\ v_x(t,1) = \beta_1''(t) - \beta_2(t). \tag{54}$$

For simplicity reasons, we will assume that the Neumann conditions (36)-(37) are homogeneous, that is, $u_x(t,0) = u_x(t,1) = 0$ and $u_{xxx}(t,0) = u_{xxx}(t,1) = 0$. So that $v_x(t,0) = v_x(t,1) = 0$.
Multiplying both sides of Eq. (52) by v_t, employing integration by parts over

D^τ, and taking into account the initial-Neumann conditions (53) and $v_x(t,0) = v_x(t,1) = 0$, we obtain

$$\int_{D^\tau} (v_t)^2(t,x)dtdx + \frac{1}{2}\int_0^1 (v_x)^2(\tau,x)dx = \int_{D^\tau} h(t,x)v_t dtdx + \frac{1}{2}\int_0^1 v'^2(0,x)dx. \quad (55)$$

Applying the $\epsilon-$ inequality to the second right side of (55), we obtain

$$\begin{aligned} &2\int_{D^\tau} (v_t)^2(t,x)dtdx + \int_0^1 (v_x)^2(\tau,x)dx \\ \leq &\epsilon_2 \int_{D^\tau} h^2 dtdx + \frac{1}{\epsilon_2}\int_{D^\tau} (v_t)^2 dtdx + \int_0^1 v'^2(0,x)dx, \end{aligned} \quad (56)$$

where $\epsilon_2 > 0$. Consequently,

$$(2 - \frac{1}{\epsilon_2})\int_{D^\tau} (v_t)^2(t,x)dtdx + \int_0^1 (v_x)^2(\tau,x)dx \leq \epsilon_2 \int_{D^\tau} h^2 dtdx + \int_0^1 v'^2(0,x)dx. \quad (57)$$

If we sum side to side (45) and (57) we obtain

$$\begin{aligned} &\int_0^1 v^2(\tau,x)dx + (2 - \frac{1}{\epsilon_2} - k)\int_{D^\tau} (v_t)^2(t,x)dtdx + \int_0^1 (v_x)^2(\tau,x)dx + \epsilon_2 \int_{D^\tau} h^2 dtdx \\ \leq &k\left(\int_{D^\tau} v^2(\tau,x)dtdx + \int_0^1 v^2(0,x)dx\right) + \int_0^1 v'^2(0,x)dx. \end{aligned} \quad (58)$$

We now apply the Gronwall's inequality to (58) and taking the upper bound with respect to τ in the interval $(0,T)$, we obtain

$$\|v\|_{E_1} \leq c_1 \left(\|h\|_{L_2(D)} + \|\xi\|_{L_2[0,1]} + \|\xi'\|_{L_2[0,1]}\right), \quad (59)$$

where $c_1 = \frac{\min(1, 2-\frac{1}{\epsilon_2}-k)}{\max(1,\epsilon_2,k)}$ and $\xi(x) = \omega(x) - \phi''(x)$.
We now go back to the initial-Neumann problem (51), where $u(0,x) = \phi$, $u_t(0,x) = \psi$ and $u_x(t,0) = u_x(t,1) = 0$.
Proceeding as before, we obtain

$$\|u\|_{E_2} \leq c_2 \left(\|v\|_{L_2(D)} + \|\phi\|_{L_2[0,1]} + \|\phi'\|_{L_2[0,1]} + \|\psi\|_{L_2[0,1]}\right), \ c_2 > 0. \quad (60)$$

This means that u is a unique solution to the given problem. This completes the proof. □

Example 4. Consider the initial-boundary value problem

$$\begin{cases} \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right) = 0, \\ u(0,x) = -x^4, \ u_t(0,x) = 0, \ u_{tt}(0,x) = 0, \\ u_x(t,0) = 0, \ u_x(t,1) = -4, \\ u_{xxx}(t,0) = 0, \ u_{xxx}(t,1) = -24. \end{cases} \quad (61)$$

We have to mention that we can change the inhomogeneous Neumann conditions to homogeneous conditions by using the transformation formula

$$w(t,x) = u(t,x) + x^4.$$

A simple computation yields $H(t) = -24$ and $\chi(t) = 4t^3 - \frac{1}{5}$. Thus

$$d_0 = -\frac{1}{5},\ d_1 = d_2 = 0,\ d_3 = 4,\ d_n = 0,\ n \geq 4,$$

$$a_1(x) = a_2(x) = 0,\ a_n(x) = 0,\ n \geq 4,$$

$$a_0(x) = \begin{cases} d_0 \dfrac{\Theta_1(x)}{K}, \text{ if } K \neq 0, \\ d_0\left(1 + C_1\Theta_1(x)\right), \text{ if } K = 0 \end{cases} \text{ and } a_3(x) = \begin{cases} d_3 \dfrac{\Theta_2(x)}{K}, \text{ if } K \neq 0, \\ d_3\left(1 + C_2\Theta_2(x)\right), \text{ if } K = 0 \end{cases}$$

where $\Theta_i(x),\ i = 1, 2$ are arbitrary functions.
The solution is then given by

$$u(t, x) = \sum_{n=0}^{\infty} a_n(x)t^n = a_0(x) + a_3(x)t^3.$$

Using the boundary condition $u(0, x) = -x^4$ we find $a_0(x) = -x^4$. In view of $a_3'(0) = a_3'(1) = 0$ and $a_3'''(0) = a_3'''(1) = 0$, we can choose $\Theta_2(x) = 1$, thus $a_3(x) = d_3 = 4$. Therefore, the solution is given by

$$u(t, x) = a_0(x) + a_3(x)t^3 = -x^4 + 4t^3,$$

which is the exact solution to this particular problem.

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