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ISOPERIMETRIC INEQUALITY IN α-PLANE

MIN SEONG KIM, IL SEOG KO AND BYUNG HAK KIM*

Abstract. Taxicab plane geometry and Cinese-Checker plane geometry are non-Euclidean and more practical notion than Euclidean geometry in the real world. The α-distance is a generalization of the Taxicab distance and Chinese-Checker distance. It was first introduced by Songlin Tian in 2005, and generalized to n-dimensional space by Ozcan Gelisgen in 2006. In this paper, we studied the isoperimetric inequality in α-plane.

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1. Introduction

The distance between two points in the Euclidean geometry is defined as the length of the segment between two points. Although it is the most popular distance function, it is not practical when we measure the distance which we actually move in the real world. To compensate this defect, the taxicab distance and Chinese-Checker distance were introduced. Taxicab distance [4] and Chinese-Checker distance [3] are the distance function similar to moving with a car or Chinese chess in the real world. As a generalization of these two distance functions, α-distance in the plane was introduced in 2005 by Songlin Tian [7]. In 2006, O.Gelisgen and R.Kaya [2] studied α-distance in an n-dimensional space. There are not so many results for the geometry with α-distance function in the contrary many papers for the taxicab geometry.

Isoperimetric inequality is a fundamental and important geometric concept. It is well known that equality holds if and only if the curve is a circle in Euclidean geometry, and equality holds if and only if the curve is a square in Taxicab geometry [5]. However, there is no research in the plane with α-distance funcion. Regarding this meaningful topic, we verified that an octagon holds equality of isoperimetric inequality in the plane with α-distance function.

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2. α-distance function in the plane

Let $A(x_1, y_1)$ and $B(x_2, y_2)$be two points in R^2. Denote
$\triangle_{AB}$ =max$\{|x_1 - x_2|, |y_1 - y_2|\}$ and δ_{AB} =min$\{|x_1 - x_2|, |y_1 - y_2|\}$.
Let $d_E(A, B)$, $d_T(A, B)$, $d_C(A, B)$ and $d_\alpha(A, B)$ be the Euclidean distance function, Taxicab distance function, Chinese-Checker distance function and α-distance function between two points A and B respectively. Then they are respectively given by [1, 3, 4, 6]

$$d_E(A, B) = \sqrt{\triangle_{AB}^2 + \delta_{AB}^2},$$
$$d_T(A, B) = \triangle_{AB} + \delta_{AB},$$
$$d_C(A, B) = \triangle_{AB} + (\sqrt{2} - 1)\delta_{AB},$$
$$d_\alpha(A, B) = \triangle_{AB} + (\sec\alpha - \tan\alpha)\delta_{AB} \ (\alpha \in [0, \tfrac{\pi}{4}]).$$

It follows that if $\delta_{AB} \neq 0$, then

$$d_T(A, B) \geq d_\alpha(A, B) \geq d_C(A, B) > d_E(A, B).$$

Because there has been a lot of discussions about isoperimetric inequality in the Taxicab geometry [5], we only explored the isoperimetric inequality in the plane with $\alpha-$ distance function when $\alpha \in (0, \frac{\pi}{4}]$. Through this paper, α-plane means that the plane with $\alpha-$distance function.

3. The shortest route between two points on α-plane

For the isoperimetric inequality in α-distance, we consider the shortest route between two points $A(0, 0)$ and $B(x, y)$ $(x, y \geq 0, x \geq y)$ on α-plane.

Furthermore, we study the locus of the shortest route between A and B. Before the proof of theorems, we prepared some definitions.

Definition 3.1. The length of a curve or segment in α-plane is called α-length

Definition 3.2. The difference between two x-coordinates of the end points of the segments is called horizontal length

Definition 3.3. The difference between two y-coordinates of the end points of the segments is called vertical length.

Theorem 3.4. *The shortest route between A and B can be written as monotone and continuous function.*

Proof. Assume that the shortest route between A and B which cannot be written as monotone function exists.

Split the curve into $n(n \to \infty)$ pieces by defining points $A_0, A_1, A_2, \cdots, A_{n-1}$, $A_n (A_0 = A, A_n = B)$ on the curve whose x-coordinates are $0, x_1, x_2, \cdots, x_{n-1}, x$ and y-coordinates are $0, y_1, y_2, \cdots, y_{n-1}, y$. Then, there exist i, j such that $x_i > x_j$ or $y_i > y_j (i > j)$. Let x_i be bigger than x_j. The curve and a line $x = x_j$ have at least one more intersection point except A_j. Let an intersection point be X.

Let α-length of the curve between X and A_j be d_α. Then

$$\begin{aligned} d_\alpha &\geq d_\alpha(X, A_i) + d_\alpha(A_i, A_j) \\ &> d_E(X, A_i) + d_E(A_i, A_j) \\ &> d_E(X, A_j) = d_\alpha(X, A_j). \end{aligned}$$

Another curve shorter than the curve exists, which is contradict to the assumption. Hence, the shortest route between A and B can be written as monotone and continuous function. □

Theorem 3.5. *The curve between A and B is the shortest route if and only if slope of the tangent segment on arbitrary point on the curve is from 0 to 1.*

Proof. Split the curve into $n(n \to \infty)$ pieces by defining points $A_0, A_1, A_2, \cdots, A_{n-1}, A_n(A_0 = A, A_n = B)$ on the curve whose x-coordinates are $0, x_1, x_2, \cdots, x_{n-1}, x$ and y-coordinates are $0, y_1, y_2, \cdots, y_{n-1}, y$. Then, the slope of the tangent segment at arbitrary point on the curve is equal to the slope of $\overline{A_iA_{i+1}}$ which contains the point. Hence it is sufficient to prove that curve between A and B is the shortest route if and only if the slope of arbitrary $\overline{A_iA_{i+1}}$ is from 0 to 1 for the proof of this theorem. Let us calculate the α-length of the curve which is monotone and the shortest route between A and B.

Because the curve is monotone, the slope of arbitrary $\overline{A_iA_{i+1}}$ is bigger than or equal to 0. Classify $\overline{AA_1}, \overline{A_1A_2}, \cdots, \overline{A_{n-1}B}$ into two sets, which one is the set whose elements$'$ slopes are smaller than or equal to 1, and the other is the set whose slopes are bigger than 1. In this chapter, the former is called set P, and the latter is called set Q.

Let h_p and h_q be the sum of horizontal length of the segments belong to P and Q respectively.

Let v_p and v_q be the sum of vertical length of the segments belong to P and Q respectively.

Then, the sum of α-length of the segments belong to P is $h_p + v_p(\text{sec}\alpha - \text{tan}\alpha)$, and the sum of α-length of the segments belong to Q is $v_q + h_q(\text{sec}\alpha - \text{tan}\alpha)$. Therefore, the α-length of the curve between A and B is $(h_p + v_q) + (h_q + v_p)(\text{sec}\alpha - \text{tan}\alpha)$ and the α-length of the shortest route between A and B is $(h_p + h_q) + (v_p + v_q)(\text{sec}\alpha - \text{tan}\alpha)$.

Since the curve is the shortest route, we see that

$$(h_p + v_q) + (h_q + v_p)(\sec\alpha - \tan\alpha) = (h_p + h_q) + (v_p + v_q)(\sec\alpha - \tan\alpha).$$

Therefore, $(h_q - v_q)(\text{sec}\alpha - \text{tan}\alpha - 1)$ is equal to 0. $\text{sec}\alpha - \text{tan}\alpha - 1$ is equal to 0 if and only if α is 0. Thus, h_q is equal to v_q. If h_q is bigger than 0, $h_q < v_q$ is obvious because of the definition of set Q. It is contradiction to h_q is equal to v_q. Hence, h_q is equal to 0. Then, v_q is equal to 0, too. Therefore, Q is an empty set. That is, there is no segment whose slope is bigger than 1 among $\overline{AA_1}, \overline{A_1A_2}, \cdots, \overline{A_{n-1}B}$. Thus, Curve between A and B is the shortest route if and only if the slope of arbitrary $\overline{A_iA_{i+1}}$ is from 0 to 1.

Curve between A and B is the shortest route if and only if slope of the tangent segment on arbitrary point on the curve is from 0 to 1. $\square$

Due to Theorems 3.4 and 3.5, locus of the shortest route between A and B is parallelogram.

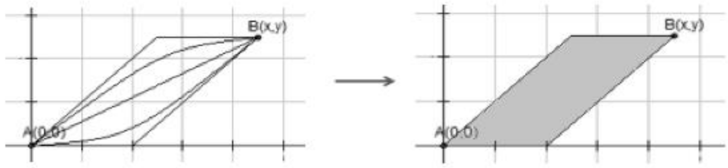

FIGURE 1. Locus of the shortest route between A and B

Locus of the shortest route between A and B is parallelogram whose internal angles are $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

4. Isoperimetric inequality in α-plane

In this chapter, we study the shape of closed curve which holds equality of isoperimetric inequality in α-plane. Before the proof of the isoperimetric inequality in α-plane, we introduce the definition of oval.

Definition 4.1. For every pair of points on the simple closed curve, if every point on the straight line segment that joins them is within the region, then the curve is called oval, and the curve which is not oval is called non-oval.

Since the isoperimetric inequality does not hold for a non-oval curve in α-plane, it is sufficient to only consider about the oval.

Draw the tangent line l and m whose slopes are 0 and 1 on the leftside and topside of an arbitrary oval curve C. Let the points of contact be A and H .

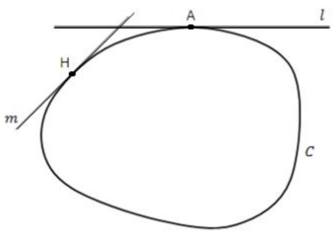

FIGURE 2. An oval curve C.

Let us think about the route to increase its area while the perimeter is constant by changing the curve between the points A and H. Exploring the characteristic of the curve between the point A and H is necessary.

Theorem 4.2. *The curve between the point A and H is the shortest route between the two points.*

Proof. Split the curve between H and A into $n(n \to \infty)$ pieces by defining points $A_0, A_1, A_2, \cdots, A_{n-1}, A_n$ $(A_0 = H, A_n = A)$ on the curve whose x-coordinates are $x_0, x_1, x_2, \cdots, x_{n-1}, x$ and y-coordinates are $0, y_1, y_2, \cdots, y_{n-1}, y$. Then, A_i $(i = 1, 2, \cdots, n-1)$ is under l and m , and above $\overline{AH}$.

Let the coordinates of the points H and A be $(0, 0)$ and (x, y). If x is bigger than y, then A is above the line m. It is a contradiction to A is under the line m. Thus, x is bigger than or equal to y.

If the slope of $\overline{HA_1}$ is bigger than 1, then it is contradict to A_1 is under m. If the slope of $\overline{HA_1}$ is less than 0, then it is contradict to A_1 is above $\overline{AH}$.

Hence, the slope of $\overline{HA_1}$ is from 0 to 1. Let us prove that if the slope of $\overline{A_{i-1}A_i}(i = 1, 2, \cdots, k)$ is from 0 to 1, then the slope of $\overline{A_kA_{k+1}}$ is also from 0 to 1.

If the slope of $\overline{A_kA_{k+1}}$ is bigger than 1, then $\overline{A_{k-1}A_{k+1}}$ is out of the curve C. It is contradict to the assumption that the curve C is an oval-curve. If the slope of $\overline{A_kA_{k+1}}$ is less than 0, then $\overline{A_kA}$ is out of the curve C. It is contradict to the assumption that the curve C is an oval-curve. Hence the slope of $\overline{A_kA_{k+1}}$ is from 0 to 1, so the slope of $\overline{A_0A_1}, \overline{A_1A_2}, \overline{A_2A_3}, \cdots \overline{A_{n-1}A}$ are all from 0 to 1.

Therefore, due to Theorem 3.5., the curve between the point A and H is the shortest route between the two points. □

Moreover we have

Theorem 4.3. *The shape of a curve which holds equality of isoperimetric inequality is an octagon whose angles are all $\frac{3}{4}\pi$.*

Proof. By the using the locus of the shortest route, the curve between A and H can be changed like as Figure 3. The thick curve increases the curve's area while the perimeter is constant.

FIGURE 3. Changing the curve between A and H.

By the same way, whole curve can be changed like Figure 4.

That is, about an arbitrary oval-curve, there is an octagon whose angles are all $\frac{3}{4}\pi$ and which has larger area while the perimeter is constant.

Therefore, shape of a curve which holds equality of isoperimetric inequality is octagon whose angles are all $\frac{3}{4}\pi$. □

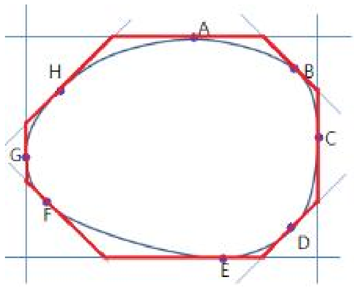

FIGURE 4. The changed curve.

Due to the Theorem 4.3., we only think about the octagon whose angles are all $\frac{3}{4}\pi$. Let α-length of the sides be $y_1, x_1, y_2, x_2, y_3, x_3, y_4$, and x_4, and a be $y_1 + y_2 + y_3 + y_4$ and b be $x_1 + x_2 + x_3 + x_4$. Let L be the perimeter of the octagon, S be the area and $k = \sec\alpha - \tan\alpha + 1$.

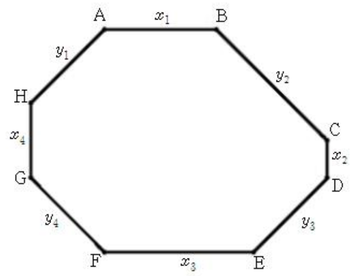

FIGURE 5. The octagon.

Theorem 4.4. *The isoperimetric inequality in α-plane is given by*

$$L^2 \geq 8(-k^2 + 4k - 2)S.$$

Proof. Since the width and the height of an octagon should be constant, we can obtain the following relations.

$$\frac{y_1 + y_2}{k} + x_1 = \frac{y_3 + y_4}{k} + x_3, \tag{1}$$

$$\frac{y_1 + y_4}{k} + x_4 = \frac{y_2 + y_3}{k} + x_2. \tag{2}$$

By (1) and (2), we have

$$\begin{aligned}
S &= (x_1 + \frac{y_1+y_2}{k})(x_4 + \frac{y_1+y_4}{k}) - \frac{y_1^2+y_2^2+y_3^2+y_4^2}{2k^2} \\
&= \frac{(a+k(x_1+x_3))(a+k(x_2+x_4))}{4k^2} - \frac{y_1^2+y_2^2+y_3^2+y_4^2}{2k^2} \\
&\le \frac{1}{4k^2}\left(a+\frac{kb}{2}\right)^2 - \frac{a^2}{8k^2} \\
&= \frac{2-4k+k^2}{16k^2}\left[\left(a-\frac{2Lk-Lk^2}{-k^2+4k-2}\right)^2 + \frac{L^2k^2}{2-4k+k^2} - \left(\frac{2Lk-Lk^2}{2-4k+k^2}\right)^2\right].
\end{aligned}$$

Since $k \in [\sqrt{2}, 2)$, it is obvious that $2-4k+k^2 < 0$ and $2Lk - Lk^2 > 0$.

If we apply the result above to the inequality for S, then we can see that S is convex- up and the x-coordinate of the vertex is bigger than or equal to 0. Therefore, S has its maximum at $a = \frac{2Lk-Lk^2}{-k^2+4k-2}$. In this case, we get

$$\begin{aligned}
S &\le \frac{2-4k+k^2}{16k^2}\left(\frac{L^2k^2}{2-4k+k^2} - \left(\frac{2Lk-Lk^2}{2-4k+k^2}\right)^2\right) \\
&= \frac{L^2}{8(-k^2+4k-2)}.
\end{aligned}$$

The equality holds if and only if the following three relations hold

$$x_1 + x_3 = x_2 + x_4 = \frac{b}{2},$$

$$y_1 = y_2 = y_3 = y_4 = \frac{a}{4},$$

$$a = \frac{2Lk-Lk^2}{-k^2+4k-2}.$$

By (1), (2) and upper results, we can see that x_1, x_2, x_3 and x_4 are all equal to $\frac{b}{4}$, $\frac{a}{4} = \frac{2Lk-Lk^2}{4(-k^2+4k-2)}$, and $\frac{b}{4} = \frac{2Lk-2L}{4(-k^2+4k-2)}$. Hence, the equality holds if and only if x_1, x_2, x_3 and x_4 are all $\frac{2Lk-2L}{4(-k^2+4k-2)}$, and y_1, y_2, y_3 and y_4 are all $\frac{2Lk-Lk^2}{4(-k^2+4k-2)}$. Hence the proof of the theorem is completed. $\square$

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Min Seong Kim is a student of Gyeonggi Science High School.

Gyeonggi Science High School, Suwon 440-800, Korea.
`e-mail: mskim1143@@naver.com`

Il Seog Ko received Ed.M. from Seoul National University. He is a Teacher of Gyeonggi Science High School. His research interests are Gifted Education, Discrete Mathematical Structures.

Gyeonggi Science High School, Suwon 440-800, Korea.
`e-mail: iskois@@naver.com`

Byung Hak Kim received Ph.D at Hiroshima University. His research area include differential geometry and global analysis.

Department of Applied Mathematics, Kyung Hee University, Yong-In 446-701, Korea.
`e-mail: bhkim@khu.ac.kr`