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NUMERICAL SOLUTION OF A CLASS OF THE NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we develop the operational Tau method for solving nonlinear Volterra integro-differential equations of the second kind. The existence and uniqueness of the problem is provided. Here, we show that the nonlinear system resulted from the operational Tau method has a semi triangular form, so it can be solved easily by the forward substitution method.
Finally, the accuracy of the method is verified by presenting some numerical computations.

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1. Introduction

Consider the nonlinear Volterra integro-differential equation (NVIDE)

$$Dy(x) + \int_0^x K(x,t,y(t))dt = f(x), \qquad x \in [0,b], \tag{1}$$

with m initial conditions

$$y^{(i)}(0) = \alpha_i, \quad i = 0,1,\ldots,m-1, \tag{2}$$

where m is the order of the differential operator D, f and K are given smooth functions and K is nonlinear in y.
The field of integral and integro-differential equations is a very important subject in applied mathematics, because mathematical formulation of many physical phenomena contains integro-differential equations. These equations also arises in many other fields like fluid dynamics, biological models and chemical kinetics [1].

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On the other hand, integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution [1]. Therefore, the numerical solution of such equations have been highly studied by many researchers. Khani and Shahmorad [2], used Adomian decomposition method to solve NVIDEs of the second kind. Khani et.al. [3], proposed a method for system of NVIDEs, in which corresponding unknown coefficients of the method are determined using computational aspects of matrices. Darania and Ebadian [4], developed the Taylor expansion approach for nonlinear integro-differential equations. The variational iteration method is applied for solving nonlinear integro-differential equations by Batiha, et.al. in Ref. [1].

In the recent years, the operational Tau method has been shown to be a successful technique, leading to reasonably simple algorithms for numerical solution of ordinary and partial differential equations. This method was introduced by Lanczos in 1938 and extended for getting numerical solution of ODEs by Ortiz [5]. In 1981, Ortiz and Samara studied the operational Tau method for solving nonlinear differential equations. The operational Tau method is employed to solve a system of nonlinear Volterra integro-differential equations with nonlinear differential part by Abbasbandy and Taati [6]. More recently, Bhrawy et.al [7] developed a direct solution technique for solving multi-order fractional differential equatipns with variable coefficients using a quadrature shifted Legendre Tau (Q-SLT) method. A Tau approach was developed for solving the space fractional diffusion equation by Karimi and Aminataei [8]. Ghoreishi and Yazdani [9] provided an efficient numerical approach for the fractional differential equations based on a spectral Tau method. EL-Daou [10] developed a method to solve a class of second-order ordinary differential equations with highly oscillatory solutions. Saadatmandi and Dehghan [11], presented approximation techniques based on the shifted Legendre-Tau idea to solve a class of initial-boundary value problems for the fractional diffusion equations with variable coefficients on a finite domain.

In this work, the operational Tau method is developed to present the numerical solution of Eq. (1). This leads to a semi lower triangular nonlinear system of equations which can be easily solved using a forward substitution method.

The rest of this paper is organized as follows:
In Section 2, we discuss about existence and uniqueness of the solution of the Eq. (1). In Section 3, we briefly describe Tau method for solving nonlinear Volterra integro-differential equations and formulate the problem. In Section 4, we give some examples to show the accuracy and efficiency of the presented method. Finally, a conclusion is given in Section 5.

2. Existence and uniqueness of the solution

In this section, the existence and uniqueness of the solution for Eq. (1) are presented. First we give the following theorem from [12].

Theorem 2.1. *Consider the system of equations*

$$y(t) = f(t) + \int_0^x K(x,t,y(t))dt. \tag{3}$$

Assume that
(i) $f(x)$ is continuous (i.e., every component is continuous),
(ii) $K(x,t,y)$ is a continuous function for $0 \le x \le t \le b$ and $-\infty \le \| y \| \le \infty$,
(iii) the kernel satisfies the Lipschitz condition

$$\|K(x,t,y_1) - K(x,t,y_2)\| \le L\|y_1 - y_2\| \tag{4}$$

wherer L is independent of x, t, y_1 and y_2. Then the Eq. (3) *has a unique continuous solution in $0 \le x \le b$.*

Now we consider some cases of the integro-differential equations and investigate existence and uniqueness of the solutions of them.

Corollary 2.2. *Consider the equations of the form*

$$y'(x) = f(x) + \int_0^x K\left(x,t,y(t)\right)dt, \tag{5}$$

with initial condition $y(0) = \alpha$ where f and K are continuous functions and K satisfies the Lipschitz condition

$$\| K(x,t,y_1) - K(x,t,y_2) \| \le L_1 \| y_1 - y_2 \| . \tag{6}$$

Then this problem has a unique continuous solution.

Proof. Replacing x by s in (5), leads to

$$y'(s) = f(s) + \int_0^s K\left(s,t,y(t)\right)dt \tag{7}$$

by integrating from (7) and using $y(0) = \alpha$, we obtain

$$y(x) = \alpha + \int_0^x f(s)ds + \int_0^x \int_0^s K\left(s,t,y(t)\right)dtds,$$

or

$$y(x) = \alpha + \int_0^x \left(f(s) + \int_0^s K(s,t,y(t))dt \right) ds.$$

Assuming $H(s,y(s)) = f(s) + \int_0^s K(s,t,y(t))dt$, gives

$$y(x) = \alpha + \int_0^x H(s,y(s))ds \tag{8}$$

which is in the form of Eq. (3), where obviously α and $H(s,y(s))$ are continuous. Therefore, for the existence and uniqueness of a continuous solution of the Eq. (5)

it is sufficient to show that Eq. (8) satisfies the Lipschitz condition. To this end, we have

$$\begin{aligned} \| \, H(s,y_1(s)) - H(s,y_2(s)) \, \| &= \left\| \int_0^s \left(K(s,t,y_1(t)) - K(s,t,y_2(t)) \right) dt \right\| \\ &\leq \int_0^s \| \, K(s,t,y_1(t)) - K(s,t,y_2(t)) \, \| \, dt \\ &\leq L_1 \, \| \, y_1 - y_2 \, \| \int_0^s dt \\ &\leq L_1 b \, \| \, y_1 - y_2 \, \| \, . \end{aligned}$$

So by Theorem 2.1, the Eq. (5) has a unique continuous solution. □

Corollary 2.3. *If in the equation*

$$y'(x) + cy(x) = f(x) + \int_0^x K(x,t,y(t))dt, \tag{9}$$

with initial condition $y(0) = \alpha$*, the functions* f *and* K *are continuous and* K *satisfies the Lipschitz condition* (6)*, then the equation* (9) *with given condition has a unique continuous solution.*

Proof. Similar to corollary 2.2, replacing x by s in (9), leads to

$$y'(s) = f(s) - cy(s) + \int_0^s K(s,t,y(t))dt, \tag{10}$$

by integrating from (10) and using $y(0) = \alpha$, we obtain

$$y(x) = \alpha + \int_0^x f(s)ds - c\int_0^x y(s)ds + \int_0^x \left(\int_0^s K(s,t,y(t))dt \right) ds.$$

Therefore,

$$y(x) = \alpha + \int_0^x \left\{ f(s) - cy(s) + \int_0^s K(s,t,y(t))dt \right\} ds$$

and by setting $H(s,y(s)) = f(s) - cy(s) + \int_0^s K(s,t,y(t))dt$, similar to the previous corollary we only investigate the Lipschitz condition. To this end, we have

$$\begin{aligned} \|H(s,y_1) - H(s,y_2)\| &= \left\| c\,[y_2(s) - y_1(s)] + \int_0^s [K(s,t,y_1(t) - K(s,t,y_2(t)]\, dt \right\| \\ &\leq | \, c \, | \, \|y_1 - y_2\| + \int_0^s L_1 \|y_1 - y_2\| dt \\ &\leq (| \, c \, | + bL_1)\|y_1 - y_2\|. \end{aligned}$$

Again, by Theorem 2.1, Eq. (9) has a unique continuous solution. □

In the following corollary we consider the NVIDEs of the second order.

Corollary 2.4. *Assume in the equation*

$$y''(x) + c_1 y'(x) + c_2 y(x) = f(x) + \int_0^x K(x,t,y(t))dt, \tag{11}$$

with initial conditions $y(0) = \alpha,\ y'(0) = \beta$*, the functions* f *and* K *are continuous functions and* K *satisfies the Lipschitz condition* (6)*. Then the mentioned problem has a unique continuous solution.*

Proof. With the same manner, replacing x by s in (11), gives

$$y''(s) + c_1 y'(s) + c_2 y(s) = f(s) + \int_0^s K(s,t,y(t))dt. \tag{12}$$

By two times integrating from (12) and using conditions $y(0) = \alpha,\ y'(0) = \beta$, we have

$$y(z) = \alpha + (\beta - c_1\alpha)z + \int_0^z \left\{ -c_1 y(x) + \int_0^x \left(f(s) - c_2 y(s) + \int_0^s K(s,t,y(s))dt \right) ds \right\} dx.$$

If we put

$$H(x, y(x)) = -c_1 y(x) + \int_0^x \left(f(s) - c_2 y(s) + \int_0^s K(s,t,y(s))dt \right) ds,$$

then we obtain

$$\begin{aligned}
&\|H(x,y_1(x)) - H(x,y_2(x))\| \\
&= \left\| c_1(y_2(x) - y_1(x)) + \int_0^x \left(c_2(y_2(s) - y_1(s)) + \int_0^s (K(s,t,y_1(s) - K(s,t,y_2(t))\, dt \right) ds \right\| \\
&\leq | \ c_1 \ | \ .\|y_1 - y_2\| + b \ | \ c_2 \ | \ .\|y_1 - y_2\| + \int_0^x \int_0^s L_1 \|y_1 - y_2\| dt ds \\
&\leq \left(| \ c_1 \ | + b \ | \ c_2 \ | + b^2 L_1 \right) \|y_1 - y_2\|.
\end{aligned}$$

Similar to previous cases, by Theorem 2.1 Eq. (11) has a unique continuous solution. □

The same conclusion can be drawn for the differential operator D in Eq. (1). For example we can consider the equation

$$y^{(n)}(x) = f(x) + \int_0^x k(x,t,y(t))dt$$

with conditions $y^{(i)}(0) = \alpha_i, i = 0, 1, \ldots, n-1$, and similar to the previous corollaries we can convert this problem to an equation of the form (1).

3. Formulation of the problem

Assume that the differential operator D in the NVIDE (1) has the following form

$$D := \sum_{r=0}^{m} p_r(x) \frac{d^r}{dx^r}, \qquad p_r(x) = \sum_{j=0}^{d_r} p_{r_j} x^j = \mathbf{p_r X}, \tag{13}$$

where d_r is degree of $p_r(x)$, $\mathbf{p_r} = (p_{r_0}, p_{r_1}, \ldots, p_{r_{d_r}}, 0, 0, \ldots)$ and $\mathbf{X} = (1, x, x^2, \ldots)^T$. To introduce the operational Tau method, we need to define the following basic matrices [13]

$$\mu = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \ \eta = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \ \iota = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which have interesting properties, as outlined below:
If $y(x) = \mathbf{aX}$, where $\mathbf{a} = (a_0, a_1, \ldots, a_n, 0, \ldots, 0)$, $\mathbf{X} = (1, x, x^2, \ldots)^T$, then

$$\frac{d}{dx} y(x) = \mathbf{a}\eta\mathbf{X}, \tag{14}$$

$$xy(x) = \mathbf{a}\mu\mathbf{X}, \tag{15}$$

$$\int_0^x y(x)dx = \mathbf{a}\iota\mathbf{X}. \tag{16}$$

To represent the differential part of the Eq. (1) in the matrix form, we need the following Theorem [14].

Theorem 3.1. *For any linear differential operator D defined by* (13)*, we have*

$$Dy(x) = \mathbf{a}\Pi_D\mathbf{X}, \tag{17}$$

where

$$\Pi_D = \sum_{i=0}^{m} \eta^i p_i(\mu). \tag{18}$$

To represent the initial conditions in matrix form, we introduce the vector $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{m-1}, 0, \ldots)$ and the diagonal matrix $S = (s_{ii})$ such that $s_{ii} = y^{(i)}(0)$ for $i = 0, 1, \ldots, m-1$. Then, the initial conditions (2) take the following form

$$\mathbf{a}S = \alpha. \tag{19}$$

To represent the integral part of the Eq. (1) in matrix form, one needs the following lemma and theorem [15].

Lemma 3.2. *Let*

$$y(x) = \sum_{i=0}^{\infty} a_i x^i = \mathbf{aX}$$

be a polynomial where $\mathbf{X} = [1, x, x^2, \ldots]^T$ is a standard basis vector and $\mathbf{a} = [a_0, a_1, a_2, \ldots]$, then for any $p \in N$, we have

$$y^p(x) = \mathbf{a}\mathbf{B}^{p-1}\mathbf{X},$$

where $\mathbf{B}$ is an infinite upper triangular Toeplitz matrix having the following structure

$$\mathbf{B} = \begin{bmatrix} a_0 & a_1 & a_2 & \dots \\ 0 & a_0 & a_1 & \dots \\ 0 & 0 & a_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Theorem 3.3. *Suppose that the analytic functions $y(x)$ and $k(x,t)$ can be expressed as:*

$$y(x) = \sum_{i=0}^{\infty} a_i x^i = \mathbf{aX}, \qquad K(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} k_{i,j} x^i t^j.$$

Then we have

$$\int_0^x K(x,t) y^p(t) dt = \mathbf{aB}^{p-1}\mathbf{MX},$$

where

$$\mathbf{M} = \begin{bmatrix} 0 & k_{0,0} & k_{0,1} + \frac{1}{2}k_{1,0} & k_{0,2} + \frac{1}{2}k_{1,1} + \frac{1}{3}k_{2,0} & \dots \\ 0 & 0 & \frac{1}{2}k_{0,0} & \frac{1}{2}k_{0,1} + \frac{1}{3}k_{1,0} & \dots \\ 0 & 0 & 0 & \frac{1}{3}k_{0,0} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & \dots & 0 & \frac{1}{n}k_{0,0} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Now, we apply the previous results for constructing the operational Tau approximate solution of Eq. (1).
The right hand side of Eq. (1) can be written as

$$f(x) = \sum_{i=0}^{\infty} f_i x^i = \mathbf{fX}, \tag{20}$$

where $\mathbf{f} = [f_0, f_1, \dots]$.
Using the given relations in theorems 3.1 and 3.3 and Eq. (20), Eq. (1) can be transformed to the following matrix form:

$$\mathbf{a\Pi_D X} + \mathbf{aB}^{p-1}\mathbf{MX} = \mathbf{fX},$$

which can be written as

$$\mathbf{a}(\mathbf{\Pi_D} + \mathbf{B^{p-1}M})X = \mathbf{fX}.$$

Therefore, we have

$$\mathbf{a}\left(\mathbf{\Pi_D} + \mathbf{B^{p-1}M}\right) = \mathbf{f}, \tag{21}$$

which is matrix representation of Eq. (1). It is clear that, the Tau matrix representation for the problem is a semi lower triangular nonlinear system of equations, which can be solved using a forward substitution method.

Let $\mathbf{\Pi}$ stand for the collected matrix $\mathbf{\Pi_D} + \mathbf{B^{p-1}M}$ and π_i for the i-th column of $\mathbf{\Pi}$. Then the coefficient vector $\mathbf{a}$ which is the exact solution of $y = \mathbf{aX}$ for problem (1) and (2), satisfies the following infinite algebraic system of equations

$$\begin{aligned} &\mathbf{a}S_j = \alpha_j, \qquad && j = 0, 1, \ldots, m-1, \\ &\mathbf{a}\pi_i = f_i, && i = 0, 1, \ldots, d_f, \\ &\mathbf{a}\pi_i = 0, && i \geq d_f + 1, \end{aligned} \tag{22}$$

where S_j is the j-th column vector of matrix S. Let us to introduce $G = (S_1, S_2, \ldots, S_m, \pi_0, \pi_1, \ldots)$ and $\delta = (\alpha, f, 0, 0, \ldots)$. Then the infinite algebraic system of equations (22) can be written as

$$\mathbf{a}G = \delta. \tag{23}$$

Definition 3.1. Let G_n is the matrix including the first $n+1$ rows and columns of G and δ_n is a vector containing the first $n+1$ elements of δ. Then $\mathbf{a}_n$ is the solution of the algebraic system of equations

$$\mathbf{a}_n G_n = \delta_n, \tag{24}$$

where $\mathbf{a}_n = [a_0, a_1, \ldots, a_n]$ and $\mathbf{X}_n = [1, x, x^2, \ldots, x^n]^T$. The polynomial $y_n = \mathbf{a}_n \mathbf{X}_n$ is called an operational Tau approximate solution of Eq. (23).

Remark 3.1. Since the matrix B contains unknowns $a_0, a_1, \ldots, a_n$, Eq. (21) and so Eq. (24) are nonlinear in $\mathbf{a} = [a_0, a_1, \ldots, a_n]$.

Remark 3.2. Since the presented method leads to a semi lower triangular nonlinear system of equations, can choose n arbitrary large to obtain the solution y_n with the desired accuracy and easily can be solved.

4. Numerical examples

To elucidate this presentation and test the accuracy of the presented method some example are investigated.

Example 1. Consider the following NVIDE [16]:

$$y'(x) + \int_0^x 3\cos(x-t) y^2(t) dt = 2\sin(x)\cos(x), \qquad x \in [0, 1], \tag{25}$$

with the initial condition $y(0) = 1$ whose exact solution is $y(x) = \cos(x)$.
By applying the operational Tau method (OTM) to this example one may obtain the following nonlinear system of equations

$$\begin{aligned} a_0 =& 1, \;\; a_1 = 0, \;\; a_2 = \frac{-3a_0}{2} + 1, \;\; a_3 = -\frac{a_1}{2}, a_4 = -\frac{a_0}{8} - \frac{a_2}{4} - \frac{1}{3} + \frac{3a_0^2}{8} \\ a_5 =& -\frac{a_1}{8} - \frac{3a_3}{20} + \frac{3a_0 a_1}{10}, \;\; a_6 = \frac{3a_0}{80} - \frac{11a_2}{120} - \frac{a_4}{10} + \frac{2}{45} + \frac{a_1^2}{20} + \frac{7a_2 a_0}{40} - \frac{3a_0^3}{80}, \\ a_7 =& \frac{3a_1}{112} - \frac{19a_3}{280} - \frac{a_5}{14} - \frac{27a_1 a_0^2}{560} + \frac{3a_0 a_1}{280} + \frac{3a_1 a_2}{56} + \frac{33a_3 a_0}{280}, \end{aligned}$$

$$a_8 = -\frac{41a_0}{13440} + \frac{131a_2}{6720} - \frac{29a_4}{560} - \frac{3a_6}{56} - \frac{1}{315} + \frac{3a_4a_0}{35} + \frac{13a_1^2}{2240} + \frac{39a_1a_3}{1120}$$
$$+ \frac{19a_2a_0}{2240} - \frac{3a_0a_1^2}{160} + \frac{3a_0^3}{4480} + \frac{3a_2^2}{224} - \frac{33a_2a_0^2}{1120} - \frac{9a_0^2}{4480} + \frac{9a_0^4}{4480},$$
$$\vdots$$

Thus

$$\mathbf{a} = \left[1, 0, -\frac{1}{2}, 0, \frac{1}{24}, 0, -\frac{1}{720}, 0, \frac{1}{40320}, \dots\right] = \left[1, 0, -\frac{1}{2!}, 0, \frac{1}{4!}, 0, -\frac{1}{6!}, 0, \frac{1}{8!}, \dots\right]$$

and the polynomial solution is

$$y_n(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \dots$$

which is exactly as accurate as the analytical solution. Computational results in Table 1 show that high accuracy is achieved for $n = 8$ in comparison to the absolute error in Ref. [16].

Table 1. Computational results of Example 1 for different n at some nodes.

n	8	10	12	8
x	*Error (OTM)*			*Error* [16]
0.0	0.000000	0.000000	0.000000	0.0000
0.1	0.275552e-16	0.208756e-20	0.114703e-24	0.1167e-2
0.2	0.282101e-13	0.854924e-17	0.187905e-20	0.1571e-2
0.3	0.162612e-11	0.110893e-14	0.548436e-18	0.1231e-2
0.4	0.288609e-10	0.349946e-13	0.307710e-16	0.22e-3
0.5	0.268605e-9	0.508987e-12	0.699390e-15	0.1338e-2
0.6	0.166175e-8	0.453544e-11	0.897548e-14	0.734e-3
0.7	0.775544e-8	0.288185e-10	0.776387e-13	0.766e-3
0.8	0.294465e-7	0.142961e-9	0.503146e-12	0.1302e-2
0.9	0.954994e-7	0.587006e-9	0.261530e-11	0.2176e-2
1.0	0.273497e-6	0.207625e-8	0.114231e-10	0.3189e-2

Example 2. As the second example consider the following NVIDE [17]:

$$y'(x) = -2\sin(x) - \frac{1}{3}\cos(x) - \frac{2}{3}\cos(2x) + \int_0^x \cos(x-t)y^2(t)dt \qquad (26)$$

for $x \in [0, 1]$ with the initial condition $y(0) = 1$, which has the exact solution $y(x) = \cos(x) - \sin(x)$. Using OTM, we obtain the following semi lower triangular nonlinear system of equations

$$\begin{aligned}
&a_0 = 1, \;\; a_1 = -1, \;\; a_2 = \frac{a_0}{2} - 1, \;\; a_3 = \frac{a_1}{6} - \frac{a_0}{6} + \frac{1}{2},\\
&a_4 = -\frac{a_0}{8} + \frac{a_2}{12} - \frac{a_1}{12} + \frac{1}{12} + \frac{a_0^2}{24},\\
&a_5 = -\frac{7a_1}{120} + \frac{a_3}{20} + \frac{a_0}{30} - \frac{a_2}{20} - \frac{11}{120} + \frac{a_1 a_0}{30} - \frac{a_0^2}{120},\\
&\quad\vdots
\end{aligned}$$

Thus

$$\mathbf{a} = \left[1, -1, -\frac{1}{2}, \frac{1}{6}, \frac{1}{24}, -\frac{1}{120}, \ldots\right] = \left[1, -1, -\frac{1}{2!}, \frac{1}{3!}, \frac{1}{41}, -\frac{1}{5!}, \ldots\right]$$

and the solution can be present as

$$y_n = \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots\right) - \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots\right),$$

which is exactly what we should expect. Table 2 shows a comparison between the OTM and the approximate solution that is given in Ref. [17] for $n = 8$.

TABLE 2. Computational results of Example 2 for different n at some nodes.

n	8	10	12	8
x		*Error (OTM)*		*Error* [17]
0.0	0.000000	0.000000	0.000000	0.10e-5
0.1	0.278304e-14	0.252592e-18	0.161730e-22	0.16e-4
0.2	0.143863e-11	0.521485e-15	0.133410e-18	0.26e-3
0.3	0.558228e-10	0.454624e-13	0.261408e-16	0.84e-4
0.4	0.750210e-9	0.108468e-11	0.110765e-14	0.94e-3
0.5	0.563868e-8	0.127219e-10	0.202794e-13	0.12e-4
0.6	0.293425e-7	0.952144e-10	0.218359e-12	0.28e-3
0.7	0.118465e-6	0.522628e-9	0.162996e-11	0.47e-4
0.8	0.397171e-6	0.228612e-8	0.930486e-11	0.11e-4
0.9	0.115530e-5	0.840796e-8	0.432782e-10	0.78e-4
1.0	0.300434e-5	0.269685e-7	0.171252e-9	0.81e-3

Example 3. Consider the following NVIDE [18]:

$$y'(x) = -1 + \int_0^x y^2(t)dt, \qquad x \in [0, 1] \tag{27}$$

with the initial condition $y(0) = 0$. Using OTM, we obtain the following nonlinear system of equations

$$a_0 = 0,\ \ a_1 = -1,\ \ a_2 = 0,\ \ a_3 = -\frac{a_0}{6},$$
$$a_4 = -\frac{a_1}{12},\ \ a_5 = -\frac{a_2}{20} - \frac{a_0^2}{120},\ \ a_6 = -\frac{a_3}{30} - \frac{a_1 a_0}{120},$$
$$\vdots$$

Thus

$$\mathbf{a} = [0, -1, 0, 0, \frac{1}{12}, 0, 0, \ldots].$$

TABLE 3. Computational results of Example 3 for different n at some nodes.

n		12	15	15
x	*Exact solution*	*OTM*	*OTM*	*WGM*
0.0000	0.00000	0.000000	0.000000	0.0000
0.0625	-0.06250	-0.062499	-0.062499	-0.0625
0.1250	-0.12498	-0.124980	-0.124980	-0.1250
0.1875	-0.18740	-0.187397	-0.187397	-0.1874
0.2500	-0.24967	-0.249675	-0.249675	-0.2497
0.3125	-0.31171	-0.311706	-0.311706	-0.3117
0.3750	-0.37336	-0.373356	-0.373356	-0.3734
0.4375	-0.43446	-0.434459	-0.434459	-0.4345
0.5000	-0.49482	-0.494822	-0.494822	-04948
0.5625	-0.55423	-0.554227	-0.554227	-0.5542
0.6250	-0.61243	-0.612431	-0.612431	-0.6124
0.6875	-0.66917	-0.669167	-0.669167	-0.6692
0.7500	-0.72415	-0.724153	-0.724153	-0.7242
0.8125	-0.77709	-0.777090	-0.777090	-0.7771
0.8750	-0.82767	-0.827666	-0.827667	-0.8277
0.9375	-0.87557	-0.875566	-0.875569	-0.8756
1.0000	-0.92048	-0.920469	-0.920476	-0.9205

Avudainayagam and Vani [18] solved this problem using Wavelet-Galerkin method (WGM) by $n = 15$. Table 3 contains a numerical comparison between OTM and the solution of (VGM).
The reported results of the proposed method and WGM for $n = 15$ show that both methods have produced nearly equivalent approximate solutions.

5. Conclusion

In this paper, we obtained an interesting form for the operational Tau representation of the NVIDEs with initial conditions. It was shown that, the NVIDEs with initial conditions can be converted to a semi lower triangular nonlinear system which has an important advantage that we can solve the problem with a desired accuracy. The numerical examples shown this truth.

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