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SOLUTION OF THE SYSTEM OF FOURTH ORDER BOUNDARY VALUE PROBLEM USING REPRODUCING KERNEL SPACE

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Abstract. In this paper, a general technique is proposed for solving a system of fourth-order boundary value problems. The solution is given in the form of series and its approximate solution is obtained by truncating the series. Advantages of the method are that the representation of exact solution is obtained in a new reproducing kernel Hilbert space and accuracy of numerical computation is higher. Numerical results show that the method employed in the paper is valid. Numerical evidence is presented to show the applicability and superiority of the new method..

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1. Introduction

In this paper, reproducing kernel space is applied to develop numerical method for obtaining smooth approximation to the solution of a system of fourth-order boundary-value problem of the form:

$$u^{(4)}(x) = \begin{cases} f(x), & a \le x \le c, \\ f(x) + u(x)g(x) + r, & c \le x \le d, \\ f(x), & d \le x \le b, \end{cases} \tag{1}$$

along with the boundary conditions

Case 1

$$u(a) = u(b) = \alpha_1,\ u^{(2)}(a) = u^{(2)}(b) = \alpha_2,\ u(c) = u(d) = \alpha_3,\ u^{(2)}(c) = u^{(2)}(d) = \alpha_4 \tag{2}$$

Case 2

$$u(a) = u(b) = \alpha_1,\ u^{(1)}(a) = u^{(1)}(b) = \alpha_5,\ u(c) = u(d) = \alpha_3,\ u^{(1)}(c) = u^{(1)}(d) = \alpha_6 \tag{3}$$

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where $f(x)$ and $g(x)$ are continuous functions on $[a, b]$ and $[c, d]$, respectively. The parameters $r,\ \alpha_i,\ i = 1, 2, ..., 6$ are real constants. Such type of systems have been used to study a wide class of odd order and nonsymmetric obstacle, unilateral, moving and equilibrium problems arising in various branches of pure and applied in a unified and general framework. During the past few years, this has emerged as an interesting and important branch of applied mathematics. In [1, 2, 3, 15] the solution of a system of second order boundary value problems associated with obstacle, unilateral and contact problems is developed using finite difference and spline techniques. There are also many research papers [7, 12, 13] for the solution of third order system of boundary value problem using finite difference and spline methods. In [8] Khalifa and Noor discussed the system of fourth order boundary value problem using quintic spline collocation method. Khan *et al*. [9] developed parametric quintic splines to derive some consistency relations to develop a numerical method for computing the solution of a system of fourth-order boundary-value problems associated with obstacle, unilateral, and contact problems. Momani *et al*. [11] applied decomposition method and a modified form of this method for the solution of a system of fourth-order boundary value problems. Siddiqi and Akram [16] used non polynomial spline for the solution of system of fourth order boundary value problem associated with obstacle, unilateral and contact problems. Al-Said and his coworkers [4, 5, 6] applied spline method for the solution of system of fourth order boundary value problem associated with obstacle, unilateral and contact problems. Recently, reproducing kernel Hilbert space method is used for constructing approximate solutions of boundary value problems [17, 18]

This paper is organized as follows: In Section 2, definition and a derivation of reproducing kernel is presented. The solution of the problem in reproducing kernel Hilbert space is given in Section 3. In Section 4, numerical results and comparison with other methods are presented.

2. Reproducing Kernel Spaces

(i) The reproducing kernel space $W_2^1[a_0, b_0]$ is defined by $W_2^1[a_0, b_0]= \{u(x)|\ u$ is absolutely continuous real valued function in $[a_0, b_0],\ u^{(1)} \in L^2[a_0, b_0]\}$ also the inner product and norm are defined by

$$\langle u(x), v(x)\rangle = \int_{a_0}^{b_0} (u(x)v(x) + u^{(1)}(x)v^{(1)}(x))dx,\ u(x), v(x) \in W_2^1[a_0, b_0] \quad (4)$$

$$\|u\| = \sqrt{\langle u(x), u(x)\rangle}, \quad u(x) \in W_2^1[a_0, b_0] \quad (5)$$

In [10], authors proved that $W_2^1[a_0, b_0]$ is a complete reproducing kernel space and its reproducing kernel is given by

$$Q_x(y) = \frac{1}{2\sinh(b_0 - a_0)}[\cosh(x + y - b_0 - a_0) + \cosh(|x - y| - b_0 + a_0)].$$

(ii) The reproducing kernel space $W_2^5[a_0, b_0]$ is defined by $W_2^5[a_0, b_0] = \{u(x)|$ $u^{(i)}$, $i = 0, 1, ..., 4$ are absolutely continuous real valued functions in $[a_0, b_0]$, $u^{(5)} \in L^2[a_0, b_0]\}$. The inner product and norm in $W_2^5[a_0, b_0]$ are given by

$$\begin{aligned}\langle u(x), v(x)\rangle = &\sum_{i=0}^{2} u^{(i)}(a_0)v^{(i)}(a_0) + \sum_{i=0}^{1} u^{(i)}(b_0)v^{(i)}(b_0) \\ &+ \int_{a_0}^{b_0} u^{(5)}(x)v^{(5)}(x)dx, \qquad u(x), v(x) \in W_2^5[a_0, b_0]\end{aligned} \tag{6}$$

$$\|u\| = \sqrt{\langle u(x), u(x)\rangle}\ , \quad u(x) \in W_2^5[a_0, b_0] \tag{7}$$

Construction of reproducing kernel: Reproducing kernel $R_x(y)$ of $W_2^5[a_0, b_0]$ is the requirement of the algorithm so, it can be obtained by the following method. Applying Eq. (6), gives

$$\langle u(y), R_x(y)\rangle = \sum_{i=0}^{2} u^{(i)}(a_0)R_x^{(i)}(a_0) + \sum_{i=0}^{1} u^{(i)}(b_0)R_x^{(i)}(b_0) + \int_{a_0}^{b_0} u^{(5)}(x)R_x^{(i)}(y)dy \tag{8}$$

Case 1 Conditions of case 1 and Eq. (8), gives

$$\begin{cases} R_x^{(1)}(a_0) - R_x^{(8)}(a_0) = 0,\ R_x^{(1)}(b_0) - R_x^{(8)}(b_0) = 0, \\ R_x^{(i)}(a_0) = 0,\ i = 0, 2, 5, 6,\ R_x^{(i)}(b_0) = 0,\ i = 0, 2, 5, 6 \end{cases} \tag{9}$$

Case 2 Conditions of case 2 and Eq. (8), gives

$$\begin{cases} R_x^{(2)}(a_0) - R_x^{(7)}(a_0) = 0,\ R_x^{(i)}(a_0) = 0,\ i = 0, 1, 5, 6 \\ R_x^{(i)}(b_0) = 0,\ i = 0, 1, 5, 6, 7 \end{cases} \tag{10}$$

then Eq. (8) also implies that

$$\langle u(y), R_x(y)\rangle = \int_{a_0}^{b_0} u(y)(-R_x^{(10)}(y))dy$$

For all $x \in [a_0, b_0]$, if $R_x(y)$ also satisfies

$$-R_x^{(10)}(y) = \delta(y - x) \tag{11}$$

then

$$\langle u(y), R_x(y)\rangle = u(x). \tag{12}$$

Eq. (12), shows $R_x(y)$ is reproducing kernel in $W_2^5[a_0, b_0]$, for any fixed $y \in [a_0, b_0]$ and any $u(x) \in W_2^5[a_0, b_0]$.

The characteristic equation of Eq. (11) is given by $\lambda^{10} = 0$. Then the characteristic values $\lambda = 0$ can be determined whose multiplicity is 10. The reproducing kernel $R_x(y)$ can be defined as

$$R_x(y) = \begin{cases} \sum_{i=1}^{10} c_i y^{i-1}, & y \le x. \\ \sum_{i=1}^{10} d_i y^{i-1}, & y > x. \end{cases} \tag{13}$$

and let $R_x(y)$ satisfies

$$R_x^{(k)}(x+a_0) = R_x^{(k)}(x-a_0),\ k = 0,1,...,8 \tag{14}$$

and

$$R_x^{(9)}(x-a_0) - R_x^{(9)}(x+a_0) = 1. \tag{15}$$

The conditions in Eq. (9) corresponding to system (1), for the case 1 can be considered, as

$$\begin{cases} R_x^{(1)}(a) - R_x^{(8)}(a) = 0,\ R_x^{(1)}(c) - R_x^{(8)}(c) = 0, \\ R_x^{(i)}(a) = 0,\ i = 0,2,5,6,\ R_x^{(i)}(c) = 0,\ i = 0,2,5,6 \end{cases} \tag{16}$$

$$\begin{cases} R_x^{(1)}(c) - R_x^{(8)}(c) = 0,\ R_x^{(1)}(d) - R_x^{(8)}(d) = 0, \\ R_x^{(i)}(c) = 0,\ i = 0,2,5,6,\ R_x^{(i)}(d) = 0,\ i = 0,2,5,6 \end{cases} \tag{17}$$

$$\begin{cases} R_x^{(1)}(d) - R_x^{(8)}(d) = 0,\ R_x^{(1)}(b) - R_x^{(8)}(b) = 0, \\ R_x^{(i)}(d) = 0,\ i = 0,2,5,6,\ R_x^{(i)}(b) = 0,\ i = 0,2,5,6 \end{cases} \tag{18}$$

The coefficients c_i and d_i $(i = 1,2,...,10)$ can be determined from Eqns. (14), (15) and (16) for the interval $[a,c]$. For the interval $[c,d]$, the coefficients c_i and d_i $(i = 1,2,...,10)$ can be determined from Eqns. (14), (15) and (17). The coefficients c_i and d_i $(i = 1,2,...,10)$ can be determined from Eqns. (14), (15) and (18) for the interval $[d,b]$.

The conditions in Eq. (10) corresponding to system (1), for the case 2 can be considered, as

$$\begin{cases} R_x^{(2)}(a) - R_x^{(7)}(a) = 0,\ R_x^{(i)}(a) = 0,\ i = 0,1,5,6 \\ R_x^{(i)}(c) = 0,\ i = 0,1,5,6,7 \end{cases} \tag{19}$$

$$\begin{cases} R_x^{(2)}(c) - R_x^{(7)}(c) = 0,\ R_x^{(i)}(c) = 0,\ i = 0,1,5,6 \\ R_x^{(i)}(d) = 0,\ i = 0,1,5,6,7 \end{cases} \tag{20}$$

$$\begin{cases} R_x^{(2)}(d) - R_x^{(7)}(d) = 0,\ R_x^{(i)}(d) = 0,\ i = 0,1,5,6 \\ R_x^{(i)}(b) = 0,\ i = 0,1,5,6,7 \end{cases} \tag{21}$$

The coefficients c_i and d_i $(i = 1,2,...,10)$ can be determined from Eqns. (14), (15) and (19) for the interval $[a,c]$. For the interval $[c,d]$, the coefficients c_i and d_i $(i = 1,2,...,10)$ can be determined from Eqns. (14), (15) and (20). The coefficients c_i and d_i $(i = 1,2,...,10)$ can be determined from Eqns. (14), (15) and (21) for the interval $[d,b]$.

In the following lemma it is proved that reproducing kernel is symmetric.

Lemma 2.1. $R_x(y) = R_y(x)$

Proof. By the reproducing property,

$$R_x(y) = \langle R_x(\delta), R_y(\delta)\rangle = \langle R_y(\delta), R_x(\delta)\rangle = R_y(x).$$

□

3. Exact and Approximate Solution

A bounded linear operator: The Eq. (1) can be written in the form

$$u^{(4)}(x) - H(x)u(x) = F(x) \tag{22}$$

where

$$H(x) = \begin{cases} 0, & a \le x \le c, \\ g(x), & c \le x \le d, \\ 0, & d \le x \le b, \end{cases} \tag{23}$$

and

$$F(x) = \begin{cases} f(x), & a \le x \le c, \\ f(x) + r, & c \le x \le d, \\ f(x), & d \le x \le b, \end{cases} \tag{24}$$

Let bounded linear operator $L : W_2^5[a_0, b_0] \to W_2^1[a_0, b_0]$ be defined as

$$(Lu)(x) = u^{(4)}(x) - H(x)u(x) \tag{25}$$

then transformed into equivalent operator equation

$$\begin{cases} (Lu)(x) = F(x) \\ u(a) = u(b) = \alpha_1,\ u^{(2)}(a) = u^{(2)}(b) = \alpha_2, \\ u(c) = u(d) = \alpha_3,\ u^{(2)}(c) = u^{(2)}(d) = \alpha_4 \\ \text{or} \\ u(a) = u(b) = \alpha_1,\ u^{(1)}(a) = u^{(1)}(b) = \alpha_2, \\ u(c) = u(d) = \alpha_5,\ u^{(1)}(c) = u^{(1)}(d) = \alpha_6 \end{cases} \tag{26}$$

Choose a countable dense subset $D = \{x_i\}_{i=1}^{\infty}$ in the domain $[a_0, b_0]$, and let

$$\varphi_i(x) = Q_{x_i}(y),\ i \in N \tag{27}$$

where $Q_{x_i}(y) \in W_2^1[a_0, b_0]$ is reproducing kernel of $W_2^1[a_0, b_0]$. Further assume that $\psi_i(x) = (L^*\varphi_i)(x)$, where $L^* : W_2^1[a_0, b_0] \to W_2^5[a_0, b_0]$ is the adjoint operator of L.

Theorem 3.1. *$\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete system of $W_2^5[a_0, b_0]$.*

Proof. For each fixed $u(x) \in W_2^5[a, b]$, let $\langle u(x), \psi_i(x)\rangle = 0$ $(i = 1, 2, \dots)$, which implies

$$\langle u(x), (L^*\varphi_i)(x)\rangle = \langle Lu(x), \varphi_i(x)\rangle = (Lu)(x_i) = 0\,. \tag{28}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense in $[a_0, b_0]$, so $(Lu)(x) = 0$, which implies $u \equiv 0$ from the existence of L^{-1}. □

Theorem 3.2. $\psi_i(x) = R_x^4(y)|_{y=x_i} - H(y)R_x(y)|_{y=x_i}$.

Proof. From Eq. (12), it can be written as

$$\begin{aligned}\psi_i(x) &= \langle \psi_i(y), R_x(y)\rangle = \langle (L^*\varphi_i)(x), R_x(y)\rangle = \langle \varphi_i(y), LR_x(y)\rangle \\ &= L_y R_x(y)|_{y=x_i} = R_x^4(y)|_{y=x_i} - H(y)R_x(y)|_{y=x_i}.\end{aligned}$$

□

To orthonormalize the sequence $\{\psi_i\}_{i=1}^{\infty}$ in the reproducing kernel space $W_2^5[a_0, b_0]$ Gram-Schmidt process can be used as

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad i = 1, 2, \dots \tag{29}$$

Theorem 3.3. *If $\{x_i\}_{i=1}^{\infty}$ is dense in $[a_0, b_0]$ and the solution of Eq. (26) is unique, for all $u(x) \in W_2^5[a_0, b_0]$, the series is convergent in the norm of $\|.\|_{W_2^5}$. If $u(x)$ is exact solution then the solution of Eq. (26) has the form, as*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x).$$

Proof. Since $u(x) \in W_2^5[a_0, b_0]$ and can be expanded in the form of Fourier series about normal orthogonal system $\{\psi_i\}_{i=1}^{\infty}$ as

$$u(x) = \sum_{i=1}^{\infty} \langle (u(x), \overline{\psi}_i(x)\rangle \overline{\psi}_i(x). \tag{30}$$

Since the space $W_2^5[a_0, b_0]$ is Hilbert space so the series $\sum_{i=1}^{\infty}\langle (u(x), \overline{\psi}_i(x)\rangle\overline{\psi}_i(x)$ is convergent in the norm of $\|.\|_{W_2^5}$. From Eqns. (29) and (30), it can be written

$$\begin{aligned}u(x) &= \sum_{i=1}^{\infty}\langle (u(x), \overline{\psi}_i(x)\rangle\overline{\psi}_i(x) = \sum_{i=1}^{\infty}\langle u(x), \sum_{k=1}^{i}\beta_{ik}\psi_k(x)\rangle\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x), \psi_k(x)\rangle\overline{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x), (L^*\varphi_k)(x)\rangle\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle Lu(x), \varphi_k(x)\rangle\overline{\psi}_i(x).\end{aligned}$$

If $u(x)$ is the exact solution of Eq. (26) and $Lu = f$, then

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle f(x), \varphi_k(x)\rangle\overline{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik} f(x_k)\overline{\psi}_i(x).$$

The approximate solution of $u(x)$ can be obtained in n-term of Fourier series by truncating the above equation, as

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik} f(x_k)\overline{\psi}_i(x)\,. \tag{31}$$

□

Theorem 3.4. *For each $u(x) \in W_2^5[a_0, b_0]$, and ε_n is the error between the approximate solution $u_n(x)$ and exact solution $u(x)$. Let $\varepsilon_n^2 = \|u(x) - u_n(x)\|^2$, then sequence $\{\varepsilon_n\}$ is monotone decreasing and $\varepsilon_n \to 0$ $(n \to \infty)$.*

Proof. Given

$$\varepsilon_n^2 = \|u(x) - u_n(x)\|^2 = \left\| \sum_{i=n+1}^{\infty} \langle u(x), \overline{\psi}_i(x)\rangle \overline{\psi}_i(x) \right\|^2 = \sum_{i=n+1}^{\infty} (\langle u(x), \overline{\psi}_i(x)\rangle)^2.$$

Similarly,

$$\varepsilon_{n-1}^2 = \|u(x) - u_{n-1}(x)\|^2 = \sum_{i=n}^{\infty} (\langle u(x), \overline{\psi}_i(x)\rangle)^2.$$

Clearly $\varepsilon_{n-1} \geq \varepsilon_n$. $\{\varepsilon_n\}$ is monotone decreasing and from Theorem 3.4, it is noted that Eq. (31) is convergent in the norm of $\|.\|_{W_2^5}$ *i.e.* $\varepsilon_n \to 0$ $(n \to \infty)$. □

4. Numerical Examples

In order to test the utility of the proposed method, two examples are considered in this section. All the computation are performed using Mathematica 5.2.

Example 4.1. Consider the following system of differential equation [16]

$$u^{(4)}(x) = \begin{cases} 1, & -1 \leq x \leq -\frac{1}{2},\ \frac{1}{2} \leq x \leq 1, \\ 2 - 4u, & -\frac{1}{2} \leq x \leq \frac{1}{2} \end{cases}$$

with the boundary conditions corresponding to the case 1. The exact solution for the above problem is

$$u(x) = \begin{cases} \frac{1}{24}x^4 + \frac{1}{8}x^3 + \frac{13}{96}x^2 + \frac{1}{16}x + \frac{1}{96}, \\ u(-1) = u(-\frac{1}{2}) = 0,\ u^{(1)}(-1) = u^{(1)}(-\frac{1}{2}) = 0,\ -1 \leq x \leq -\frac{1}{2}, \\ (\frac{1}{2}\beta_1)[\beta_1 - \exp(\frac{1}{2} + x)(\beta_2 + \exp(1))\beta_3 - \exp(\frac{1}{2} - x)(\beta_4 + \exp(1)\beta_5)], \\ u(-1/2) = u(\frac{1}{2}) = 0,\ u^{(1)}(-\frac{1}{2}) = u^{(1)}(\frac{1}{2}) = 0,\ -\frac{1}{2} \leq x \leq \frac{1}{2}, \\ \frac{1}{24}x^4 + \frac{1}{8}x^3 + \frac{13}{96}x^2 + \frac{1}{16}x + \frac{1}{96}, \\ u(\frac{1}{2}) = u(1) = 0,\ u^{(1)}(\frac{1}{2}) = u^{(1)}(1) = 0,\ \frac{1}{2} \leq x \leq 1, \end{cases}$$

where $\beta_1 = \exp(2) - 1 + 2\exp(1)\sin(1)$, $\beta_2 = \sin(\frac{1}{2} + x) - \cos(\frac{1}{2} + x)$, $\beta_3 = \sin(\frac{1}{2} - x) + \cos(\frac{1}{2} - x)$, $\beta_4 = \sin(\frac{1}{2} - x) - \cos(\frac{1}{2} - x)$ and $\beta_5 = \sin(\frac{1}{2} + x) + \cos(\frac{1}{2} + x)$. The maximum absolute error in the solution is summarized in Table 1 to compare with other method in [16] .

Example 4.2. Consider the following system of differential equation [4, 5, 6]

$$u^{(4)}(x) = \begin{cases} 1, & -1 \leq x \leq -\frac{1}{2},\ \frac{1}{2} \leq x \leq 1, \\ 2 - 4u, & -\frac{1}{2} \leq x \leq \frac{1}{2} \end{cases}$$

Table 1. Maximum absolute error for Example 4.1

h	present method	[16](For $\alpha=\frac{1}{120},\beta=\frac{13}{60}$ $\gamma=\frac{11}{20}$)	[16](For $\alpha=\frac{1}{280},\beta=\frac{26}{280}$, $\gamma=\frac{226}{280}$)
1/12	2.59×10^{-7}	7.65×10^{-7}	5.61×10^{-7}
1/24	6.65×10^{-8}	1.14×10^{-7}	7.81×10^{-8}
1/48	1.67×10^{-8}	1.86×10^{-8}	1.32×10^{-8}

Table 2. Maximum absolute error for Example 4.2

h	Present method	[6]	[5]	[4]
1/12	3.57×10^{-6}	6.8×10^{-6}	7.8×10^{-6}	$1,2\times10^{-5}$
1/24	1.11×10^{-6}	1.6×10^{-6}	1.9×10^{-6}	2.8×10^{-6}
1/48	3.07×10^{-7}	4.2×10^{-7}	4.9×10^{-7}	6.9×10^{-7}

with the boundary conditions corresponding to the case 2. The exact solution for the above problem is

$$u(x)=\begin{cases}\frac{1}{24}x^4+\frac{1}{8}x^3+\frac{1}{8}x^2+\frac{3}{64}x+\frac{1}{192},\\ u(-1)=u(-\frac{1}{2})=0,\ u^{(2)}(-1)=u^{(2)}(-\frac{1}{2})=0,\ -1\le x\le -\frac{1}{2},\\ \frac{1}{2}-\frac{1}{\phi_3}[\phi_1\sin x\sinh x+\phi_2\cos x\cosh x],\\ u(-\frac{1}{2})=u(\frac{1}{2})=0,\ u^{(2)}(-\frac{1}{2})=u^{(2)}(\frac{1}{2})=0,\ -\frac{1}{2}\le x\le \frac{1}{2},\\ \frac{1}{24}x^4-\frac{1}{8}x^3+\frac{1}{8}x^2-\frac{3}{64}x+\frac{1}{192},\\ u(\frac{1}{2})=u(1)=0,\ u^{(2)}(\frac{1}{2})=u^{(2)}(1)=0,\ \frac{1}{2}\le x\le 1,\end{cases}$$

where $\phi_1=\sin(\frac{1}{2})\sinh(\frac{1}{2})$, $\phi_2=\cos(\frac{1}{2})\cosh(\frac{1}{2})$ and $\phi_3=\cos 1+\cosh 1$. The maximum absolute error in the solution is summarized in Table 2 to compare with other methods in [4, 5, 6] .

5. Conclusion

In this paper, an iterative RKHSM is used to find the approximate solution of the fourth order boundary value problems in the reproducing kernel space. It is noted that approximate solution obtained by present method converges to the exact solution. The numerical comparison of the method with other methods is shown in Tables 1, 2. It is clear that the present method gives better results than the methods developed in [4, 5, 6, 16].

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