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APPLICATION OF CONVOLUTION SUM $\sum_{k=1}^{N-1}\sigma_1(k)\sigma_1(2^nN-2^nk)$

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Abstract. Let $S^{\pm}_{(n,k)} := \{(a,b,x,y) \in \mathbb{N}^4 : ax+by = n, x \equiv \pm y \pmod{k}\}$. From the formula $\sum_{(a,b,x,y)\in S^{\pm}_{(n,k)}} ab = 4\sum_{\substack{m\in\mathbb{N}\\ m<n/k}} \sigma_1(m)\sigma_1(n-km) + \frac{1}{6}\sigma_3(n) - \frac{1}{6}\sigma_1(n) - \sigma_3(\frac{n}{k}) + n\sigma_1(\frac{n}{k})$, we find the Diophantine solutions for modulo $2^{m'}$ and $3^{m'}$, where $m' \in \mathbb{N}$.

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1. Introduction

We defined a divisor function, which is applied in many areas of number theory, as

$$\sigma_s(N) = \sum_{d|N} d^s.$$

A Diophantine equation is an algebraic equation with the constraint that only integers are allowed as variables. In general, it is extremely difficult to solve Diophantine equations, even though many approaches have been adopted. Fermat's Last Theorem is a well-known Diophantine equation that remained unsolved for over 350 years.

In Section 2, we determine $h_m(2^nN)$ as follows: Let m $(0 \le m \le n)$ be any integer and $n \in \mathbb{N}\cup\{0\}$, where $N \in \mathbb{N}$. Then, we obtain

$$\begin{aligned} h_m(2^nN) &:= \sum_{(a,b,x,y)\in S^{\pm}_{(2^nN,2^m)}} ab \\ &= \frac{1}{6}[2(2^m-1)\sigma_3(2^{n-m+1}N) - 2(2^m-3)\sigma_3(2^{n-m}N) + \sigma_3(2^nN) \end{aligned}$$

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$$+ (2^m - 1)(1 - 3 \cdot 2^{n-m+1}N)\sigma_1(2^{n-m+1}N)$$
$$- \{2^m - 3 + 3 \cdot 2^{n+1}N(2^{-m+1} - 1)\}\sigma_1(2^{n-m}N) - \sigma_1(2^nN)]$$

(see Corollary 2.3 below). Similarly, in Section 3, we have

$$\begin{aligned} k_m(3^nN) &:= \sum_{(a,b,x,y)\in S^{\pm}_{(3^nN,3^m)}} ab \\ &= \frac{1}{12}[(3^m - 1)\sigma_3(3^{n-m+1}N) - (3^m - 9)\sigma_3(3^{n-m}N) + 2\sigma_3(3^nN) \\ &\quad - (3^m - 1)(2 \cdot 3^{n-m+1}N - 1)\sigma_1(3^{n-m+1}N) \\ &\quad - \{3^m - 5 + 2 \cdot 3^{n+1}N(3^{-m+1} - 1)\}\sigma_1(3^{n-m}N) - 2\sigma_1(3^nN)] \end{aligned}$$

(see Corollary 3.1 below).

2. Application of convolution sum $\sum_{k=1}^{N-1} \sigma_1(k)\sigma_1(2^nN - 2^nk)$

We deduce a formula after directly computing $\sum_{(a,b,x,y)\in S^{\pm}_{(4,1)}} ab$ and $\sum_{(a,b,x,y)\in S^{\pm}_{(8,2)}} ab$ by numbering. First, for $\sum_{(a,b,x,y)\in S^{\pm}_{(4,1)}} ab$, since $a, b, x, y \in \mathbb{N}$, a, b can be an element of the set $\{1,2,3\}$. For such a, b, we can create ordered pairs and obtain feasible (x, y) satisfying $x \equiv \pm y \pmod 1$, as listed in Table 1.

Table 1. $\sum_{(a,b,x,y)\in S^{\pm}_{(4,1)}} ab$

(a, b)	(x, y)	$\sum ab$
$(1, 1)$	$(1, 3), (2, 2), (3, 1)$	6
$(1, 2)$	$(2, 1)$	4
$(1, 3)$	$(1, 1)$	6
$(2, 1)$	$(1, 2)$	4
$(2, 2)$	$(1, 1)$	8
$(3, 1)$	$(1, 1)$	6

From Table 1, we get

$$\sum_{(a,b,x,y)\in S^{\pm}_{(4,1)}} ab = 6 + 4 + 6 + 4 + 8 + 6 = 34.$$

Similarly, we can construct a table of possible (x, y) for $\sum_{(a,b,x,y)\in S^{\pm}_{(8,2)}} ab$ (see Table 2). Therefore $\sum_{(a,b,x,y)\in S^{\pm}_{(8,2)}} ab = 14 + 4 + \cdots + 24 + 14 = 258$. To obtain

Table 2. $\sum_{(a,b,x,y)\in S^{\pm}_{(8,2)}} ab$

(a,b)	(x,y)	$\sum ab$
$(1,1)$	$(1,7),(2,6),(3,5),(4,4),(5,3),(6,2),(7,1)$	14
$(1,2)$	$(4,2)$	4
$(1,3)$	$(5,1),(2,2)$	12
$(1,5)$	$(3,1)$	10
$(1,7)$	$(1,1)$	14
$(2,1)$	$(2,4)$	4
$(2,2)$	$(1,3),(2,2),(3,1)$	24
$(2,6)$	$(1,1)$	24
$(3,1)$	$(1,5),(2,2)$	12
$(3,5)$	$(1,1)$	30
$(4,4)$	$(1,1)$	32
$(5,1)$	$(1,3)$	10
$(5,3)$	$(1,1)$	30
$(6,2)$	$(1,1)$	24
$(7,1)$	$(1,1)$	14

a formula related to these summations, we can see that

$$\begin{aligned}\sum_{\substack{m\in\mathbb{N}\\ m<n/k}} \sigma_1(m)\sigma_1(n-km) = &-\frac{1}{24}\sigma_3(n)+\frac{1}{24}\sigma_1(n)+\frac{1}{4}\sigma_3(\frac{n}{k})-\frac{n}{4}\sigma_1(\frac{n}{k})\\ &+\frac{1}{4}\sum_{(a,b,x,y)\in S^{+}_{(n,k)}} ab+\frac{1}{4}\sum_{(a,b,x,y)\in S^{-}_{(n,k)}} ab \end{aligned} \tag{1}$$

in [1], [4, p 156]. Where $S^{+}_{n,k}=\{(a,b,x,y)\in\mathbb{N}^4 : ax+by=n, x\equiv y \pmod k\}$ and $S^{-}_{n,k}=\{(a,b,x,y)\in\mathbb{N}^4 : ax+by=n, x\equiv -y \pmod k\}$ As an application, we have the following corollary.

Corollary 2.1. *Let $n,N\in\mathbb{N}$. Then, we obtain*

$$\begin{aligned} h_{n-1}(2^nN) &:= \sum_{(a,b,x,y)\in S^{\pm}_{(2^nN,2^{n-1})}} ab\\ &= \frac{1}{12}[2\{\sigma_3(2^nN)+2^n\sigma_3(4N)-2\sigma_3(4N)-(2^n-6)\sigma_3(2N)-\sigma_1(2^nN)\}\\ &\quad -(2^n-2)(12N-1)\sigma_1(4N)+\{12(2^n-4)N-2^n+6\}\sigma_1(2N)]. \end{aligned}$$

In particular, for odd and prime N we can construct Table 3.

Proof. By substituting m with k, n with 2^nN, and k with 2^{n-1} in equation (1), and thus, by using

TABLE 3. Formula $h_{n-1}(2^nN)$ for odd and prime N

N	$\sum_{(a,b,x,y)\in S^{\pm}_{(2^nN,2^{n-1})}} ab$
odd number	$\frac{1}{42}[(8^{n+1}+7\cdot 2^{n+6}-645)\sigma_3(N)-21\{(2^{n+3}-4)N-1\}\sigma_1(N)]$
odd prime	$\frac{1}{42}(N+1)(8^{n+1}N^2+7\cdot 2^{n+6}N^2-645N^2-8^{n+1}N-77\cdot 2^{n+3}N$ $+729N+8^{n+1}+7\cdot 2^{n+6}-624)$

$$\begin{aligned}
&\sum_{k=1}^{N-1}\sigma_1(2^mk)\sigma_1(2^n(N-k))\\
&=\frac{1}{24}[(3\cdot 2^{n+m}-2^m-2^n-1)\sigma_3(2N)-(3\cdot 2^{n+m}-2^m-2^n-11)\sigma_3(N)\\
&\quad+\{2^n+2^m-2-6(2^{n+m+1}-2^m-2^n)N\}\sigma_1(2N)\\
&\quad-\{2^n+2^m-4-12(2^{n+m}-2^m-2^n)N\}\sigma_1(N)]
\end{aligned}\tag{2}$$

in [3, Table 1], we obtain the proof. Finally, we replace m with 0, n with $n-1$, and N with $2N$ in Eq. (2). □

For $n\in\mathbb{N}\cup\{0\}$ and $N\in\mathbb{N}$, let M_{2^nN} be the number of solutions (a,b,x,y) of the diophantine equations

$$ax+by=2^nN \text{ and } x\equiv \pm y \pmod{2^n}$$

where a, b, x, $y\in\mathbb{N}$.

Example 2.2. Some values for $h_{n-1}(2^nN)$ are listed in Table 4.

TABLE 4. Some values of $h_{n-1}(2^nN)$

n \ N	2	3	4	5	6	7	8	9
1	34	140	370	768	1372	2248	3410	4982
2	258	940	2290	4560	8156	12808	19282	28270
3	1218	4332	10226	20208	36060	55944	83794	123294
4	7234	25452	58866	116016	206556	318344	474962	700926

Let us generalize $h_m(2^nN)$ in the following corollary.

Corollary 2.3. *Let m $(0\le m\le n)$ be any integer and $n\in\mathbb{N}\cup\{0\}$, where $N\in\mathbb{N}$. Then, we obtain*

$$\begin{aligned}
h_m(2^nN) &:= \sum_{(a,b,x,y)\in S^{\pm}_{(2^nN,2^m)}} ab\\
&=\frac{1}{6}[2(2^m-1)\sigma_3(2^{n-m+1}N)-2(2^m-3)\sigma_3(2^{n-m}N)+\sigma_3(2^nN)
\end{aligned}$$

$$+ (2^m - 1)(1 - 3 \cdot 2^{n-m+1}N)\sigma_1(2^{n-m+1}N)$$
$$- \{2^m - 3 + 3 \cdot 2^{n+1}N(2^{-m+1} - 1)\}\sigma_1(2^{n-m}N) - \sigma_1(2^nN)].$$

In particular, for odd and prime N we can construct Table 5.

TABLE 5. Formula $h_m(2^nN)$ for odd and prime N

N	$\sum_{(a,b,x,y)\in S^{\pm}_{(2^nN,2^m)}} ab$
odd number	$\frac{1}{42\cdot 2^{3m}}[(8^{n+m+1} - 5\cdot 8^m + 7\cdot 2^{3n+m+4} - 5\cdot 2^{3n+4})\sigma_3(N) - 7\{8^m - 2^{n+2m+1} + 3\cdot 2^{n+2m+1}(2^{n+1} - 1)N\}\sigma_1(N)]$
odd prime	$\frac{1}{42\cdot 2^{3m}}(N+1)(8^{n+m+1}N^2 - 5\cdot 8^mN^2 + 7\cdot 2^{3n+m+4}N^2 - 5\cdot 2^{3n+4}N^2 - 8^{n+m+1}N + 5\cdot 8^mN - 7\cdot 2^{3n+m+4}N + 5\cdot 2^{3n+4}N - 21\cdot 2^{2n+2m+2}N + 21\cdot 2^{n+2m+1}N + 8^{n+m+1} + 7\cdot 2^{3n+m+4} - 5\cdot 2^{3n+4} + 7\cdot 2^{n+2m+1} - 3\cdot 2^{3m+2})$

Proof. By substituting m with k, n with 2^nN, and k with 2^m in equation (1), and thus, by using (2), we can obtain the proof. □

Remark 2.1. We can validate Corollary 2.1 from Corollary 2.3 by putting $m = n - 1$.

Example 2.4. Figure 1 shows $h_m(2^nN)$ with $N = 3$ and $n = 7$ (left panel), and with $N = 5$ and $n = 10$ (right panel).

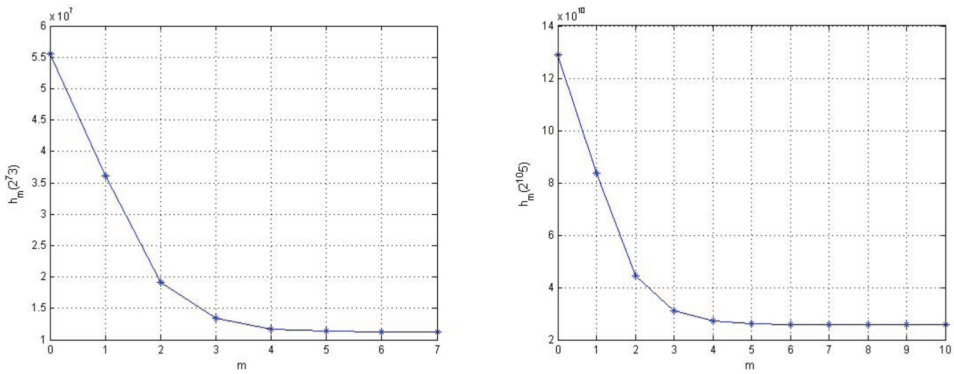

FIGURE 1. Graphs for $h_m(2^nN)$ with $N = 3$ and $n = 7$, and with $N = 5$ and $n = 10$

Lemma 2.5. *Let us consider the difference of $h_m(2^nN)$ and $h_n(2^nN)$.*

$$h_m(2^nN) - h_n(2^nN) = \sum_{(a,b,x,y)\in S^{\pm}_{(2^nN,2^m)}} ab - \sum_{(a,b,x,y)\in S^{\pm}_{(2^nN,2^n)}} ab$$
$$= \frac{1}{6}[2(2^m-1)\sigma_3(2^{n-m+1}N) - 2(2^m-3)\sigma_3(2^{n-m}N)$$
$$- 2(2^n-1)\sigma_3(2N) + 2(2^n-3)\sigma_3(N)$$
$$- (2^m-1)(3\cdot 2^{n-m+1}N-1)\sigma_1(2^{n-m+1}N)$$
$$- \{2^m-3-3\cdot 2^{n-m+1}(2^m-2)N\}\sigma_1(2^{n-m}N)$$
$$+ (2^n-1)(6N-1)\sigma_1(2N) - \{3-2^n+6(2^n-2)N\}\sigma_1(N)].$$

Remark 2.2. The value of $h_m(2^nN) - h_n(2^nN)$ implies that the solutions $x \equiv \pm y \pmod M$ of the Diophantine equation $ax + by = 2^nN$ are congruent modulo $M = 2^m$ but 2^n. And the value of $h_m(2^nN) - h_n(2^nN)$ is helpful to give us the information which shows how many different solutions existing by modulo M in Eq. $ax + by = 2^nN$. Specifically, we can find non-overlapping points according to $h_m(2^nN) - h_n(2^nN)$.

Example 2.6. We can confirm Lemma 2.5 by giving instances as follows. From Table 6, we get $\sum_{(a,b,x,y)\in S^{\pm}_{(4,1)}} ab - \sum_{(a,b,x,y)\in S^{\pm}_{(4,2)}} ab = 8$. From Table 7, we

TABLE 6. $h_0(2^1\cdot 2)$ and $h_1(2^1\cdot 2)$

(a,b)	(x,y)	Condition	$\sum ab$	Condition	$\sum ab$
$(1,1)$	$(1,3)$	$1 \equiv \pm 3 \pmod 1$	6	$1 \equiv \pm 3 \pmod 2$	6
	$(2,2)$	$2 \equiv \pm 2 \pmod 1$		$2 \equiv \pm 2 \pmod 2$	
	$(3,1)$	$3 \equiv \pm 1 \pmod 1$		$3 \equiv \pm 1 \pmod 2$	
$(1,2)$	$(2,1)$	$2 \equiv \pm 1 \pmod 1$	4	$2 \not\equiv \pm 1 \pmod 2$	0
$(1,3)$	$(1,1)$	$1 \equiv \pm 1 \pmod 1$	6	$1 \equiv \pm 1 \pmod 2$	6
$(2,1)$	$(1,2)$	$1 \equiv \pm 2 \pmod 1$	4	$1 \not\equiv \pm 2 \pmod 2$	0
$(2,2)$	$(1,1)$	$1 \equiv \pm 1 \pmod 1$	8	$1 \equiv \pm 1 \pmod 2$	8
$(3,1)$	$(1,1)$	$1 \equiv \pm 1 \pmod 1$	6	$1 \equiv \pm 1 \pmod 2$	6

get $\sum_{(a,b,x,y)\in S^{\pm}_{(6,1)}} ab - \sum_{(a,b,x,y)\in S^{\pm}_{(6,2)}} ab = 40$. Finally, from Table 8, we get $\sum_{(a,b,x,y)\in S^{\pm}_{(8,2)}} ab - \sum_{(a,b,x,y)\in S^{\pm}_{(8,4)}} ab = 120$.

Figure 2 shows $h_m(2^nN) - h_n(2^nN)$ with $N = 3$. The x-axis represents n from 3 to 10. The pink curve represents $h_{n-3}(2^n\cdot 3) - h_n(2^n\cdot 3)$, the blue curve represents $h_{n-2}(2^n\cdot 3) - h_n(2^n\cdot 3)$, and the red curve represents $h_{n-1}(2^n\cdot 3) - h_n(2^n\cdot 3)$.

3. Convolution sum $\sum_{k=1}^{N-1}\sigma_1(k)\sigma_1(3^nN - 3^nk)$

Let us generalize $k_m(3^nN)$ in the following corollary.

TABLE 7. $h_0(2^1 \cdot 3)$ and $h_1(2^1 \cdot 3)$

(a,b)	(x,y)	Condition	$\sum ab$	Condition	$\sum ab$
$(1,1)$	$(1,5)$	$1 \equiv \pm 5 \pmod 1$		$1 \equiv \pm 5 \pmod 2$	
	$(2,4)$	$2 \equiv \pm 4 \pmod 1$		$2 \equiv \pm 4 \pmod 2$	
	$(3,3)$	$3 \equiv \pm 3 \pmod 1$	10	$3 \equiv \pm 3 \pmod 2$	10
	$(4,2)$	$4 \equiv \pm 2 \pmod 1$		$4 \equiv \pm 2 \pmod 2$	
	$(5,1)$	$5 \equiv \pm 1 \pmod 1$		$5 \equiv \pm 1 \pmod 2$	
$(1,2)$	$(2,2)$	$2 \equiv \pm 2 \pmod 1$		$2 \equiv \pm 2 \pmod 2$	
	$(4,1)$	$4 \equiv \pm 1 \pmod 1$	8	$4 \not\equiv \pm 1 \pmod 2$	4
$(1,3)$	$(3,1)$	$3 \equiv \pm 1 \pmod 1$	6	$3 \equiv \pm 1 \pmod 2$	6
$(1,4)$	$(2,1)$	$2 \equiv \pm 1 \pmod 1$	8	$2 \not\equiv \pm 1 \pmod 2$	0
$(1,5)$	$(1,1)$	$1 \equiv \pm 1 \pmod 1$	10	$1 \equiv \pm 1 \pmod 2$	10
$(2,1)$	$(1,4)$	$1 \equiv \pm 4 \pmod 1$		$1 \not\equiv \pm 4 \pmod 2$	
	$(2,2)$	$2 \equiv \pm 2 \pmod 1$	8	$2 \equiv \pm 2 \pmod 2$	4
$(2,2)$	$(1,2)$	$1 \equiv \pm 2 \pmod 1$		$1 \not\equiv \pm 2 \pmod 2$	
	$(2,1)$	$2 \equiv \pm 1 \pmod 1$	16	$2 \not\equiv \pm 1 \pmod 2$	0
$(2,4)$	$(1,1)$	$1 \equiv \pm 1 \pmod 1$	16	$1 \equiv \pm 1 \pmod 2$	16
$(3,1)$	$(1,3)$	$1 \equiv \pm 3 \pmod 1$	6	$1 \equiv \pm 3 \pmod 2$	6
$(3,3)$	$(1,1)$	$1 \equiv \pm 1 \pmod 1$	18	$1 \equiv \pm 1 \pmod 2$	18
$(4,1)$	$(1,2)$	$1 \equiv \pm 2 \pmod 1$	8	$1 \not\equiv \pm 2 \pmod 2$	0
$(4,2)$	$(1,1)$	$1 \equiv \pm 1 \pmod 1$	16	$1 \equiv \pm 1 \pmod 2$	16
$(5,1)$	$(1,1)$	$1 \equiv \pm 1 \pmod 1$	10	$1 \equiv \pm 1 \pmod 2$	10

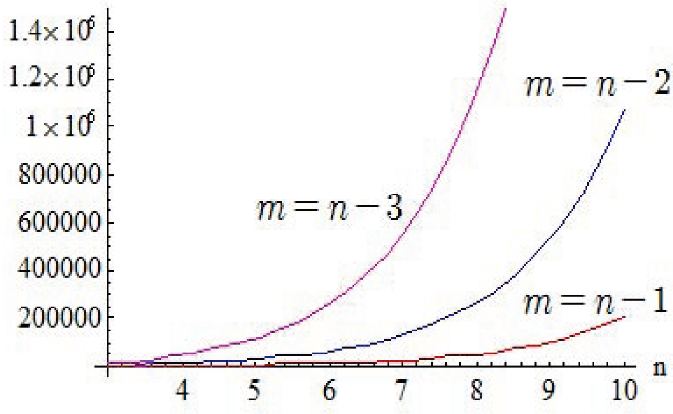

FIGURE 2. Graphs for $h_m(2^nN) - h_n(2^nN)$ with $m = n-3,\ n-2,\ n-1$

Table 8. $h_1(2^2 \cdot 2)$ and $h_2(2^2 \cdot 2)$

(a,b)	(x,y)	Condition	$\sum ab$	Condition	$\sum ab$
$(1,1)$	$(1,7)$	$1 \equiv \pm 7 \pmod 2$		$1 \equiv -7 \pmod 4$	
	$(2,6)$	$2 \equiv \pm 6 \pmod 2$		$2 \equiv \pm 6 \pmod 4$	
	$(3,5)$	$3 \equiv \pm 5 \pmod 2$		$3 \equiv -5 \pmod 4$	
	$(4,4)$	$4 \equiv \pm 4 \pmod 2$	14	$4 \equiv \pm 4 \pmod 4$	10
	$(5,3)$	$5 \equiv \pm 3 \pmod 2$		$5 \equiv -3 \pmod 4$	
	$(6,2)$	$6 \equiv \pm 2 \pmod 2$		$6 \equiv \pm 2 \pmod 4$	
	$(7,1)$	$7 \equiv \pm 1 \pmod 2$		$7 \equiv -1 \pmod 4$	
$(1,2)$	$(2,3)$	$2 \not\equiv \pm 3 \pmod 2$		$2 \not\equiv \pm 3 \pmod 4$	
	$(4,2)$	$4 \equiv \pm 2 \pmod 2$	4	$4 \not\equiv \pm 2 \pmod 4$	0
	$(6,1)$	$6 \not\equiv \pm 1 \pmod 2$		$6 \not\equiv \pm 1 \pmod 4$	
$(1,3)$	$(2,2)$	$2 \equiv \pm 2 \pmod 2$		$2 \equiv \pm 2 \pmod 4$	
	$(5,1)$	$5 \equiv \pm 1 \pmod 2$	12	$5 \equiv 1 \pmod 4$	9
$(1,5)$	$(3,1)$	$3 \equiv \pm 1 \pmod 2$	10	$3 \equiv -1 \pmod 4$	5
$(1,7)$	$(1,1)$	$1 \equiv \pm 1 \pmod 2$	14	$1 \equiv 1 \pmod 4$	7
$(2,1)$	$(1,6)$	$1 \not\equiv \pm 6 \pmod 2$		$1 \not\equiv \pm 6 \pmod 4$	
	$(2,4)$	$2 \equiv \pm 4 \pmod 2$	4	$2 \not\equiv \pm 4 \pmod 4$	0
	$(3,2)$	$3 \not\equiv \pm 2 \pmod 2$		$3 \not\equiv \pm 2 \pmod 4$	
$(2,2)$	$(1,3)$	$1 \equiv \pm 3 \pmod 2$		$1 \equiv -3 \pmod 4$	
	$(2,2)$	$2 \equiv \pm 2 \pmod 2$	24	$2 \equiv \pm 2 \pmod 4$	16
	$(3,1)$	$3 \equiv \pm 1 \pmod 2$		$3 \equiv -1 \pmod 4$	
$(2,6)$	$(1,1)$	$1 \equiv \pm 1 \pmod 2$	24	$1 \equiv 1 \pmod 4$	12
$(3,1)$	$(1,5)$	$1 \equiv \pm 5 \pmod 2$		$1 \equiv 5 \pmod 4$	
	$(2,2)$	$2 \equiv \pm 2 \pmod 2$	12	$2 \equiv \pm 2 \pmod 4$	9
$(3,5)$	$(1,1)$	$1 \equiv \pm 1 \pmod 2$	30	$1 \equiv 1 \pmod 4$	15
$(4,4)$	$(1,1)$	$1 \equiv \pm 1 \pmod 2$	32	$1 \equiv 1 \pmod 4$	16
$(5,1)$	$(1,3)$	$1 \equiv \pm 3 \pmod 2$	10	$1 \equiv -3 \pmod 4$	5
$(5,3)$	$(1,1)$	$1 \equiv \pm 1 \pmod 2$	30	$1 \equiv 1 \pmod 4$	15
$(6,2)$	$(1,1)$	$1 \equiv \pm 1 \pmod 2$	24	$1 \equiv 1 \pmod 4$	12
$(7,1)$	$(1,1)$	$1 \equiv \pm 1 \pmod 2$	14	$1 \equiv 1 \pmod 4$	7

Corollary 3.1. *Let m $(0 \le m \le n)$ be any integer and $n \in \mathbb{N} \cup \{0\}$, where $N \in \mathbb{N}$. Then, we obtain*

$$\begin{aligned} k_m(3^n N) &:= \sum_{(a,b,x,y) \in S^{\pm}_{(3^n N, 3^m)}} ab \\ &= \frac{1}{12}[(3^m - 1)\sigma_3(3^{n-m+1}N) - (3^m - 9)\sigma_3(3^{n-m}N) + 2\sigma_3(3^n N) \\ &\quad - (3^m - 1)(2 \cdot 3^{n-m+1}N - 1)\sigma_1(3^{n-m+1}N) \\ &\quad - \{3^m - 5 + 2 \cdot 3^{n+1}N(3^{-m+1} - 1)\}\sigma_1(3^{n-m}N) - 2\sigma_1(3^n N)]. \end{aligned}$$

In particular, for N $(3 \nmid N)$ and prime N with $3 \nmid N$, we can construct Table 9.

TABLE 9. Formula $k_m(3^n N)$ for N with $3 \nmid N$ and prime N

N	$\sum_{(a,b,x,y)\in S^{\pm}_{(3^n N, 3^m)}} ab$
$3 \nmid N$	$\frac{1}{156\cdot 3^{3m}}[(27^{n+m+1} - 5\cdot 27^m + 13\cdot 3^{3n+m+3}$ $-3^{3n+5})\sigma_3(N) - 13\{2\cdot 3^{n+2m+1}(3^{n+1}-1)N$ $+27^m - 3^{n+2m+1}\}\sigma_1(N)]$
$3 \nmid N$ *and prime* N	$\frac{1}{156\cdot 3^{3m}}(N+1)(27^{n+m+1}N^2 - 5\cdot 27^m N^2$ $+13\cdot 3^{3n+m+3}N^2 - 3^{3n+5}N^2 - 27^{n+m+1}N + 5\cdot 27^m N$ $-13\cdot 3^{3n+m+3}N + 3^{3n+5}N - 26\cdot 3^{2n+2m+2}N$ $+26\cdot 3^{n+2m+1}N + 27^{n+m+1} + 13\cdot 3^{3n+m+3}$ $-3^{3n+5} + 13\cdot 3^{n+2m+1} - 2\cdot 3^{3m+2})$

Proof. By substituting m with k, n with $3^n N$, and k with 3^m in equation (1), and thus, by using

$$\begin{aligned}&\sum_{k=1}^{N-1}\sigma_1(3^m k)\sigma_1(3^n(N-k))\\&=\frac{1}{144}[-(2+3^m+3^n-4\cdot 3^{m+n})\sigma_3(3N)+(62+3^m+3^n-4\cdot 3^{m+n})\sigma_3(N)\\&\quad+3\{-2+3^m+3^n-6(-3^m-3^n+2\cdot 3^{m+n})N\}\sigma_1(3N)\\&\quad+3\{6-3^m-3^n+6(-3^{1+m}-3^{1+n}+2\cdot 3^{m+n})N\}\sigma_1(N)]\end{aligned} \tag{3}$$

in [2, Theorem 3.6], we obtain the proof. Finally, we replace m with 0, n with m, and N with $3^{n-m}N$ in Eq. (3). □

Example 3.2. Figure 3 shows $k_m(3^n N)$ with $N = 2$ and $n = 5$ (left panel), and with $N = 7$ and $n = 5$ (right panel).

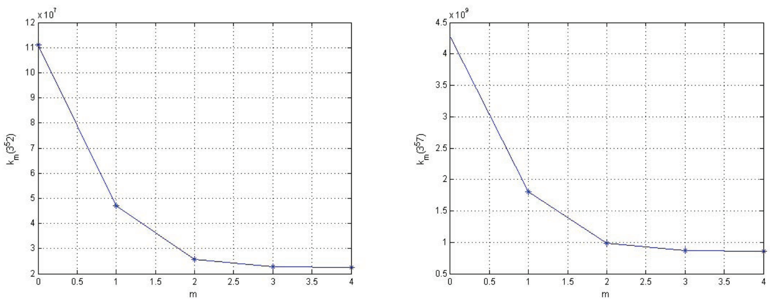

FIGURE 3. Graphs for $k_m(3^n N)$ with $N = 2$ and $n = 5$, and with $N = 7$ and $n = 5$

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