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t-CONVEX VAGUE SETS

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ABSTRACT. In this paper, we introduce the notion of t-convex vague sets and study their properties in details.

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1. Introduction and preliminaries

Convex sets play a key role in quantum logics and quantum information science. For instance, in quantum mechanics and classical theory, the state of a quantum mechanical system forms a convex set. Also, a range of fuzzy values for an event can be expressed as a convex set. A fuzzy interpretation of convexity is that any mixture of two distributions in a set is also in the set.

Zadeh proposed the theory of fuzzy sets [4]. Since then it has been applied in wide varieties of fields like Computer Science, Management Science, Medical Sciences, Engineering problems etc. to list a few only.

In [4], Zadeh introduced the concept of convex fuzzy sets, which is an important kind of extension of classical convex sets from the viewpoint of cut set. After that the theory and applications about convex fuzzy sets have been studied intensively.

The notion of vague set theory introduced by W. L. Gau and D. J. Buehrer [2], as a generalizations of Zadeh's fuzzy set theory [4]. Vague sets are studied in many branch of mathematics [1, 3]. In [2], the concept of convex vague sets are introduced and studied.

Here we review some concepts of vague set theory.

Let $U = \{u_1, u_2, ..., u_n\}$ be the universe of discourse. The membership function for fuzzy sets can take any value from the closed interval $[0; 1]$. Fuzzy set A is defined as the set of ordered pairs $A = \{(u; \mu_A(u)) \mid u \in U\}$ where $\mu_A(u)$ is

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the grade of membership of element u in set A. The greater $\mu_A(u)$, the greater is the truth of the statement that 'the element u belongs to the set A'. But Gau and Buehrer [2] pointed out that this single value combines the 'evidence for u' and the 'evidence against u'. It does not indicate the 'evidence for u' and the 'evidence against u', and it does not also indicate how much there is of each. Consequently, there is a genuine necessity of a different kind of fuzzy sets which could be treated as a generalization of Zadeh's fuzzy sets [4].

Definition 1.1 ([2])**.** A vague set A in the universe of discourse U is characterized by two membership functions given by:

1- A truth membership function

$$t_A : U \to [0,1]$$

and

2- A false membership function

$$f_A : U \to [0,1]$$

where $t_A(u)$ is a lower bound of the grade of membership of u derived from the evidence for u, and $f_A(u)$ is a lower bound of the negation of u derived from the evidence against u and

$$t_A(u) + f_A(u) \leq 1.$$

Thus the grade of membership of u in the vague set A is bounded by a sub interval $[t_A(u), 1 - f_A(u)]$ of $[0,1]$. This indicates that if the actual grade of membership is $\mu(u)$, then

$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$

The vague set A is written as

$$A = \{(u, [t_A(u), 1 - f_A(u)]) \mid u \in U\},$$

where the interval $[t_A(u), 1 - f_A(u)]$ is called the vague value of u in A and is denoted by $V_A(u)$.

It is worth to mention here that interval-valued fuzzy sets (i-v fuzzy sets) [5] are not vague sets. In i-v fuzzy sets, an interval valued membership value is assigned to each element of the universe considering the evidence for u only, without considering evidence against u. In vague sets both are independently proposed by the decision maker. This makes a major difference in the judgment about the grade of membership.

Definition 1.2 ([2])**.** A vague set A of a set U is called

1- the zero vague set of U if $t_A(u) = 0$ and $f_A(u) = 1$ for all $u \in U$,

2- the unit vague set of U if $t_A(u) = 1$ and $f_A(u) = 0$ for all $u \in U$,

3- the α-vague set of U if $t_A(u) = \alpha$ and $f_A(u) = 1 - \alpha$ for all $u \in U$, where $\alpha \in (0,1)$.

Let $D[0,1]$ denotes the family of all closed sub-intervals of $[0,1]$. Now we define refined minimum (briefly, $rmin$) and order " $\leq$ " on elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of $D[0,1]$ as:

$$rmin(D_1, D_2) = [min\{a_1, a_2\}, min\{b_1, b_2\}]$$

$$D_1 \leq D_2 \Longleftrightarrow a_1 \leq a_2 \ \wedge \ b_1 \leq b_2$$

Similarly we can define $\geq$, $=$ and $rmax$. Then concept of $rmin$ and $rmax$ could be extended to define $rinf$ and $rsup$ of infinite number of elements of $D[0,1]$.

It is that $L = \{D[0,1], rinf, rsup, \leq\}$ is a lattice with universal bounds $[0,0]$ and $[1,1]$.

For $\alpha, \beta \in [0,1]$ we now define (α,β)-cut and α-cut of a vague set.

Definition 1.3 ([2])**.** Let A be a vague set of a universe X with the true-membership function t_A and false-membership function f_A. The (α,β)-cut of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set X given by

$$A_{(\alpha,\beta)} = \{x \in X \mid V_A(x) \geq [\alpha, \beta]\},$$

where $\alpha \leq \beta$.

Clearly $A_{(0,0)} = X$. The (α,β)-cuts are also called vague-cuts of the vague set A.

Definition 1.4 ([2])**.** The α-cut of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha,\alpha)}$.

Note that $A_0 = X$ and if $\alpha \geq \beta$ then $A_\beta \subseteq A_\alpha$ and $A_{(\beta,\alpha)} = A_\alpha$. Equivalently, we can define the α-cut as

$$A_\alpha = \{x \in X \mid t_A(x) \geq \alpha\}.$$

We assume for concreteness that X is a real Euclidean space E^n.

Definition 1.5 ([2])**.** A vague set A is convex if and only if

$$t_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(t_A(x_1), t_A(x_2))$$

$$1 - f_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(1 - f_A(x_1), 1 - f_A(x_2))$$

for all x,$x_2 \in X$ and all $\lambda \in [0,1]$.

Definition 1.6 ([2])**.** A vague set A is strongly convex if for any two distinct point x_1 and x_2 in X and $\lambda \in (0,1)$

$$t_A(\lambda x_1 + (1-\lambda)x_2) > \min(t_A(x_1), t_A(x_2))$$

$$1 - f_A(\lambda x_1 + (1-\lambda)x_2) > \min(1 - f_A(x_1), 1 - f_A(x_2))$$

Theorem 1.1 ([2])**.** *If A and B are (strongly) convex, so is their intersections.*

2. t-convex vague sets

In a natural way, in this section, we generalized the notion of convex vague set from the viewpoint of t-norm based fuzzy logic, and proposed the notions of t-convex (concave) vague sets. In what follows, T always denote a left continuous t-norm,

Definition 2.1. A vague set A, is said to be t-convex vague set if

$$t_A(\lambda x_1 + (1-\lambda)x_2) \geq T(t_A(x_1), t_A(x_2))$$

$$1 - f_A(\lambda x_1 + (1-\lambda)x_2) \geq T(1 - f_A(x_1), 1 - f_A(x_2))$$

for all $x, y \in E$ and $\lambda \in [0,1]$. In some sense, this is consistent with the definition of vague sets.

In the same way, we can also define the notion of t-concave vague set.

Definition 2.2. A vague set A, is said to be t-concave vague set if

$$t_A(\lambda x_1 + (1-\lambda)x_2) \leq S(t_A(x_1), t_A(x_2))$$

$$1 - f_A(\lambda x_1 + (1-\lambda)x_2) \leq S(1 - f_A(x_1), 1 - f_A(x_2))$$

for all $x, y \in E$ and $\lambda \in [0,1]$.

By definition, the following two conclusions are obvious.

Theorem 2.1. *Let T_1 and T_2 be two left continuous t-norms and T_1 is weaker then T_2, i.e. $T_1(x,y) \leq T_2(x,y)$ for any $(x,y) \in [0,1]^2$, then every T_2-convex (or concave) vague set is also T_1-convex (or concave) vague set.*

Remark 2.1. Since min is the strongest t-norm on $[0,1]$ and the drastic product T_Δ is the weakest t-norm on $[0,1]$, so for any t-norm T, every min-convex (or concave) vague set is an t-convex (or concave) vague set, every t-convex (or concave) vague set is also an T_Δ-convex (or concave) vague set, where the drastic product T_Δ is defined as follows:

$$T_\Delta(a,b) = \begin{cases} a & \text{if } b = 1, \\ b & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In the following example we give a vague set which is not a min-convex, but there exists a t-norm, different from drastic t-norm such that it will be t-convex.

Example 2.2. Let $A = \{(x; [t_A(x), 1 - f_A(x)]) \mid x \in E\}$ be a vague set defined by:

$$t_A(x) = \begin{cases} x & \text{if } 0 \leq x \leq 0.2, \\ 0 & \text{if } 0.2 < x < 0.5, \\ x & \text{if } 0.5 < x \leq 0.7 \end{cases}$$

and

$$1 - f_A(x) = \begin{cases} -x + 0.7 & \text{if } 0 \le x \le 0.2, \\ 0.4 & \text{if } 0.2 < x < 0.5, \\ -x + 0.7 & \text{if } 0.5 \le x \le 0.7 \end{cases}$$

Then A is not a min-convex vague set, but if consider bounded sum t-norm, $T_b(x, y) = \max(0, x + y - 1)$, then A is a T_b-convex vague set.

In crisp case, the intersection of any two convex sets is also convex set, the convex fuzzy sets defined by Zadeh preserve this property as well. In the following, we generalize the intersection, union and complement of vague sets using triangular norms, it's dual conorms and inverse order and involutive operators, define $\cap_T$, $\cup_S$ and A^c as follows:

$$A^c = \{(u; [f_A(u), 1 - t_A(u)]) \mid u \in X\}$$
$$A \cap_T B = \{(u; [T(t_A(u), t_B(u)), 1 - S(f_A(u), f_B(u))]) \mid u \in X\}$$
$$A \cup_S B = \{(u; [S(t_A(u), t_B(u)), 1 - T(f_A(u), f_B(u))]) \mid u \in X\}$$

Theorem 2.3. *If A and B are t-convex vague sets, then $A \cap_T B$ is t-convex vague set.*

Proof. Let $C = A \cap_T B$. Then

$$t_C(\lambda x_1 + (1 - \lambda)x_2) = T(t_A(\lambda x_1 + (1 - \lambda)x_2), t_B(\lambda x_1 + (1 - \lambda)x_2))$$

and

$$1 - f_C(\lambda x_1 + (1 - \lambda)x_2) = T(1 - f_A(\lambda x_1 + (1 - \lambda)x_2), 1 - f_B(\lambda x_1 + (1 - \lambda)x_2)).$$

By hypothesis we have

$$t_A(\lambda x_1 + (1 - \lambda)x_2) \ge T(t_A(x_1), t_A(x_2))$$
$$t_B(\lambda x_1 + (1 - \lambda)x_2) \ge T(t_B(x_1), t_B(x_2))$$
$$1 - f_A(\lambda x_1 + (1 - \lambda)x_2) \ge T(1 - f_A(x_1), 1 - f_A(x_2))$$
$$1 - f_B(\lambda x_1 + (1 - \lambda)x_2) \ge T(1 - f_B(x_1), 1 - f_B(x_2))$$

therefore

$$\begin{aligned} t_C(\lambda x_1 + (1 - \lambda)x_2) &= T(t_A(\lambda x_1 + (1 - \lambda)x_2), t_B(\lambda x_1 + (1 - \lambda)x_2)) \\ &\ge T(T(t_A(x_1), t_A(x_2)), T(t_B(x_1), t_B(x_2))) \\ &= T(t_C(x_1), t_C(x_2)) \end{aligned}$$

also

$$\begin{aligned} 1 - f_C(\lambda x_1 + (1 - \lambda)x_2) &= T(1 - f_A(\lambda x_1 + (1 - \lambda)x_2), 1 - f_B(\lambda x_1 + (1 - \lambda)x_2)) \\ &\ge T(T(1 - f_A(x_1), 1 - f_A(x_2)), T(1 - f_B(x_1), 1 - f_B(x_2)) \\ &= T(1 - f_C(x_1), 1 - f_C(x_2)). \end{aligned}$$

□

In the same way, we can prove the following conclusion.

Theorem 2.4. *If A and B are t-concave vague sets, then $A \cup_S B$ is t-concave vague set.*

The following conclusion are obvious.

Theorem 2.5. *If A is a t-convex vague set, then A^c is t-concave vague set, and vice versa.*

Proposition 2.6. *If t is a continuous t-norm, then the t-convexity is preserved under the point-wise limit operation, i.e., the limit of a point-wise convergent sequence A_n of t-convex vague sets is a t-convex vague set.*

Corollary 2.7. *Let $\{A_n\}$ be a finite sequence of t-convex vague sets on R. Define*

$$A : R^d \to D[0,1]$$
$$(x_1, ...x_d) \mapsto [T(t_{A_1}(x_1), ..., t_{A_d}(x_d)), 1 - S(f_{A_1}(x_1), ..., f_{A_d}(x_d)]$$

for all d-tuples $(x_1, ..., x_d) \in R^d$. Then A is a t-convex vague set on R^d.

Corollary 2.8. *If A and B are* min*-convex vague sets, then $A \cap_T B$ is t-convex vague set for any t-norm T.*

Corollary 2.9. *If A and B are* min*-concave vague set, then $A \cup_S B$ is t-concave vague set for any t-norm T, where S is the dual conorm of T.*

It is well known that every level set of Zadeh's convex fuzzy set is a convex subset. In what follows, we will discuss the properties of various level sets of t-convex (or concave) vague set.

Theorem 2.10. *Let A be a vague set on E. If for any $a, b \in [0,1]$, the sets $A_a^1 = \{x \in E \mid t_A(x) \geq a\}$ and $A_b^2 = \{x \in E \mid 1 - f_A(x) \geq b\}$ are convex subsets in E, then A is a t-convex vague set.*

Proof. Assume that for any $a \in [0,1]$, both A_a^1 and A_a^2 are convex subsets in E. Let $a = T(t_A(x), t_A(y))$. For any $x, y \in E$, then $t_A(x) \geq a$, $t_A(y) \geq a$, thus $x, y \in A_a^1$. Since A_a^1 is a convex set, thus for any $\lambda \in [0,1]$ we have, $\lambda x + (1-\lambda)y \in A_a^1$. Therefore, $t_A(\lambda x + (1-\lambda)y) \geq a = T(t_A(x), t_A(y))$. On the other hand, let $b = T(1 - f_A(x), 1 - f_A(y))$, then $1 - f_A(x) \geq b$, $1 - f_A(y) \geq b$, thus $x, y \in A_b^2$. Since A_b^2 is a convex set, thus for any $\lambda \in [0,1]$ we have, $\lambda x + (1-\lambda)y \in A_b^2$. Hence $1 - f_A(\lambda x + (1-\lambda)y) \geq b = T(1 - f_A(x), 1 - f_A(y))$. Therefore A is a t-convex vague set. □

The converse of above theorem is not true.

Example 2.11. Let $A = \{(x; [t_A(x), 1 - f_A(x)]) \mid x \in E\}$ be a vague set defined by:

$$t_A(x) = \begin{cases} 0.4x - 0.4 & \text{if } 1 \leq x < 2, \\ 0.4x - 0.2 & \text{if } 2 \leq x < 3 \end{cases}$$

and

$$1 - f_A(x) = \begin{cases} 1.4 - 0.4x & \text{if } 1 \leq x < 2, \\ 1.2 - 0.4x & \text{if } 2 \leq x < 3 \end{cases}$$

It is clear that A is a T_Δ-convex vague set, but the sets $A_a^1 = \{x \in E \mid t_A(x) \geq a\}$ and $A_b^2 = \{x \in E \mid 1 - f_A(x) \geq b\}$ are not convex subsets in E, for any $a, b \in [0, 1]$.

We prove the converse of Theorem 2.10 with a special t-norm.

Theorem 2.12. *If A is a* min*-convex vague set on E, then A_a^1 and A_b^2 are convex subsets in E.*

Proof. Assume that A is a min-convex vague set. Let $x, y \in A_a^1$ and $\lambda \in [0, 1]$, then $t_A(x) \geq a$, $t_A(y) \geq a$. Since $t_A(\lambda x + (1-\lambda)y) \geq \min(t_A(x), t_A(y))$, thus $\min(t_A(x), t_A(y)), \geq \min(a, a) = a$. Hence $t_A(\lambda x + (1-\lambda)y) \geq a$, this shows that $\lambda x + (1-\lambda)y \in A_a^1$. Hence A_a^1 is a convex subset in E.

Similarly, we can prove that A_b^2 is a convex subset in E. □

Corollary 2.13. *If A is an* min*-convex vague set on E, then A_a is convex subset in E.*

Remark 2.2. The convexity of level sets is necessary and sufficient for min-convexity and this is not true for t-convexity. It is only sufficient for t-convexity.

For graded concave vague sets, some analogous conclusions can be obtained.

Theorem 2.14. *Let A be a vague set on E. If for any $a, b \in [0, 1]$, both the sets of $B_a^1 = \{x \in E \mid t_A(x) \leq 1 - a\}$ and $B_b^2 = \{x \in E \mid f_A(x) \geq b\}$ are convex subsets in E, then A is a t-concave vague set in E.*

Theorem 2.15. *If A is a* min*-concave vague set on E, then B_a^1 and B_b^2 are convex subsets in E.*

Corollary 2.16. *If A is a* min*-concave vague set on E, then $B_a = \{x \in E \mid t_A(x) \leq 1 - a$ and $f_A(x) \geq a\}$ is convex subset in E.*

Theorem 2.17. *If A is a* min*-convex vague set on E, then the following lower level set*

$$A_\lambda(x) = \begin{cases} 1 & t_A(x) \geq \lambda, \\ \frac{1}{2} & t_A(x) < \lambda \leq 1 - f_A(x), \\ 0 & \lambda > 1 - f_A(x) \end{cases}$$

is a convex vague set in E.

Proof. Assume that A is a t-convex vague set. We only need to prove that for any $x, y \in E$, $\lambda \in [0, 1]$, $A_\lambda(\gamma x + (1-\gamma)y) \geq \min(A_\lambda(x), A_\lambda(y))$. If $A_\lambda(x) = 0$ or $A_\lambda(y) = 0$, the conclusion is obvious.

If $A_\lambda(x) = 1$ and $A_\lambda(y) = 1$, then $t_A(x) \geq \lambda$ and $t_A(y) \geq \lambda$, it follows that

$$\begin{aligned} t_A(\gamma x + (1-\gamma)y) &\geq \min(t_A(x), t_A(y)) \\ &\geq \min(\lambda, \lambda) \\ &= \lambda. \end{aligned}$$

thus $A_\lambda(\gamma x + (1-\gamma)y) = 1 \geq \min(A_\lambda(x), A_\lambda(y))$.

If $A_\lambda(x) = 1$ and $A_\lambda(y) = \frac{1}{2}$, then $t_A(x) \geq \lambda$ and $1 - f_A(y) \geq \lambda$. Since

$$1 - f_A(\gamma x + (1-\gamma)y) \geq \min(1 - f_A(x), 1 - f_A(y))$$

by hypothesis we have $t_A(x) \leq 1 - f_A(x)$, then

$$\begin{aligned} 1 - f_A(\gamma x + (1-\gamma)y) &\geq \min(1 - f_A(x), 1 - f_A(y)) \\ &\geq \min(\lambda, \lambda) \\ &= \lambda. \end{aligned}$$

Therefore $A_\lambda(\gamma x + (1-\gamma)y) = \frac{1}{2} \geq \min(A_\lambda(x), A_\lambda(y))$.

If $A_\lambda(x) = \frac{1}{2}$ and $A_\lambda(y) = \frac{1}{2}$, then $1 - f_A(x) \geq \lambda$ and $1 - f_A(y) \geq \lambda$. Similarly, we can see $A_\lambda(\gamma x + (1-\gamma)y) = \frac{1}{2} \geq \min(A_\lambda(x), A_\lambda(y))$. Sum up, A_λ is a convex vague set in E. □

Similarly, we can prove the following conclusions.

Theorem 2.18. *If A is a* min-*concave vague set on E, then the lower level set A_λ is a concave fuzzy set in E.*

Theorem 2.19. *If A is a* min-*concave vague set on E, then the following upper level set*

$$A^\lambda(x) = \begin{cases} 1 & f_A(x) \geq \lambda, \\ \frac{1}{2} & f_A(x) < \lambda \leq 1 - t_A(x), \\ 0 & \lambda > 1 - t_A(x) \end{cases}$$

is a convex vague set in E.

Theorem 2.20. *If A is a* min-*convex vague set on E, then the upper level set A^λ is a concave vague set in E.*

3. Conclusion

In this paper, a kind of graded convex (concave) vague sets have been established based on triangular norms, some properties of their various cut sets have also been presented. As we have seen, the main conclusions obtained are suitable for all t-norms.

It is our hope that this work would other foundations for further study of the theory of vague sets and convex sets.

In our future study of structure of vague, may be the following topics should be considered:

(1) To get more results in convex sets and application;

(2) If we take other t-norms as the underlying t-norm, what properties can be obtained?

(3) To define another types of convex vague sets;

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References

1. A. Borumand Saeid, *Vague BCK/BCI-algebras*, Opuscula Math., Vol. 29, no.2, (2009), 177-186.
2. W. L. Gau and D. J. Buehrer, *Vague sets*, IEEE Transactions on Systems, Man and Cybernetics 23 (1993), 610-614.
3. S. M. Taheri and R. Zarei, *Extension Principle for Vague Sets and Its Applications*, Advances in Fuzzy Math. Vol. 6, No.1 (2011), pp. 1726.
4. L. A. Zadeh, *Fuzzy sets,* Inform. and Control, 8 (1965), 338–353.
5. L. A. Zadeh, *The concept of a linguistic variable and its application to approximate reasoning-I*, Inform. and Control. 8, (1975), 199-249.

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