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MODULAR TRANSFORMATION FORMULAE FOR GENERALIZED NON-HOLOMORPHIC EISENSTEIN SERIES

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Abstract. In this paper, we compute transformation formulae for generalized non-holomorphic Eisenstein series.

1. Introduction

J. Lewittes [3] proved transformation formulae for a large class of Eisenstein series which is defined by

$$G(z,s,r_1,r_2) = \sum_{m,n=-\infty}^{\infty}{}' ((m+r_1)z+n+r_2)^{-s},$$

where the dash $'$ means $(m,n) \neq (-r_1,-r_2)$, Im $z>0$, $r_1,\ r_2$ are real numbers and Re $s>2$. However, Lewittes's results were too complicated to deduce desired results. It was B. C. Berndt [1] who has obtained considerably simpler formulae than Lewittes's results. Berndt [2] also proved transformation formulae for a more general class of Eisenstein series than Lewittes's case, which is defined by

$$G(z,s;r_1,r_2,h_1,h_2) := \sum_{m,n=-\infty}^{\infty}{}' \frac{e^{2\pi i(mh_1+nh_2)}}{((m+r_1)z+n+r_2)^s},$$

where the dash $'$ means $(m,n) \neq (-r_1,-r_2)$, Im $z>0$, $r_1,\ r_2,\ h_1,\ h_2$ are real and Re $s>2$.
In this paper, we consider a class of generalized non-holomorphic Eisenstein series. In fact, H. Maass [5] considered non-holomorphic Eisenstein series(when $r=h=(0,0)$ in section 3) as an important thing of the theory of non-holomorphic modular forms and determined the Fourier

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series of this non-holomorphic Eisenstein series. We obtain modular transformation formulae for generalized non-holomorphic Eisenstein series.

2. Notations

If w is a complex, we choose the branch of the argument defined by $-\pi \leq \arg w < \pi$. We write $e(w)$ for $e^{2\pi i w}$. Let λ denote the characteristic function of the integers. In the sequel, $V\tau = V(\tau) = \frac{a\tau+b}{c\tau+d}$ always denotes a modular transformation with $c > 0$ for every complex τ. Let $r = (r_1, r_2)$ and $h = (h_1, h_2)$ denote real vectors, and define the associated vectors R and H by

$$R = (R_1, R_2) = (ar_1 + cr_2, br_1 + dr_2)$$

and

$$H = (H_1, H_2) = (dh_1 - bh_2, -ch_1 + ah_2).$$

For real x, α and $\operatorname{Re} s > 1$, let

$$\psi(x, \alpha, s) := \sum_{n+\alpha>0} \frac{e(nx)}{(n+\alpha)^s}. \tag{2.1}$$

For a real number x, $[x]$ denotes the greatest integer less than or equal to x and $\{x\} := x - [x]$. Let ${}_2F_1(\alpha, \beta; \gamma; z)$ be a hypergeometric function defined by

$${}_2F_1(\alpha, \beta; \gamma; z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n,$$

where $(x)_n$ denotes the rising factorial defined by

$$(x)_n := x(x+1)\cdots(x+n-1) \text{ for } n > 0, \ (x)_0 := 1.$$

Euler's integral representation of ${}_2F_1$-hypergeometric function says that, for $\operatorname{Re} \gamma > \operatorname{Re} \alpha > 0$ and $z \in \mathbb{C}\backslash[1, \infty)$,

$${}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-zt)^{-\beta}\, dt. \tag{2.2}$$

Thus we see that [4, 6]

$$\frac{1}{\Gamma(\gamma)}\ {}_2F_1(\alpha, \beta; \gamma; z)$$

can be analytically continued to all $\alpha,\ \beta,\ \gamma \in \mathbb{C}$ and all $z \in \mathbb{C}\backslash[1, \infty)$.

3. Generalized non-holomorphic Eisenstein series

Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, the upper half-plane.

Definition 3.1. Let $\tau \in \mathbb{H}$ and $s,\ s_1 \in \mathbb{C}$. For Re $s > 2$, the generalized non-holomorphic Eisenstein series $\mathbf{G}(\tau, \bar{\tau}, s, s_1; r, h)$ is defined to be

$$\mathbf{G}(\tau, \bar{\tau}, s, s_1; r, h) := \sum_{m,n=-\infty}^{\infty}{}' \frac{e(mh_1 + nh_2)}{((m + r_1)\tau + n + r_2)^{s_1} ((m + r_1)\bar{\tau} + n + r_2)^{s-s_1}},$$

where the dash $'$ means $(m, n) \neq (-r_1, -r_2)$.

Theorem 3.2. *Let $Q = \{\tau \in \mathbb{H} \mid \mathrm{Re}\ \tau > -d/c\}$ and $\varrho = c\{R_2\} - d\{R_1\}$. Then for $\tau \in Q$, $s_1 \in \mathbb{C}$ and $s \in \mathbb{C}$ with* Re $s > 2$,

$$\begin{aligned}
&(c\tau + d)^{-s_1}(c\bar{\tau} + d)^{-s+s_1}\mathbf{G}(V\tau, V\bar{\tau}, s, s_1; r, h) \\
&= \mathbf{G}(\tau, \bar{\tau}, s, s_1; R, H) - 2i \sin(\pi s)\lambda(R_1)e(-R_1H_1)\psi(-H_2, -R_2, s) \\
&\quad + \frac{e\left(-\frac{s}{2}\right)}{\Gamma(s_1)\Gamma(s - s_1)}\mathbf{L}(\tau, \bar{\tau}, s, s_1, ; R, H),
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
&\mathbf{L}(\tau, \bar{\tau}, s, s_1; R, H) \\
&:= \sum_{j=1}^{c} e(-H_1(j + [R_1] - c) - H_2([R_2] + 1 + [(jd + \varrho)/c] - d)) \\
&\cdot \int_0^1 v^{s_1-1}(1 - v)^{s-s_1-1} \int_C u^{s-1} \frac{e^{-((c\tau+d)v+(c\bar{\tau}+d)(1-v))(j-\{R_1\})u/c}}{e^{-((c\tau+d)v+(c\bar{\tau}+d)(1-v))u} - e(cH_1 + dH_2)} \\
&\qquad \cdot \frac{e^{\{(jd+\varrho)/c\}u}}{e^u - e(-H_2)} du dv,
\end{aligned}$$

where C is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in the positive direction so that $u = 0$ is the only zero of

$$\left(e^{-((c\tau+d)v+(c\bar{\tau}+d)(1-v))u} - e(cH_1 + dH_2)\right)(e^u - e(-H_2))$$

lying "inside" the loop, and then returning to $+\infty$ in the lower half plane. Here, we choose the branch of u^s with $0 <$ arg $u < 2\pi$.

Proof. For $M = am + cn$, $N = bm + dn$,

$$\begin{aligned}
((m + r_1)V\tau + n + r_2)^s &= \left((m + r_1)\frac{a\tau + b}{c\tau + d} + n + r_2\right)^s \\
&= \left(\frac{(M + R_1)\tau + N + R_2}{c\tau + d}\right)^s.
\end{aligned} \tag{3.2}$$

If m and n run over all integers, then M and N run over all integer except for $(M,N)=(-R_1,-R_2)$ since $ad-bc=1$. Using (3.2), we see that

$$\mathbf{G}(V\tau,V\bar{\tau},s,s_1;r,h)=\sum_{M,N=-\infty}^{\infty}{}' \frac{e(MH_1+NH_2)}{\left(\frac{(M+R_1)\tau+N+R_2}{c\tau+d}\right)^{s_1}\left(\frac{(M+R_1)\bar{\tau}+N+R_2}{c\bar{\tau}+d}\right)^{s-s_1}}.$$

Thus, by Lemma 1 in [1],

$$\begin{aligned}
&(c\tau+d)^{-s_1}(c\bar{\tau}+d)^{-s+s_1}\mathbf{G}(V\tau,V\bar{\tau},s,s_1;r,h)\\
&=e(-s)\sum_{(m,n)\in I}\frac{e(mH_1+nH_2)}{((m+R_1)\tau+n+R_2)^{s_1}\,((m+R_1)\bar{\tau}+n+R_2)^{s-s_1}}\\
&\quad+\sum_{(m,n)\notin I}{}'\frac{e(mH_1+nH_2)}{((m+R_1)\tau+n+R_2)^{s_1}\,((m+R_1)\bar{\tau}+n+R_2)^{s-s_1}}\\
&=\mathbf{G}(\tau,\bar{\tau},s,s_1;R,H)+(e(-s)-1)\mathbf{g}(\tau,\bar{\tau},s,s_1;R,H), \qquad (3.3)
\end{aligned}$$

where

$$\begin{aligned}
&\mathbf{g}(\tau,\bar{\tau},s,s_1;R,H)\\
&:=\sum_{(m,n)\in I}\frac{e(mH_1+nH_2)}{((m+R_1)\tau+n+R_2)^{s_1}\,((m+R_1)\bar{\tau}+n+R_2)^{s-s_1}},
\end{aligned}$$

and

$$I:=\{(m,n)\in\mathbb{Z}\times\mathbb{Z}\mid m+R_1\le 0,\ d(m+R_1)>c(n+R_2)\}.$$

Replacing m by $-m$ and n by $-n$, and separating the sum for $m=-R_1$, we have

$$\begin{aligned}
\mathbf{g}(\tau,\bar{\tau},s,s_1;R,H)&=\lambda(R_1)e\left(\frac{s}{2}\right)e(-R_1H_1)\psi(-H_2,-R_2,s)\\
&\quad+e\left(\frac{s}{2}\right)\mathbf{h}(\tau,\bar{\tau},s,s_1;R,H), \qquad (3.4)
\end{aligned}$$

where

$$\begin{aligned}
&\mathbf{h}(\tau,\bar{\tau},s,s_1;R,H)\\
&:=\sum_{(m,n)\in J}\frac{e(-mH_1-nH_2)}{((m-R_1)\tau+n-R_2)^{s_1}\,((m-R_1)\bar{\tau}+n-R_2)^{s-s_1}},
\end{aligned}$$

where

$$J:=\{(m,n)\in\mathbb{Z}\times\mathbb{Z}\mid m-R_1>0,\ d(m-R_1)<c(n-R_2)\}.$$

We now use Euler's integral representation of $\Gamma(s)$ to obtain that, for $\tau \in Q$,

$$\Gamma(s-s_1)\Gamma(s_1)\mathbf{h}(\tau,\bar\tau,s,s_1;R,H) = \sum_{(m,n)\in J} e(-mH_1-nH_2) \cdot \int_0^\infty \int_0^\infty u^{s_1-1}v^{s-s_1-1}e^{-(m-R_1)(\tau u+\bar\tau v)-(n-R_2)(u+v)}\, dudv.$$

Replace indices, m by $m+[R_1]+1$ and n by $n+[R_2+(d(m-R_1)/c]+1$ to find that

$$\begin{aligned}&\Gamma(s-s_1)\Gamma(s_1)\mathbf{h}(\tau,\bar\tau,s,s_1;R,H)\\ &= \sum_{m,n=0}^{\infty} e(-H_1(m+[R_1]+1)-H_2(n+[R_2]+1+[(\varrho+md+d)/c]))\\ &\cdot \int_0^\infty \int_0^\infty u^{s_1-1}v^{s-s_1-1}e^{-(m-\{R_1\}+1)(\tau u+\bar\tau v)-(n-\{R_2\}+1+[(\varrho+md+d)/c])(u+v)}\, dudv.\end{aligned}$$

Let $m = qc+j-1$, $0\le q<\infty$, $1\le j\le c$. Involving sum of geometric series and manipulation of integral, we find that

$$\begin{aligned}&\Gamma(s-s_1)\Gamma(s_1)\mathbf{h}(\tau,\bar\tau,s,s_1;R,H)\\ &= \sum_{j=1}^{c} e(-H_1(j+[R_1])-H_2([R_2]+1+[(\varrho+jd)/c]))\\ &\quad\cdot \int_0^\infty \int_0^\infty u^{s_1-1}v^{s-s_1-1}e^{-(j-\{R_1\}+1)(\tau u+\bar\tau v)-(1-\{R_2\}+[(\varrho+jd)/c])(u+v)}\\ &\qquad \sum_{q=0}^{\infty}\sum_{n=0}^{\infty} e^{-q(2\pi i(cH_1+dH_2)+(c\tau+d)u+(c\bar\tau+d)v)-n(2\pi iH_2+u+v)}\, dudv\\ &= \sum_{j=1}^{c} e(-H_1(j+[R_1]-c)-H_2([R_2]+1+[(\varrho+jd)/c]-d))\\ &\quad\cdot \int_0^\infty \int_0^\infty u^{s_1-1}v^{s-s_1-1}\frac{e^{-(c\tau+d)(j-\{R_1\})u/c-(c\bar\tau+d)(j-\{R_1\})v/c}}{e(cH_1+dH_2)-e^{-(c\tau+d)u-(c\bar\tau+d)v}}\\ &\qquad\cdot\frac{e^{\{\frac{\varrho+jd}{c}\}(u+v)}}{e^{u+v}-e(-H_2)}\, dudv\\ &= \sum_{j=1}^{c} e(-H_1(j+[R_1])-H_2([R_2]+1+[(\varrho+jd)/c]))\\ &\quad\cdot \int_0^1 v^{s_1-1}(1-v)^{s-s_1-1}\int_0^\infty u^{s-1}\frac{e^{-((c\tau+d)v+(c\bar\tau+d)(1-v))(j-\{R_1\})u/c}}{e(cH_1+dH_2)-e^{-((c\tau+d)v+(c\bar\tau+d)(1-v))u}}\\ &\qquad\cdot\frac{e^{\{\frac{\varrho+jd}{c}\}u}}{e^{u}-e(-H_2)}\, dudv.\end{aligned}$$

The inversion of the order of summations and integrations can be justified by absolute convergence of the original series for Re $s>2$. This last integrals from 0 to ∞ can be converted to loop integrals on C by the same manner in [1]. Hence, for Re $s > 2$ and $\tau \in Q$, it follows that

$$\begin{aligned}
&\Gamma(s-s_1)\Gamma(s_1)\mathbf{h}(\tau,\bar{\tau},s,s_1;R,H) \\
&= -\frac{1}{e(s)-1}\sum_{j=1}^{c} e(-H_1(j+[R_1]) - H_2([R_2]+1+[(\varrho+jd)/c])) \\
&\quad \cdot \int_0^1 v^{s_1-1}(1-v)^{s-s_1-1}\int_C u^{s-1}\frac{e^{-((c\tau+d)v+(c\bar{\tau}+d)(1-v))(j-\{R_1\})u/c}}{e(cH_1+dH_2)-e^{-((c\tau+d)v+(c\bar{\tau}+d)(1-v))u}} \\
&\qquad \cdot \frac{e^{\{\frac{\varrho+jd}{c}\}u}}{e^u - e(-H_2)}\, dudv.
\end{aligned} \tag{3.5}$$

Putting (3.5) into (3.4) and then (3.4) into (3.3), the proof is done. □

We see that each loop integrals in (3.5) converges uniformly on any compact set of the s-plane. Thus, using the analytic continuation of ${}_2F_1(\alpha,\beta;\gamma;z)/\Gamma(\gamma)$ and (2.2) and applying the same manner in [4], p. 239, we find that

$$\frac{1}{\Gamma(s_1)\Gamma(s-s_1)}\mathbf{L}(\tau,\bar{\tau},s,s_1,;R,H)$$

can be analytically continued to all values of s_1 and s. Now if s is an integer, then integrals may be computed using the residue theorem. We use the generating function

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad (|t| < 2\pi)$$

for Bernoulli polynomials $B_n(x)$, $n \geq 0$. The n-th Bernoulli number B_n, $n \geq 0$, is defined by $B_n = B_n(0)$. Put $\bar{B}_n(x) = B_n(\{x\})$, $n \geq 0$. Let $s = -N$ for an integer N. By the same manner in [1], we find that

$$\begin{aligned}
&\int_C u^{s-1}\frac{e^{-((c\tau+d)v+(c\bar{\tau}+d)(1-v))(j-\{R_1\})u/c}}{e^{-((c\tau+d)v+(c\bar{\tau}+d)(1-v))u}-1}\frac{e^{\{\frac{\varrho+jd}{c}\}u}}{e^u-1}\, du \\
&= 2\pi i\sum_{k=0}^{N+2}\frac{B_k\left(\frac{j-\{R_1\}}{c}\right)\bar{B}_{N+2-k}\left(\frac{\varrho+jd}{c}\right)}{k!(N+2-k)!}(-cv(\tau-\bar{\tau})-c\bar{\tau}-d)^{k-1}.
\end{aligned}$$

Thus, replacing v by $1-v$, we obtain that

$$\begin{aligned}
&\int_0^1 v^{s_1-1}(1-v)^{-N-s_1-1}\int_C u^{-N-1}\frac{e^{-((c\tau+d)v+(c\bar{\tau}+d)(1-v))(j-\{R_1\})u/c}}{e^{-((c\tau+d)v+(c\bar{\tau}+d)(1-v))u}-1} \\
&\cdot\frac{e^{\{\frac{\varrho+jd}{c}\}u}}{e^u-1}\, dudv
\end{aligned}$$

$$= 2\pi i \sum_{k=0}^{N+2} \frac{B_k\left(\frac{j-\{R_1\}}{c}\right)\bar{B}_{N+2-k}\left(\frac{\varrho+jd}{c}\right)}{k!(N+2-k)!}(-c\tau-d)^{k-1}$$
$$\cdot \int_0^1 (1-v)^{s_1-1}v^{-N-s_1-1}\left(1-\frac{c(\tau-\bar{\tau})}{c\tau+d}v\right)^{k-1}dv$$
$$= 2\pi i \frac{\Gamma(s_1)\Gamma(-N-s_1)}{\Gamma(-N)} \sum_{k=0}^{N+2} \frac{B_k\left(\frac{j-\{R_1\}}{c}\right)\bar{B}_{N+2-k}\left(\frac{\varrho+jd}{c}\right)}{k!(N+2-k)!}(-c\tau-d)^{k-1}$$
$$\cdot\ {}_2F_1(-N-s_1, 1-k; -N; \frac{c(\tau-\bar{\tau})}{c\tau+d}).$$

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