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CONVOLUTION SUMS OF ODD AND EVEN DIVISOR FUNCTIONS

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Abstract. Let $\sigma_s(N)$ denote the sum of the s-th power of the positive divisors of N and $\sigma_{s,r}(N;m) = \sum_{\substack{d|N \\ d\equiv r \bmod m}} d^s$ with $N, m, r, s, d \in \mathbb{Z}$, $d, s > 0$ and $r \geq 0$. In a celebrated paper [33], Ramanujan proved $\sum_{k=1}^{N-1} \sigma_1(k)\sigma_1(N-k) = \frac{5}{12}\sigma_3(N) + \frac{1}{12}\sigma_1(N) - \frac{6}{12}N\sigma_1(N)$ using elementary arguments. The coefficients' relation in this identity $(\frac{5}{12} + \frac{1}{12} - \frac{6}{12} = 0)$ motivated us to write this article. In this article, we found the convolution sums $\sum_{k<N/m} \sigma_{1,i}(dk;2)\sigma_{1,j}(N-mk;2)$ for odd and even divisor functions with $i, j = 0, 1$, $m = 1, 2, 4$, and $d|m$. If N is an odd positive integer, $i, j = 0, 1$, $m = 1, 2, 4$, $s = 0, 1, 2$, and $d|m|2^s$, then there exist $u, a, b, c \in \mathbb{Z}$ satisfying $\sum_{k<2^sN/m} \sigma_{1,i}(dk;2)\sigma_{1,j}(2^sN - mk;2) = \frac{1}{u}[a\sigma_3(N) + bN\sigma_1(N) + c\sigma_1(N)]$ with $a + b + c = 0$ and $(u, a, b, c) = 1$(Theorem 1.1). We also give an elementary problem (O) and solve special cases of them in (O) (Corollary 3.27).

1. Introduction

J. Liouville [30], J. W. L. Glaisher [17, 18, 19, 20], and S. Ramanujan [33] studied some convolution formulas for divisor functions and various authors(see [8, 14, 15, 25, 31, 34]) have worked in related subject areas. Many recent works on convolution formulas for divisor functions can be found in B. C. Berndt [7]; H. Hahn [21]; J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams [22]; G. Melfi [32]; N. P. Skoruppa [37]; and A. Alaca, S. Alaca, and K. S. Williams [1, 2, 3, 4, 5, 6].

For $N, m, r, s, d \in \mathbb{Z}$ with $d, s > 0$ and $r \geq 0$, we define some necessary divisor functions and infinite products for later use; these divisor

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functions and infinite products appear in many areas of number theory:

$$\sigma_{s,r}(N;m) = \sum_{\substack{d|N \\ d\equiv r \bmod m}} d^s, \quad \sigma(N) := \sigma_1(N) = \sum_{d|N} d,$$

$$\tilde{\sigma_s}(N) = \sum_{d|N}(-1)^{d-1}d^s, \quad S_1 := \sum_{N \text{ odd}} \sigma_{1,1}(N;2)q^N,$$

$$S_2 := \sum_{N \geq 2 \text{ even}} \sigma_{1,1}(N;2)q^N, \quad (a;q)_\infty := (a)_\infty := \prod_{n\geq 0}(1-aq^n).$$

Melfi [32] proved seven identities of the following type:

$$\sum_{k=0}^{[n/m]} \sigma_r(k)\sigma_s(n-mk) = P\sigma_{r+s+1}(n) + Qn\sigma_{r+s-1}(n), \tag{1}$$

where $\sigma_m(0) = \frac{1}{2}\zeta(-m)$ and $\zeta(s)$ is the Riemann zeta function.

Ramanujan([33], [34, p.136-162]) proved the arithmetic identities

$$\sum_{k=1}^{N-1} \sigma_1(k)\sigma_1(N-k) = \frac{5}{12}\sigma_3(N) + \frac{1}{12}\sigma_1(N) - \frac{1}{2}N\sigma_1(N) \tag{2}$$

and

$$\sum_{k=1}^{N-1} \sigma_1(k)\sigma_3(N-k) = \frac{7}{80}\sigma_5(N) - \frac{1}{8}N\sigma_3(N) + \frac{1}{24}\sigma_3(N) - \frac{1}{240}\sigma_1(N). \tag{3}$$

The evaluation also appears in the work of Besgue, Glaisher, Lahiri, Lehmer and Skoruppa.

In all, Ramanujan obtained nine identities of the type (1).

For $N \in \mathbb{N}^*$, K. S. Williams [40] defined

$$W_N(n) := \sum_{m<n/N} \sigma_1(m)\sigma_1(n-Nm),$$

where m ran through the positive integers $< n/N$. K. S. Williams called W_N the convolution of level N (of the divisor function). Many mathematicians have found convolution formulas for divisor functions(TABLE 1).

Level N	Who	Where
1	Besge(Liouville), Glaisher, Ramanujan	[8], [17], [33]
$2,3,4$	Huard, Ou, Spearman & Williams; Alaca, Alaca & Williams	[22], [5]
$5,7$	Lemire, Williams, Cooper & Toh	[13], [29]
6	Alaca & Williams	[6]
8	Williams	[39]
9	Williams	[40]
10	Royer	[35]
11	Royer	[35]
12	Alaca, Alaca & Williams	[3]
13	Royer	[35]
14	Royer	[35]
16	Alaca, Alaca & Williams	[1]
18	Alaca, Alaca & Williams	[4]
23	Chan & Cooper	[9]
24	Alaca, Alaca & Williams	[2]

< TABLE 1. Some previous computations of W_N >

Later, Hahn found the convolution sums for $\overline{E}_{1,1}(N) := \sum_{k=1}^{N-1} \tilde{\sigma_1}(k) \tilde{\sigma_1}(N-k)$([21, (4.4)]) and $\overline{E}_{1,2}(N) := \sum_{k<N/2} \tilde{\sigma_1}(k)\tilde{\sigma_1}(N-2k)$([21, (4.14)]).

In this paper we calculate the convolution sums for odd and even divisor functions $\sum_{k<N/m} \sigma_{1,i}(dk;2)\sigma_{1,j}(N-mk;2)$ with $i,j=0,1$, $m=1,2,4$, and $d|m$. Furthermore, we evaluated

$$\overline{E}_{1,4}(4N) := \sum_{k=1}^{N-1} \tilde{\sigma_1}(k)\tilde{\sigma_1}(4N-4k) = \frac{1}{24}[6\sigma_3(N) + (7-12N)\sigma_1(2N)$$
$$+ (1-24N)\sigma_1(N) - 12\sigma_{3,1}(N;2) + 2(27N-5)\sigma_{1,1}(N;2)]$$
(Corollary 4.6.(b)),

$$\overline{E}_{2,4}(4N) := \sum_{k=1}^{N-1} \tilde{\sigma_1}(2k)\tilde{\sigma_1}(4N-4k) = \frac{1}{24}\{5\sigma_3(N) - 12\sigma_3(2N) + \sigma_3(4N)$$
$$+ \sigma_1(4N) + 12\sigma_{3,0}(2N;2) + (36N-7)\sigma_1(2N) + 2(1-6N)[5\sigma_{1,0}(2N;2)$$
$$+ \sigma_{1,0}(4N;2)]\}$$
(Corollary 4.4.(b)),

and

$$\overline{E}_{4,4}(4N) := \sum_{k=1}^{N-1} \tilde{\sigma_1}(4k)\tilde{\sigma_1}(4N-4k) = \frac{1}{24}[11\sigma_3(N) - 9\sigma_3(2N)$$
$$+ 6(8N-1)\sigma_1(4N) - 8\sigma_3(4N) + 12(1-8N)\sigma_{1,0}(4N;2)$$
$$+ 12\sigma_{3,0}(4N;2)] \text{ (Theorem 4.1.(b)).}$$

Using MATLAB, we recorded the following table and figure:

N	2	3	4	5	6	7	8	9	10	11	12
$\overline{E}_{1,4}(4N)$	−5	−8	−27	−36	−46	−96	−71	−212	−24	−360	−74
$\overline{E}_{2,4}(4N)$	5	38	105	246	454	756	1217	1682	2568	3210	4574
$\overline{E}_{4,4}(4N)$	25	130	369	810	1454	2460	3793	5470	7752	10350	13870

< TABLE 2. $\overline{E}a, 4(4N)(a = 1, 2, 4)$ for $2 \leq N \leq 12$ >

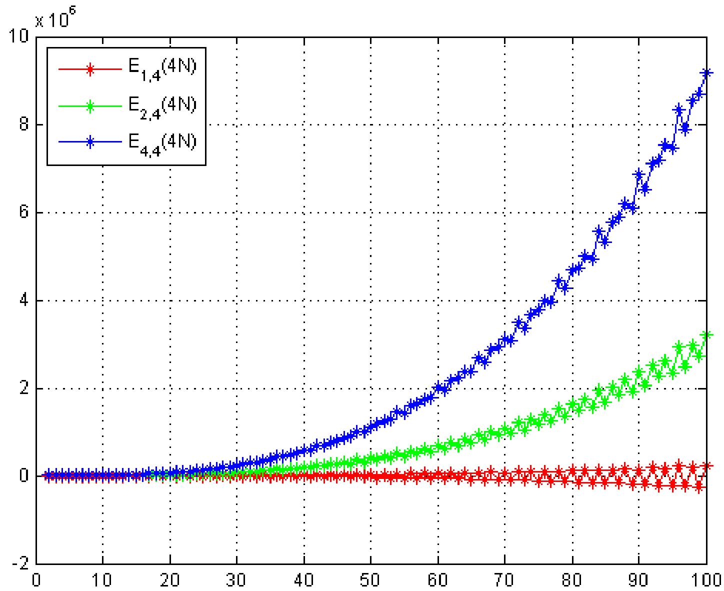

FIGURE 1. $\overline{E}_{1,4}(4N) \sim \overline{E}_{4,4}(4N)$ $(2 \leq N \leq 100)$

In TABLE 2 and Figure 1, we see that $\overline{E}_{1,4}(4N) < \overline{E}_{2,4}(4N) < \overline{E}_{4,4}(4N)$, $\overline{E}_{1,4}(4N) < 0$ and $\overline{E}_{a,4}(4N)(a = 2, 4) > 0$.

In Section 2, we recall the definitions and some properties of the Weierstrass $\wp$-functions and evaluate the values for them in terms of odd and even divisor functions in classical number theory(Ramanujan) of elliptic functions. Using these formulas, we derive some convolution sums for the fundamental divisor convolution sums (e.g., (12)). In Sections 3 and 4, we derive the convolution sums for the odd and even divisors. We found two curious convolution formulas of odd and even divisor functions (Corollary 3.27, FIGURES 10 and 11).

Thus, in this sense, we could ask a similar question regarding convolution formulas as follows: Let p be an odd prime number.

(O) Can you find $r_1, r_2, s_1, s_2, m, \alpha_1, \beta_1, \beta$ in $\mathbb{Z}$ satisfying

$$\sum_{k<\beta p/\beta_1} \sigma_{r_1,s_1}(\alpha_1 k; m)\sigma_{r_2,s_2}(\beta p - \beta_1 k; m) = \sum_{k=1}^{\frac{p-1}{2}} k^u$$

for a fixed u and for all prime p ?

We feel that this sort of problem is generally not easy to solve.

In Corollary 3.27 we concentrate only on $\sum_{k=1}^{N-1} \sigma_{1,i}(dk;2)\sigma_{1,j}(nN - mk;2)$, which are related to (O) with $u = 1$ and 2.

Using these formulas in Sections 3 and 4, we derive the main theorem as follows.

Theorem 1.1. *Let*

$$\overline{D}_{d,m}(N) := \sum_{k=1}^{N-1} \sigma_1(dk)\sigma_1(N - mk), \quad \overline{E}_{d,m}(N) := \sum_{k=1}^{N-1} \tilde{\sigma_1}(dk)\tilde{\sigma_1}(N - mk),$$

$$\overline{F}_{d,m}(N) := \sum_{k=1}^{N-1} \sigma_{1,1}(dk;2)\sigma_{1,1}(N - mk;2), \ \overline{G}_{d,m}(N) := \sum_{k=1}^{N-1} \sigma_{1,0}(dk;2)\sigma_{1,0}(N - mk;2),$$

$$\overline{H}_{d,m}(N) := \sum_{k=1}^{N-1} \sigma_{1,0}(dk;2)\sigma_{1,1}(N - mk;2), \ \tilde{H}_{d,m}(N) := \sum_{k=1}^{N-1} \sigma_{1,1}(dk;2)\sigma_{1,0}(N - mk;2),$$

$$\overline{I}_{d,m}(N) := \sum_{k=1}^{N-1} \sigma_1(dk)\tilde{\sigma_1}(N - mk), \quad \tilde{I}_{d,m}(N) := \sum_{k=1}^{N-1} \tilde{\sigma_1}(dk)\sigma_1(N - mk),$$

$$\overline{J}_{d,m}(N) := \sum_{k=1}^{N-1} \sigma_{1,1}(dk;2)\sigma_1(N - mk), \quad \tilde{J}_{d,m}(N) := \sum_{k=1}^{N-1} \sigma_1(dk)\sigma_{1,1}(N - mk;2),$$

$$\overline{K}_{d,m}(N) := \sum_{k=1}^{N-1} \sigma_{1,0}(dk;2)\sigma_1(N - mk), \quad \tilde{K}_{d,m}(N) := \sum_{k=1}^{N-1} \sigma_1(dk)\sigma_{1,0}(N - mk;2),$$

$$\overline{L}_{d,m}(N) := \sum_{k=1}^{N-1} \sigma_{1,1}(dk;2)\tilde{\sigma_1}(N - mk), \quad \tilde{L}_{d,m}(N) := \sum_{k=1}^{N-1} \tilde{\sigma_1}(dk)\sigma_{1,1}(N - mk;2),$$

$$\overline{M}_{d,m}(N) := \sum_{k=1}^{N-1} \sigma_{1,0}(dk;2)\tilde{\sigma_1}(N - mk), \text{ and } \tilde{M}_{d,m}(N) := \sum_{k=1}^{N-1} \tilde{\sigma_1}(dk)\sigma_{1,0}(N - mk;2).$$

Then the following assertions hold:

(a) *If* $f_{d,m} \in \{\overline{D}_{d,m}, ..., \tilde{M}_{d,m}\}$, *N odd,* $m = 1, 2, 4$, $s = 0, 1, 2$ *and* $d|m|2^s$, *then there exist* $a, b, c, u \in \mathbb{Z}$ *satisfying*

$$f_{d,m}(2^s N) = \frac{1}{u}[a\sigma_3(N) + bN\sigma_1(N) + c\sigma_1(N)]$$

with $a + b + c = 0$ *and* $(u, a, b, c) = 1$.

(b) *If* $f_{d,m}(2^s N) = \frac{1}{u}[a\sigma_3(N) + bN\sigma_1(N) + c\sigma_1(N)]$ *and p is an odd prime number, then* $uf_{d,m}(2^s p) = (ap + b)(p^2 - 1)$.

(c) *Let* $S(N) := \sum_{k=1}^{N} k = \frac{1}{2}N(N+1)$, $T(N) := \sum_{k=1}^{N} k^2 = \frac{1}{6}N(N+1)(2N+1)$ *and p be an odd prime number. If* $f_{d,m}(2^s N) = \frac{1}{u}[a\sigma_3(N) + bN\sigma_1(N) + c\sigma_1(N)]$ *and* $p = 2L + 1$ *is an odd prime number, then* $uf_{d,m}(2^s p) = 8(3aT(L) + bS(L))$.

Here, K. S. Williams proved a special case of Theorem 1.1 (b)([38, p.108, Theorem 10.5]).

In view of Theorem 1.1, we raise the following question.

Question 1.2. *If N is an odd positive integer,* $d, m, n \in \mathbb{Z}^+$ *and* $d|m|n$, *then there exist* $u, a_i \in \mathbb{Z}$ *and* $f_i : \mathbb{N} \to \mathbb{C}$ *satisfying*

$$f_{d,m}(nN) = \frac{1}{u}[a_1 f_1(N) + a_2 f_2(N) + \cdots + a_l f_l(N)]$$

with $a_1 + \cdots + a_l = 0$.

In Corollary 4.9, we prove that Question 1.2 is true for the case of $f_{2^s,2^l}(2^l N)(s \leq l)$. If $d \not| m$, we get a counter example of $f_{d,m}$ in Question 1.2(Example 4.11).

2. Preliminaries

N. J. Fine's list of identities of the basic hypergeometric series type appeared in [16]. While studying these identities, we found that some identities appeared more than once in the list, usually in a similar form (see [11], [12]). In this section, we state two identities that appeared in [16, p. 78, p. 79]:

$$\frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8 (q^4;q^4)_\infty^8} = 1 + 8\sum_{N=1}^{\infty} q^N (2 + (-1)^N) \sum_{\substack{\omega|N \\ \omega \text{ odd}}} \omega, \tag{4}$$

$$\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4} = \sum_{N \text{ odd}} \sigma(N) q^N. \tag{5}$$

Let $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ ($\tau \in \mathfrak{H}$ the complex upper half plane) be a lattice and $z \in \mathbb{C}$. The Weierstrass $\wp$ function relative to Λ_τ is defined by the series

$$\wp(z;\Lambda_\tau) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda_\tau\\ \omega\neq 0}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\},$$

and the Eisenstein series of weight $2k$ for Λ_τ with $k > 1$ is the series

$$G_{2k}(\Lambda_\tau) = \sum_{\substack{\omega\in\Lambda_\tau\\ \omega\neq 0}} \omega^{-2k}.$$

We use the notations $\wp(z)$ and G_{2k} instead of $\wp(z;\Lambda_\tau)$ and $G_{2k}(\Lambda_\tau)$, respectively, when the lattice Λ_τ has been fixed. Now, the Laurent series for $\wp(z)$ about $z = 0$ is given by

$$\wp(z) = z^{-2} + \sum_{k=1}^{\infty} (2k+1) G_{2k+2} z^{2k}.$$

As is customary, by setting

$$g_2(\tau) = g_2(\Lambda_\tau) = 60G_4 \quad \text{and} \quad g_3(\tau) = g_3(\Lambda_\tau) = 140G_6,$$

the algebraic relation between $\wp(z)$ and $\wp'(z)$ becomes

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau).$$

Proposition 2.1. *([26, p.251]) Let $e_1 = \wp(\frac{\tau}{2})$, $e_2 = \wp(\frac{1}{2})$ and $e_3 = \wp(\frac{\tau+1}{2})$, where $P_0 = \prod_{n=1}^{\infty}(1-q^{2n})$, $P_1 = \prod_{n=1}^{\infty}(1-q^{2n-1})$, $P_2 = \prod_{n=1}^{\infty}(1+q^{2n})$ and $P_3 = \prod_{n=1}^{\infty}(1+q^{2n-1})$. Then,*

(a) $e_2 - e_1 = \pi^2 {P_0}^4 {P_3}^8$.
(b) $e_2 - e_3 = \pi^2 {P_0}^4 {P_1}^8$.
(c) $e_3 - e_1 = 2^4 \pi^2 q {P_0}^4 {P_2}^8$.

Using (4), (5) and $\prod_{n=1}^{\infty}(1-q^{2n-1}) = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-q^{2n})}$, $\prod_{n=1}^{\infty}(1+q^{2n-1}) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})}{(1-q^n)} \frac{(1-q^{2n})}{(1-q^{4n})}$ and $\prod_{n=1}^{\infty}(1+q^{2n}) = \prod_{n=1}^{\infty} \frac{(1-q^{4n})}{(1-q^{2n})}$ then we obtain the identities for $\wp(z)$ (see [24]):

(6)

$$\begin{aligned}
\wp(\frac{\tau}{2}) &= -\frac{\pi^2}{3}(\frac{(q^2;q^2)^{20}_\infty}{(q)^8_\infty(q^4;q^4)^8_\infty} + 16\frac{q(q^4;q^4)^8_\infty}{(q^2;q^2)^4_\infty}) \\
&= -\frac{\pi^2}{3}(1 + 8\sum_{N=1}^{\infty} q^N(2+(-1)^N)\sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega + 16\sum_{N \text{ odd}}\sigma(N)q^N) \\
&= -\frac{\pi^2}{3}(1 + 8\sum_{N \text{ odd}} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega + 24\sum_{N \text{ even}} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega + 16\sum_{N \text{ odd}}\sigma(N)q^N) \\
&= -\frac{\pi^2}{3}(1 + 24\sum_{N=1}^{\infty} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega) \\
&= -\frac{\pi^2}{3}(1 + 24\sum_{N=1}^{\infty}\sigma_{1,1}(N;2)q^N) \\
&= -\frac{\pi^2}{3}(1 + 24S_1 + 24S_2) \\
&= -\frac{\pi^2}{3}\sum_{n=0}^{\infty} a_n q^n.
\end{aligned}$$

The values of a_n for $n = 0, 1, \cdots, 10$ are given in the following table.

n	0	1	2	3	4	5	6	7	8	9	10
a_n	1	24	24	96	24	144	96	192	24	312	144

$<$ TABLE 3. a_n for $0 \le n \le 10$ $>$

Similarly, the relations (4) and (5) yield the following arithmetic results [24]:

$$(7) \quad \begin{aligned}
\wp(\frac{\tau+1}{2}) &= -\frac{\pi^2}{3}(\frac{(q^2;q^2)^{20}_\infty}{(q)^8_\infty(q^4;q^4)^8_\infty} - 32\frac{q(q^4;q^4)^8_\infty}{(q^2;q^2)^4_\infty}) \\
&= -\frac{\pi^2}{3}(1 - 24S_1 + 24S_2),
\end{aligned}$$

$$(8) \quad \begin{aligned}
\wp(\frac{1}{2}) &= \frac{2\pi^2}{3}(\frac{(q^2;q^2)^{20}_\infty}{(q)^8_\infty(q^4;q^4)^8_\infty} - 8\frac{q(q^4;q^4)^8_\infty}{(q^2;q^2)^4_\infty}) \\
&= \frac{2\pi^2}{3}(1 + 24S_2).
\end{aligned}$$

Thus, we deduce the following [36, p. 59];

$$\begin{aligned} E_4(\tau) &= \frac{2^2\cdot 3}{(2\pi)^4} g_2(\tau) = \frac{2^2\cdot 3}{(2\pi)^4}(-4(e_1e_2+e_2e_3+e_3e_1)) \\ &= \frac{2^2\cdot 3}{(2\pi)^4}\cdot\frac{4\pi^4}{9}[3(1+24S_2)^2+24^2S_1^2] \end{aligned}$$

and we can change the degree of q^2 to q in $E_4(\tau')$ with $\tau' = \frac{\tau}{2}$;

(9)

$$\begin{aligned} E_4(\tau') = 1 + 240q + \sum_{M=2}^{\infty} [48\sigma_{1,1}(2M;2) + 576 \sum_{\substack{k=1\\ k+l=M}}^{M-1} \sigma_{1,1}(2k;2)\sigma_{1,1}(2l;2) \\ + 192 \sum_{\substack{k=1\\ k+l-1=M}}^{M} \sigma_{1,1}(2k-1;2)\sigma_{1,1}(2l-1;2)]q^M. \end{aligned}$$

From [36, p. 59], we have already known

$$E_4(\tau') = 1 + 240 \sum_{M\geq 1} \sigma_3(M) q^M. \tag{10}$$

3. Some convolution sums of $\sigma_{1,0}(k;2)$ and $\sigma_{1,1}(k;2)$

Formula (2) is equivalent to formula (3.1) in Lahiri [25] and originally appeared in a letter from Besge to Liouville [8]. The formula (2) also appears in the work of Glaisher [17], [18], [19], Lehmer [27, p.106], [28, p.678], Skoruppa [37], and Melfi [32] and Huard, Ou, Spearman and Williams [22].

In [14, p. 300] Glaisher proved that

(11)

$$\sigma(1)\sigma(2N-1)+\sigma(3)\sigma(2N-3)+\cdots+\sigma(2N-1)\sigma(1) = \frac{1}{8}[\sigma_3(2N)-\sigma_3(N)].$$

From (9), (10) and (11), we deduce the following formula(see [23, (11)]):

$$\begin{aligned} \overline{F}_{1,1}(N) :&= \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,1}(N-k;2) \\ &= \frac{1}{24}[11\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(N;2)] \end{aligned} \tag{12}$$

with $N \geq 2$.

Theorem 3.1. *If $N(\geq 2)$ is any integer, then*

(a)

$$\overline{G}_{1,1}(N) := \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,0}(N-k;2)$$
$$= \frac{1}{24}[-3\sigma_3(N) + \sigma_3(2N) - 6\sigma_{3,1}(N;2) + (4-12N)\sigma_{1,0}(N;2)].$$

(b)

$$\overline{H}_{1,1}(N) : = \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,0}(N-k;2) = \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,1}(N-k;2)$$
$$= \frac{1}{24}[\sigma_3(N) - \sigma_1(N) + 3\sigma_{3,1}(N;2) - 3(2N-1)\sigma_{1,1}(N;2)].$$

Proof. In [22, (3.10)], it is proved that

$$\overline{D}_{1,1}(N) := \sum_{k=1}^{N-1} \sigma_1(k)\sigma_1(N-k) = \frac{1}{12}[5\sigma_3(N) + (1-6N)\sigma_1(N)]. \tag{13}$$

We can rewrite (13) as

$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_1(k)\sigma_1(N-k) \\
&= \sum_{k=1}^{N-1} [\sigma_{1,1}(k;2) + \sigma_{1,0}(k;2)][\sigma_{1,1}(N-k;2) + \sigma_{1,0}(N-k;2)] \\
&= \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,1}(N-k;2) + \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,0}(N-k;2) \\
&\quad + \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,1}(N-k;2) + \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,0}(N-k;2) \\
&= \frac{1}{12}[5\sigma_3(N) + (1-6N)\sigma_1(N)].
\end{aligned} \tag{14}$$

In [21, (4.4)] Hahn showed that

$$4\sum_{k=1}^{N-1} \tilde{\sigma_1}(k)\tilde{\sigma_1}(N-k) = -\tilde{\sigma_3}(N) + (2N-1)\tilde{\sigma_1}(N). \tag{15}$$

Similarly, let us consider the equation (15),

$$
\begin{aligned}
\overline{E}_{1,1}(N) &:= \sum_{k=1}^{N-1} \tilde{\sigma_1}(k)\tilde{\sigma_1}(N-k) \\
&= \sum_{k=1}^{N-1} [\sigma_{1,1}(k;2) - \sigma_{1,0}(k;2)][\sigma_{1,1}(N-k;2) - \sigma_{1,0}(N-k;2)] \\
&= \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,1}(N-k;2) - \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,0}(N-k;2) \\
&\quad - \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,1}(N-k;2) + \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,0}(N-k;2) \\
&= \frac{1}{4}[-\tilde{\sigma_3}(N) + (2N-1)\tilde{\sigma_1}(N)].
\end{aligned} \tag{16}
$$

(a) From (12), (14) and (16), we find that

$$
\begin{aligned}
&2\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,1}(N-k;2) + 2\sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,0}(N-k;2) \\
&= 2 \cdot \frac{1}{24}[11\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(N;2)] + 2\sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,0}(N-k;2) \\
&= \frac{1}{12}[5\sigma_3(N) + (1-6N)\sigma_1(N)] + \frac{1}{4}[-\tilde{\sigma_3}(N) + (2N-1)\tilde{\sigma_1}(N)].
\end{aligned} \tag{17}
$$

(a) is obtained from $\tilde{\sigma_3}(N) = -\sigma_3(N) + 2\sigma_{3,1}(N;2)$ and $\tilde{\sigma_1}(N) = \sigma_1(N) - 2\sigma_{1,0}(N;2)$.

(b) Now, evaluating (14) minus (16), we obtain

$$
\begin{aligned}
&2\sum_{k=1}^{N-1} [\sigma_{1,1}(k;2)\sigma_{1,0}(N-k;2) + \sigma_{1,0}(k;2)\sigma_{1,1}(N-k;2)] \\
&= 4\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,0}(N-k;2) \\
&= \frac{1}{12}[5\sigma_3(N) + (1-6N)\sigma_1(N)] - \frac{1}{4}[-\tilde{\sigma_3}(N) + (2N-1)\tilde{\sigma_1}(N)].
\end{aligned}
$$

After employing the fact $\tilde{\sigma_3}(N) = -\sigma_3(N) + 2\sigma_{3,1}(N;2)$, we achieve the desired result.

□

Example 3.2. *Let*

$$\overline{D}_{1,1}(N) := \sum_{k=1}^{N-1} \sigma_1(k)\sigma_1(N-k), \quad \overline{E}_{1,1}(N)) := \sum_{k=1}^{N-1} \tilde{\sigma_1}(k)\tilde{\sigma_1}(N-k),$$

$$\overline{F}_{1,1}(N) := \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,1}(N-k;2), \quad \overline{G}_{1,1}(N) := \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,0}(N-k;2),$$

$$\overline{H}_{1,1}(N) := \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,1}(N-k;2).$$

We suggest TABLE 4 and FIGURE 2 for $\overline{D}_{1,1}(N) \sim \overline{H}_{1,1}(N)$.

N	2	3	4	5	6	7	8	9	10	11
$\overline{D}_{1,1}(N)$	1	6	17	38	70	116	185	258	384	490
$\overline{E}_{1,1}(N)$	1	-2	9	-18	38	-60	97	-134	192	-270
$\overline{F}_{1,1}(N)$	1	2	9	10	30	28	73	62	136	110
$\overline{G}_{1,1}(N)$	0	0	4	0	24	0	68	0	152	0
$\overline{H}_{1,1}(N)$	0	2	2	14	8	44	22	98	48	190

< *TABLE 4. Examples for* $\overline{D}_{1,1}(N) \sim \overline{H}_{1,1}(N)(2 \leq N \leq 11)$ >

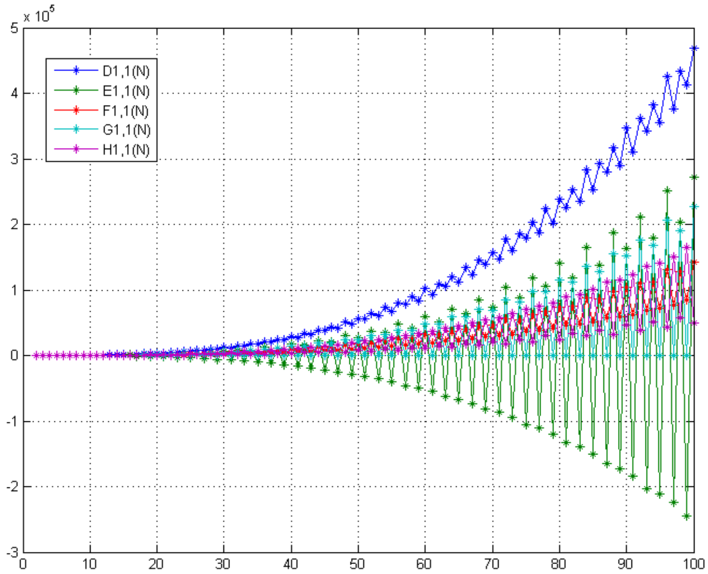

FIGURE 2. $\overline{D}_{1,1}(N) \sim \overline{H}_{1,1}(N)$ $(2 \leq N \leq 100)$

Remark 3.3. If k is odd(resp. even) then $\tilde{\sigma_1}(k) > 0$(resp. < 0). We deduce that $\tilde{\sigma_1}(k)\tilde{\sigma_1}(2N-k) > 0$, $\tilde{\sigma_1}(k)\tilde{\sigma_1}(2N+1-k) < 0$, $\overline{E}_{1,1}(2N) > 0$ and $\overline{E}_{1,1}(2N+1) < 0$(See TABLE 4, FIGURE 2).

Corollary 3.4. *Let $N(\geq 2)$ be any integer. Then we get the following.*

(a)

$$\overline{I}_{1,1}(N) := \sum_{k=1}^{N-1} \sigma_1(k)\tilde{\sigma_1}(N-k) = \sum_{k=1}^{N-1} \tilde{\sigma_1}(k)\sigma_1(N-k)$$
$$= \frac{1}{12}\left[7\sigma_3(N) - \sigma_3(2N) + 3\sigma_{3,1}(N;2) - \sigma_1(N) + (6N-1)\sigma_{1,0}(N;2)\right].$$

(b)

$$\overline{J}_{1,1}(N) := \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_1(N-k) = \sum_{k=1}^{N-1} \sigma_1(k)\sigma_{1,1}(N-k;2)$$
$$= \frac{1}{24}\left[12\sigma_3(N) - \sigma_3(2N) + 3\sigma_{3,1}(N;2) - \sigma_1(N) + (1-6N)\sigma_{1,1}(N;2)\right].$$

(c)

$$\overline{K}_{1,1}(N) := \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_1(N-k) = \sum_{k=1}^{N-1} \sigma_1(k)\sigma_{1,0}(N-k;2)$$
$$= \frac{1}{24}\left[-2\sigma_3(N) + \sigma_3(2N) - 3\sigma_{3,1}(N;2) + (2-6N)\sigma_1(N)\right.$$
$$\left. +(1-6N)\sigma_{1,0}(N;2)\right].$$

(d)

$$\overline{L}_{1,1}(N) := \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\tilde{\sigma_1}(N-k) = \sum_{k=1}^{N-1} \tilde{\sigma_1}(k)\sigma_{1,1}(N-k;2)$$
$$= \frac{1}{24}\left[10\sigma_3(N) - \sigma_3(2N) - 3\sigma_{3,1}(N;2) + \sigma_1(N) + (6N-5)\sigma_{1,1}(N;2)\right].$$

(e)

$$\overline{M}_{1,1}(N) := \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\tilde{\sigma_1}(N-k) = \sum_{k=1}^{N-1} \tilde{\sigma_1}(k)\sigma_{1,0}(N-k;2)$$
$$= \frac{1}{24}\left[4\sigma_3(N) - \sigma_3(2N) + 9\sigma_{3,1}(N;2) + (12N-5)\sigma_1(N)\right.$$
$$\left. +(7-18N)\sigma_{1,1}(N;2)\right].$$

In particular, if N is odd, then the above equation can be simplified to the following convolution formulas for the cases of $\overline{D}_{1,1}(N) \sim \overline{M}_{1,1}(N)$.

N: *odd*	*convolution formulas*	N: *odd*	*convolution formulas*
$\overline{D}_{1,1}(N)$	$\frac{1}{12}[5\sigma_3(N) + (1-6N)\sigma_1(N)]$	$\overline{I}_{1,1}(N)$	$\frac{1}{12}[\sigma_3(N) - \sigma_1(N)]$
$\overline{E}_{1,1}(N)$	$\frac{1}{4}[-\sigma_3(N) + (2N-1)\sigma_1(N)]$	$\overline{J}_{1,1}(N)$	$\frac{1}{4}[\sigma_3(N) - N\sigma_1(N)]$
$\overline{F}_{1,1}(N)$	$\frac{1}{12}[\sigma_3(N) - \sigma_1(N)]$	$\overline{K}_{1,1}(N)$	$\frac{1}{12}[2\sigma_3(N) + (1-3N)\sigma_1(N)]$
$\overline{G}_{1,1}(N)$	0	$\overline{L}_{1,1}(N)$	$\frac{1}{12}[-\sigma_3(N) + (3N-2)\sigma_1(N)]$
$\overline{H}_{1,1}(N)$	$\frac{1}{12}[2\sigma_3(N) + (1-3N)\sigma_1(N)]$	$\overline{M}_{1,1}(N)$	$\frac{1}{12}[2\sigma_3(N) + (1-3N)\sigma_1(N)]$

< TABLE 5. Convolution formulas for $\overline{D}_{1,1}(N) \sim \overline{M}_{1,1}(N)(N : odd)$ >

Proof. Consider

$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_1(k)\tilde{\sigma_1}(N-k) \\
&= \sum_{k=1}^{N-1} (\sigma_{1,1}(k;2) + \sigma_{1,0}(k;2))(\sigma_{1,1}(N-k;2) - \sigma_{1,0}(N-k;2)) \\
&= \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,1}(N-k;2) - \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,0}(N-k;2).
\end{aligned}$$

Hence, the proof is completed by using (12) and Theorem 3.1 (a). Other cases are similar to (a). □

Remark 3.5. Glaisher evaluated similar convolution formulas in Corollary 3.1, 3.4([38, p.130]).

Example 3.6. *We suggest TABLE 6 and FIGURE 3 for $\overline{I}_{1,1}(N) \sim \overline{M}_{1,1}(N)$.*

N	2	3	4	5	6	7	8	9	10	11
$\overline{I}_{1,1}(N)$	1	2	5	10	6	28	5	62	−16	110
$\overline{J}_{1,1}(N)$	1	4	11	24	38	72	95	160	184	300
$\overline{K}_{1,1}(N)$	0	2	6	14	32	44	90	98	200	190
$\overline{L}_{1,1}(N)$	1	0	7	−4	22	−16	51	−36	88	−80
$\overline{M}_{1,1}(N)$	0	2	−2	14	−16	44	−46	98	−104	190

< TABLE 6. Examples for $\overline{I}_{1,1}(N) \sim \overline{M}_{1,1}(N)(2 \leq N \leq 11)$ >

In TABLE 6 and Figure 3, we consider values of $\overline{M}_{1,1}(N)$. From the fact that $\sigma_{1,0}(odd;2) = 0$, we check $\overline{M}_{1,1}(N) = \sum_{N/2k} \sigma_{1,0}(2k;2)\tilde{\sigma_1}(N-2k)$

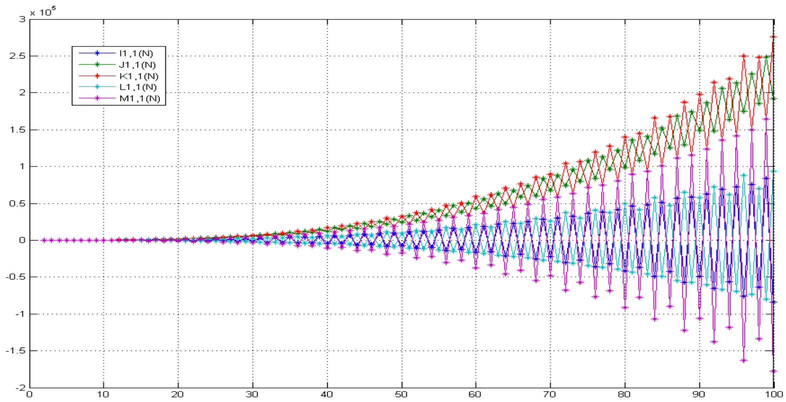

FIGURE 3. $\overline{I}_{1,1}(N) \sim \overline{M}_{1,1}(N)$ $(2 \leq N \leq 100)$

If N is an odd integer(resp., even), then $\tilde{\sigma_1}(N-2k) > 0$(resp., < 0). Hence, we obtain the following corollary:

Corollary 3.7. (a) *If $N = 2$, then $\overline{M}_{1,1}(N) = 0$.*
(b) *If $N(> 1)$ is odd, then $\overline{M}_{1,1}(N) > 0$.*
(c) *If $N(> 2)$ is even, then $\overline{M}_{1,1}(N) < 0$.*

Corollary 3.8. (a) *If $N(\geq 2)$ is any integer, then*

$$\begin{aligned}\overline{G}_{2,2}(2N) :&= \sum_{k=1}^{N-1} \sigma_{1,0}(2k;2)\sigma_{1,0}(2N-2k;2)\\ &= \frac{1}{4}\sigma_{3,0}(2N;2) - \frac{1}{3}\sigma_3(N) + (\frac{1}{6} - N)\sigma_{1,0}(2N;2).\end{aligned}$$

(b) *If $N(\geq 1)$ is any integer, then*

$$\begin{aligned}\overline{G}_{2,2}(2N+1) :&= \sum_{k=1}^{N} \sigma_{1,0}(2k;2)\sigma_{1,0}(2N+1-2k;2)\\ &= \sum_{k=1}^{2N} \sigma_{1,0}(k;2)\sigma_{1,0}(2N+1-k;2) = 0.\end{aligned}$$

Proof. (a) Using Theorem 3.1 (a), we obtain

$$\begin{aligned}
&\sum_{k=1}^{2N-1} \sigma_{1,0}(k;2)\sigma_{1,0}(2N-k;2)\\
&= \sum_{k=1}^{N-1} \sigma_{1,0}(2k;2)\sigma_{1,0}(2N-2k;2) + \sum_{k=1}^{N} \sigma_{1,0}(2k-1;2)\sigma_{1,0}(2N-(2k-1);2)\\
&= \sum_{k=1}^{N-1} \sigma_{1,0}(2k;2)\sigma_{1,0}(2N-2k;2)
\end{aligned}$$

by $\sigma_{1,0}(2k-1;2) = 0 = \sigma_{1,0}(2N-(2k-1);2)$. Thus, it has the same value of Theorem 3.1(a) by replacing N with $2N$.

(b) Since $\sigma_{1,0}(odd;2) = 0$ and $2N+1-k \not\equiv k \pmod 2$, therefore $\sigma_{1,0}(k;2)\sigma_{1,0}(2N+1-k;2)$ and $\sigma_{1,0}(2k;2)\sigma_{1,0}(2N+1-2k;2)$ are always zero for any integer k, N.

□

Theorem 3.9. *If $N(\geq 1)$ is any integer, then*

(18)

$$\begin{aligned}
&\sum_{k=1}^{2N-1} \sigma_{1,1}(k;2)\sigma_{1,0}(4N-2k;2) = \sum_{k=1}^{2N-1} \sigma_{1,1}(2k;2)\sigma_{1,0}(4N-2k;2)\\
&= \sum_{k=1}^{2N-1} \sigma_{1,0}(2k;2)\sigma_{1,1}(4N-2k;2) = \sum_{k=1}^{2N-1} \sigma_{1,0}(2k;2)\sigma_{1,1}(2N-k;2)\\
&= \frac{1}{24}[\sigma_3(4N) - \sigma_1(4N) + 3\sigma_{3,1}(4N;2) - 3(8N-1)\sigma_{1,1}(4N;2)].
\end{aligned}$$

Proof. In Theorem 3.1(b), let us consider $n = 4N$. Then ;

$$\begin{aligned}
&\sum_{k=1}^{4N-1} \sigma_{1,1}(k;2)\sigma_{1,0}(4N-k;2)\\
&= \sum_{k=1}^{2N-1} \sigma_{1,1}(2k;2)\sigma_{1,0}(4N-2k;2) + \sum_{k=1}^{2N} \sigma_{1,1}(2k-1;2)\sigma_{1,0}(4N-(2k-1);2)\\
&= \sum_{k=1}^{2N-1} \sigma_{1,1}(2k;2)\sigma_{1,0}(4N-2k;2),
\end{aligned}$$

by $\sigma_{1,0}(4N-(2k-1);2) = 0$. □

Corollary 3.10. (a) *If $N(\geq 2)$ is any integer, then*

$$\begin{aligned}
\overline{H}_{2,2}(2N) &:= \sum_{k=1}^{N-1} \sigma_{1,1}(2k;2)\sigma_{1,0}(2N-2k;2) \\
&= \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,0}(2N-2k;2) \\
&= \sum_{k=1}^{N-1} \sigma_{1,0}(2k;2)\sigma_{1,1}(2N-2k;2) = \sum_{k=1}^{N-1} \sigma_{1,0}(2k;2)\sigma_{1,1}(N-k;2) \\
&= \frac{1}{24}[\sigma_3(2N) - \sigma_1(2N) + 3\sigma_{3,1}(2N;2) - 3(4N-1)\sigma_{1,1}(2N;2)].
\end{aligned} \tag{19}$$

(b) *If $N(\geq 1)$ is any integer, then*

$$\begin{aligned}
\overline{H}_{2,2}(2N+1) &:= \sum_{k=1}^{N} \sigma_{1,1}(2k-1;2)\sigma_{1,0}(2N+1-(2k-1);2) \\
&= \sum_{k=1}^{N} \sigma_{1,0}(2k;2)\sigma_{1,1}(2N+1-2k;2) \\
&= \frac{1}{6}[\sigma_3(2N+1) - (3N+1)\sigma_1(2N+1)].
\end{aligned}$$

(c)

$$\begin{aligned}
\tilde{H}_{2,2}(2N+1) &:= \sum_{k=1}^{N} \sigma_{1,1}(2k-1;2)\sigma_{1,0}(2N-(2k-1);2) \\
&= \sum_{k=1}^{N} \sigma_{1,1}(2k;2)\sigma_{1,0}(2N+1-2k;2) \\
&= \sum_{k=1}^{N} \sigma_{1,1}(k;2)\sigma_{1,0}(2N+1-2k;2) = 0.
\end{aligned}$$

Proof. (a) Putting $N \to \frac{N}{2}$ in (18). Then we find the identity (19).
(b) By Theorem 3.1 (b), we may write

$$
\begin{aligned}
&\sum_{k=1}^{2N}\sigma_{1,1}(k;2)\sigma_{1,0}(2N+1-k;2)\\
&=\sum_{k=1}^{N}\sigma_{1,1}(2k-1;2)\sigma_{1,0}(2N+1-(2k-1);2)\\
&=\frac{1}{24}[\sigma_3(2N+1)-\sigma_1(2N+1)+3\sigma_{3,1}(2N+1;2)\\
&\qquad -3(2(2N+1)-1)\sigma_{1,1}(2N+1;2)]
\end{aligned}
\tag{20}
$$

by $\sigma_{1,0}(2N+1-(2k);2)=0$. After simplifying the identity (20), we find the result.

(c) It follows from the fact $\sigma_{1,0}(\text{odd};2)=0$.

□

Theorem 3.11. (a) *If $N(\geq 1)$ is a positive integer, then*

$$
\overline{F}_{2,2}(2N+1):=\sum_{k=1}^{N}\sigma_{1,1}(2k;2)\sigma_{1,1}(2N+1-2k;2)
$$
$$
=\sum_{k=1}^{N}\sigma_{1,1}(k;2)\sigma_{1,1}(2N+1-2k;2)=\frac{1}{24}[\sigma_3(2N+1)-\sigma_1(2N+1)].
$$

(b) *If $N(\geq 2)$ is a integer, then*

$$
\begin{aligned}
\overline{F}_{2,2}(2N)&:=\sum_{k=1}^{N-1}\sigma_{1,1}(2k;2)\sigma_{1,1}(2N-2k;2)\\
&=\sum_{k=1}^{N-1}\sigma_{1,1}(k;2)\sigma_{1,1}(2N-2k;2)\\
&=\overline{F}_{1,2}(2N)=\frac{1}{24}[11\sigma_3(N)-\sigma_3(2N)-2\sigma_{1,1}(N;2)].
\end{aligned}
$$

Proof. (a) Directly numbering the index k in (12), we have

$$
\sum_{k=1}^{N-1}\sigma_{1,1}(k;2)\sigma_{1,1}(2N+1-k;2)=2\sum_{k=1}^{N}\sigma_{1,1}(2k;2)\sigma_{1,1}(2N+1-2k;2).
$$

Thus,

$$\sum_{k=1}^{N} \sigma_{1,1}(2k;2)\sigma_{1,1}(2N+1-2k;2)$$
$$= \frac{1}{48}[11\sigma_3(2N+1) - \sigma_3(4N+2) - 2\sigma_1(2N+1)].$$

Since $\sigma_{1,1}(2k;2) = \sigma_{1,1}(k;2)$ and $\sigma_3(2N) = \sigma_3(2)\sigma_3(N)$, it is obtained.

(b) For even $2N$,

$$\sum_{k=1}^{N-1} \sigma_{1,1}(2k;2)\sigma_{1,1}(2N-2k;2) = \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,1}(N-k;2).$$

By using (12), □

Corollary 3.12. *Let $N \geq 2$ is any integer. Then*

(a)

$$\overline{D}_{2,2}(2N) := \sum_{k=1}^{N-1} \sigma_1(2k)\sigma_1(2N-2k)$$
$$= \frac{1}{24}[3\sigma_3(N) + 7\sigma_3(2N) + 2(1-12N)\sigma_1(2N)].$$

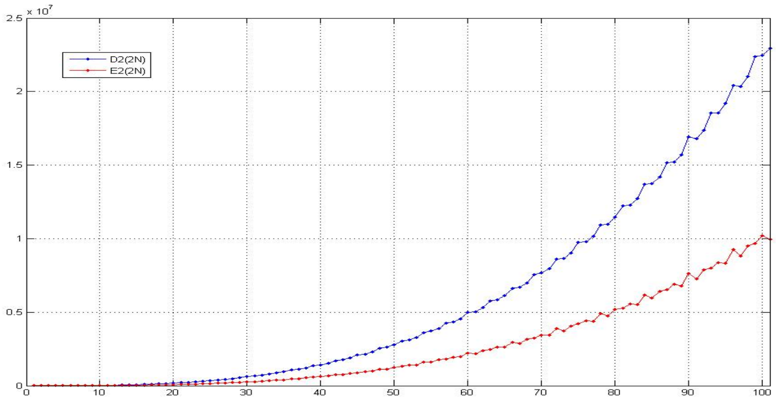

FIGURE 4. $\overline{D}_{2,2}(2N)$ and $\overline{E}_{2,2}(2N)$ $(2 \leq N \leq 100)$

(b) *(Hahn* [21], *(4.23))*

$$\begin{aligned}\overline{E}_{2,2}(2N)) :&= \sum_{k=1}^{N-1} \tilde{\sigma_1}(2k)\tilde{\sigma_1}(2N-2k) \\ &= \frac{1}{24}[3\sigma_3(N) - 9\sigma_3(2N) + 12\sigma_{3,0}(2N;2) + 2\sigma_1(2N) \\ &\quad + 8(3N-1)\sigma_{1,1}(2N;2) + 4(1-6N)\sigma_{1,0}(2N;2)].\end{aligned}$$

(c)

$$\begin{aligned}\overline{I}_{2,2}(2N) :&= \sum_{k=1}^{N-1} \sigma_1(2k)\tilde{\sigma_1}(2N-2k) \\ &= \frac{1}{24}[19\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(2N;2) - 6\sigma_{3,0}(2N;2) \\ &\quad - 4(1-6N)\sigma_{1,0}(2N;2)].\end{aligned}$$

(d)

$$\begin{aligned}\overline{J}_{2,2}(2N) :&= \sum_{k=1}^{N-1} \sigma_{1,1}(2k;2)\sigma_1(2N-2k) \\ &= \frac{1}{24}[11\sigma_3(N) - \sigma_1(2N) + 3\sigma_{3,1}(2N;2) - (12N-1)\sigma_{1,1}(2N;2)].\end{aligned}$$

(e)

$$\begin{aligned}\overline{K}_{2,2}(2N) :&= \sum_{k=1}^{N-1} \sigma_{1,0}(2k;2)\sigma_1(2N-2k) \\ &= \frac{1}{24}[4\sigma_3(2N) - 8\sigma_3(N) + 2(1-6N)\sigma_1(2N) \\ &\quad + 3\sigma_{3,0}(2N;2) + (1-12N)\sigma_{1,0}(2N;2)].\end{aligned}$$

(f)

$$\begin{aligned}\overline{L}_{2,2}(2N) :&= \sum_{k=1}^{N-1} \sigma_{1,1}(2k;2)\tilde{\sigma_1}(2N-2k) \\ &= \frac{1}{24}[11\sigma_3(N) - 2\sigma_3(2N) + \sigma_1(2N) - 3\sigma_{3,1}(2N;2) \\ &\quad + (12N-5)\sigma_{1,1}(2N;2)].\end{aligned}$$

(g)

$$\begin{aligned}\overline{M}_{2,2}(2N) :&= \sum_{k=1}^{N-1} \sigma_{1,0}(2k;2)\tilde{\sigma_1}(2N-2k) \\ &= \frac{1}{24}[-5\sigma_3(2N) + 9\sigma_{3,1}(2N;2) + 8\sigma_3(N)) \\ &\quad + (24N-5)\sigma_1(2N) + (7-36N)\sigma_{1,1}(2N;2)].\end{aligned}$$

In particular, if N is odd, then the above can be simplified to the following convolution formulas for the cases of $\overline{D}_{2,2}(2N) \sim \overline{M}_{2,2}(2N)$.

N: odd	convolution formulas	N: odd	convolution formulas
$\overline{D}_{2,2}(2N)$	$\frac{1}{4}[11\sigma_3(N) + (1-12N)\sigma_1(N)]$	$\overline{I}_{2,2}(2N)$	$\frac{1}{12}[-19\sigma_3(N) + (24N-5)\sigma_1(N)]$
$\overline{E}_{2,2}(2N)$	$\frac{1}{4}[3\sigma_3(N) + (1-4N)\sigma_1(N)]$	$\overline{J}_{2,2}(2N)$	$\frac{1}{12}[7\sigma_3(N) - (6N+1)\sigma_1(N)]$
$\overline{F}_{2,2}(2N)$	$\frac{1}{12}[\sigma_3(N) - \sigma_1(N)]$	$\overline{K}_{2,2}(2N)$	$\frac{1}{6}[13\sigma_3(N) + (2-15N)\sigma_1(N)]$
$\overline{G}_{2,2}(2N)$	$\frac{1}{3}[5\sigma_3(N) + (1-6N)\sigma_1(N)]$	$\overline{L}_{2,2}(2N)$	$\frac{1}{12}[-5\sigma_3(N) + (6N-1)\sigma_1(N)]$
$\overline{H}_{2,2}(2N)$	$\frac{1}{2}[\sigma_3(N) - N\sigma_1(N)]$	$\overline{M}_{2,2}(2N)$	$\frac{1}{6}[-7\sigma_3(N) + (9N-2)\sigma_1(N)]$

< TABLE 7. Convolution formulas for $\overline{D}_{2,2}(2N) \sim \overline{M}_{2,2}(2N)$($N$: odd) >

Proof. (a) If $N \geq 2$ is any integer, then

$$\begin{aligned}&\sum_{k=1}^{N-1} \sigma_1(2k)\sigma_1(2N-2k) \\ &= \sum_{k=1}^{N-1} [\sigma_{1,1}(2k;2) + \sigma_{1,0}(2k;2)][\sigma_{1,1}(2N-2k;2) + \sigma_{1,0}(2N-2k;2)] \\ &= \sum_{k=1}^{N-1} \sigma_{1,1}(2k;2)\sigma_{1,1}(2N-2k;2) + \sum_{k=1}^{N-1} \sigma_{1,1}(2k;2)\sigma_{1,0}(2N-2k;2) \\ &\quad + \sum_{k=1}^{N-1} \sigma_{1,0}(2k;2)\sigma_{1,1}(2N-2k;2) + \sum_{k=1}^{N-1} \sigma_{1,0}(2k;2)\sigma_{1,0}(2N-2k;2).\end{aligned}$$

Thus from Corollary 3.8 (a), Corollary 3.10 (a) and Theorem 3.11 (b), we reach the desired result. Other cases are similar.

□

Remark 3.13. If N is an odd integer, then $\overline{D}_{2,2}(2N) = \overline{E}_{2,2}(2N) + 4\overline{H}_{2,2}(2N)$(TABLE 7, 8 and FIGURE 4).

Example 3.14. *We form the following table(TABLE 8) for $\overline{D}_{2,2}(2N) \sim \overline{M}_{2,2}(2N)$.*

N	2	3	4	5	6	7	8	9	10	11	12
$\overline{D}_{2,2}(2N)$	9	42	121	258	462	780	1193	1734	2408	3270	4318
$\overline{E}_{2,2}(2N)$	1	10	33	66	158	204	433	454	936	870	1678
$\overline{F}_{2,2}(2N)$	1	2	9	10	30	28	73	62	136	110	254
$\overline{G}_{2,2}(2N)$	4	24	68	152	280	464	740	1032	1536	1960	2744
$\overline{H}_{2,2}(2N)$	2	8	22	48	76	144	190	320	368	600	660
$\overline{I}_{2,2}(2N)$	−3	−22	−59	−142	−250	−436	−667	−970	−1400	−1850	−2490
$\overline{J}_{2,2}(2N)$	3	10	31	58	106	172	263	382	504	710	914
$\overline{K}_{2,2}(2N)$	6	32	90	200	356	608	930	1352	1904	2560	3404
$\overline{L}_{2,2}(2N)$	−1	−6	−13	−38	−46	−116	−117	−258	−232	−490	−406
$\overline{M}_{2,2}(2N)$	−2	−16	−46	−104	−204	−320	−550	−712	−1168	−1360	−2084

< *TABLE 8. Examples for* $\overline{D}_{2,2}(2N) \sim \overline{M}_{2,2}(2N)(2 \leq N \leq 12)$ >

Corollary 3.15. *Let* $N \geq 1$ *be any integer. Then*

(a)

$$\begin{aligned}\overline{D}_{2,2}(2N+1) :&= \sum_{k=1}^{N} \sigma_1(2k)\sigma_1(2N+1-2k)\\ &= \frac{1}{24}[5\sigma_3(2N+1) - (12N+5)\sigma_1(2N+1)].\end{aligned}$$

(b) *(Hahn* [21], *(4.23))*

$$\begin{aligned}\overline{E}_{2,2}(2N+1) :&= \sum_{k=1}^{N} \tilde{\sigma_1}(2k)\tilde{\sigma_1}(2N+1-2k)\\ &= \frac{1}{8}[-\sigma_3(2N+1) + (4N+1)\sigma_1(2N+1)].\end{aligned}$$

(c)

$$\begin{aligned}\overline{I}_{2,2}(2N+1) :&= \sum_{k=1}^{N} \sigma_1(2k)\tilde{\sigma_1}(2N+1-2k)\\ &= \sum_{k=1}^{N} \sigma_1(2k)\sigma_1(2N+1-2k)\\ &= \frac{1}{24}[5\sigma_3(2N+1) - (12N+5)\sigma_1(2N+1)].\end{aligned}$$

(c')

$$\begin{aligned}\tilde{I}_{2,2}(2N+1) :&= \sum_{k=1}^{N} \tilde{\sigma_1}(2k)\sigma_1(2N+1-2k)\\ &= \frac{1}{8}[-\sigma_3(2N+1) + (4N+1)\sigma_1(2N+1)].\end{aligned}$$

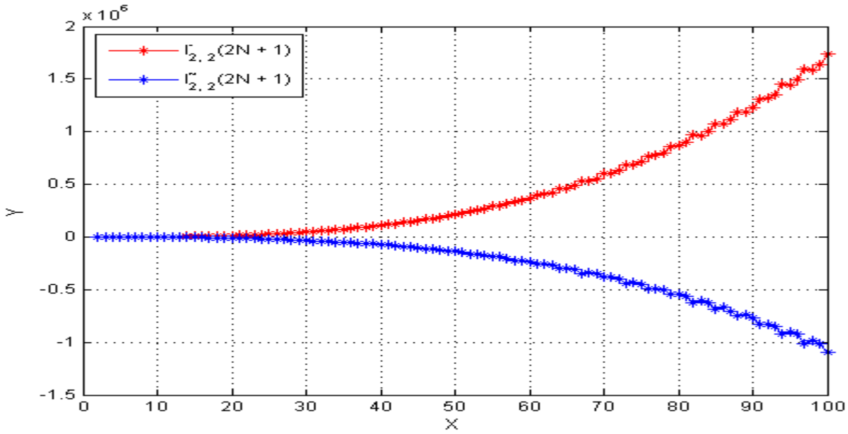

FIGURE 5. $\overline{I}_{2,2}(2N+1)$ and $\tilde{I}_{2,2}(2N+1)$ $(3 \leq N \leq 100)$

(d)

$$\begin{aligned}\overline{J}_{2,2}(2N+1) :&= \sum_{k=1}^{N} \sigma_{1,1}(2k;2)\sigma_1(2N+1-2k) \\ &= \frac{1}{24}[\sigma_3(2N+1) - \sigma_1(2N+1)].\end{aligned}$$

(d')

$$\begin{aligned}\tilde{J}_{2,2}(2N+1) :&= \sum_{k=1}^{N} \sigma_1(2k)\sigma_{1,1}(2N+1-2k;2) \\ &= \sum_{k=1}^{N} \sigma_1(2N+1-2k)\sigma_1(2k) \\ &= \frac{1}{24}[5\sigma_3(2N+1) - (12N+5)\sigma_1(2N+1)].\end{aligned}$$

(e)

$$\begin{aligned}\overline{K}_{2,2}(2N+1) :&= \sum_{k=1}^{N} \sigma_{1,0}(2k;2)\sigma_1(2N+1-2k) \\ &= \frac{1}{6}[\sigma_3(2N+1) - (3N+1)\sigma_1(2N+1)].\end{aligned}$$

(e')

$$\tilde{K}_{2,2}(2N+1) := \sum_{k=1}^{N} \sigma_1(2k)\sigma_{1,0}(2N+1-2k;2) = 0.$$

(f)

$$\begin{aligned}\overline{L}_{2,2}(2N+1) :&= \sum_{k=1}^{N} \sigma_{1,1}(2k;2)\tilde{\sigma_1}(2N+1-2k) \\ &= \frac{1}{24}[\sigma_3(2N+1) - \sigma_1(2N+1)].\end{aligned}$$

(f')

$$\begin{aligned}\tilde{L}_{2,2}(2N+1) :&= \sum_{k=1}^{N} \tilde{\sigma_1}(2k)\sigma_{1,1}(2N+1-2k;2) \\ &= \sum_{k=1}^{N} \tilde{\sigma_1}(2k)\tilde{\sigma_1}(2N+1-2k) \\ &= \frac{1}{8}[-\sigma_3(2N+1) + (4N+1)\sigma_1(2N+1)].\end{aligned}$$

(g)

$$\begin{aligned}\overline{M}_{2,2}(2N+1) :&= \sum_{k=1}^{N} \sigma_{1,0}(2k;2)\tilde{\sigma_1}(2N+1-2k) \\ &= \frac{1}{6}[\sigma_3(2N+1) - (3N+1)\sigma_1(2N+1)].\end{aligned}$$

(g')

$$\tilde{M}_{2,2}(2N+1) := \sum_{k=1}^{N} \tilde{\sigma_1}(2k)\sigma_{1,0}(2N+1-2k;2) = 0.$$

We summarize that the above equation can be simplified to the following coefficients for the cases of $\overline{D}_{2,2}(2N+1) \sim \overline{M}_{2,2}(2N+1)$ *in Theorem 1.1.*

N	u	a	b	c	N	u	a	b	c
$\overline{D}_{2,2}(2N+1)$	24	5	-6	1	$\overline{J}_{2,2}(2N+1)$	24	1	0	-1
$\overline{E}_{2,2}(2N+1)$	8	-1	2	-1	$\tilde{J}_{2,2}(2N+1)$	24	5	-6	1
$\overline{F}_{2,2}(2N+1)$	24	1	0	-1	$\overline{K}_{2,2}(2N+1)$	12	2	-3	1
$\overline{G}_{2,2}(2N+1)$	1	0	0	0	$\tilde{K}_{2,2}(2N+1)$	1	0	0	0
$\overline{H}_{2,2}(2N+1)$	12	2	-3	1	$\overline{L}_{2,2}(2N+1)$	24	1	0	-1
$\tilde{H}_{2,2}(2N+1)$	1	0	0	0	$\tilde{L}_{2,2}(2N+1)$	8	-1	2	1
$\overline{I}_{2,2}(2N+1)$	24	5	-6	1	$\overline{M}_{2,2}(2N+1)$	12	2	-3	1
$\tilde{I}_{2,2}(2N+1)$	8	-1	2	-1	$\tilde{M}_{2,2}(2N+1)$	1	0	0	0

< TABLE 9. Coefficients of a, b, c for $\overline{D}_{2,2}(2N+1) \sim \overline{M}_{2,2}(2N+1)$ >

Proof. (a) If $N \geq 1$ is any integer, then

$$\begin{aligned}
&\sum_{k=1}^{N} \sigma_1(2k)\sigma_1(2N+1-2k) \\
&= \sum_{k=1}^{N} [\sigma_{1,1}(2k;2) + \sigma_{1,0}(2k;2)][\sigma_{1,1}(2N+1-2k;2) + \sigma_{1,0}(2N+1-2k;2)] \\
&= \sum_{k=1}^{N} \sigma_{1,1}(2k;2)\sigma_{1,1}(2N+1-2k;2) + \sum_{k=1}^{N} \sigma_{1,1}(2k;2)\sigma_{1,0}(2N+1-2k;2) \\
&\quad + \sum_{k=1}^{N} \sigma_{1,0}(2k;2)\sigma_{1,1}(2N+1-2k;2) + \sum_{k=1}^{N} \sigma_{1,0}(2k;2)\sigma_{1,0}(2N+1-2k;2).
\end{aligned}$$

It follows from Corollary 3.8 (b), Corollary 3.10 (b), (c) and Theorem 3.11 (a).

□

Example 3.16. *A short table(TABLE 10) and figure(FIGURE 5) of values of* $\overline{D}_{2,2}(2N+1) \sim \tilde{M}_{2,2}(2N+1)$ *follows.*

N	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
$\overline{D}_{2,2}(2N+1)$	3	19	58	129	245	413	646	948	1335	1840	2398
$\overline{E}_{2,2}(2N+1)$	−1	−9	−30	−67	−135	−231	−354	−540	−765	−1040	−1386
$\overline{F}_{2,2}(2N+1)$	1	5	14	31	55	91	146	204	285	400	506
$\overline{G}_{2,2}(2N+1)$	0	0	0	0	0	0	0	0	0	0	0
$\overline{H}_{2,2}(2N+1)$	2	14	44	98	190	322	500	744	1050	1440	1892
$\tilde{H}_{2,2}(2N+1)$	0	0	0	0	0	0	0	0	0	0	0
$\overline{I}_{2,2}(2N+1)$	3	19	58	129	245	413	646	948	1335	1840	2398
$\tilde{I}_{2,2}(2N+1)$	−1	−30	−135	−354	−765	−1386	−2290	−3480	−5004	−7154	−9471
$\overline{J}_{2,2}(2N+1)$	1	5	14	31	55	91	146	204	285	400	506
$\tilde{J}_{2,2}(2N+1)$	3	19	58	129	245	413	646	948	1335	1840	2398
$\overline{K}_{2,2}(2N+1)$	2	14	44	98	190	322	500	744	1050	1440	1892
$\tilde{K}_{2,2}(2N+1)$	0	0	0	0	0	0	0	0	0	0	0
$\overline{L}_{2,2}(2N+1)$	1	5	14	31	55	91	146	204	285	400	506
$\tilde{L}_{2,2}(2N+1)$	−1	−9	−30	−67	−135	−231	−354	−540	−765	−1040	−1386
$\overline{M}_{2,2}(2N+1)$	2	14	44	98	190	322	500	744	1050	1440	1892
$\tilde{M}_{2,2}(2N+1)$	0	0	0	0	0	0	0	0	0	0	0

< *TABLE 10. Examples for* $\overline{D}_{2,2}(2N+1) \sim \tilde{M}_{2,2}(2N+1)(2 \leq N \leq 12)$ >

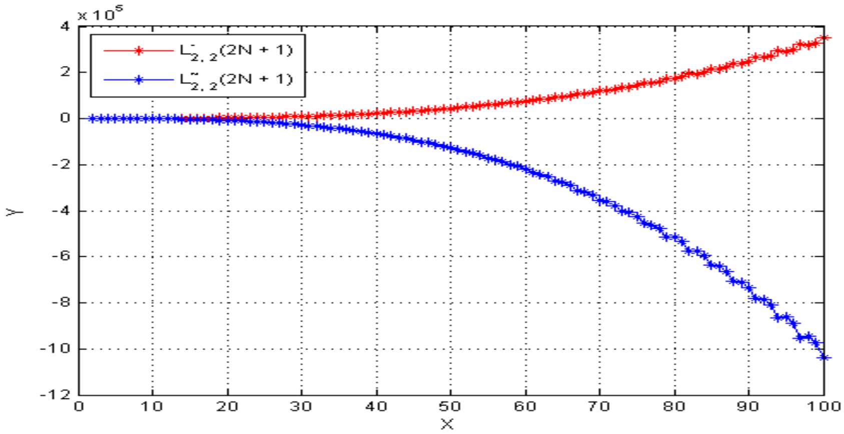

FIGURE 6. $\overline{L}_{2,2}(2N+1)$ and $\tilde{L}_{2,2}(2N+1)$ $(3 \le N \le 100)$

Remark 3.17. Corollary 3.12 (a) and Corollary 3.15 (a) are in J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S.Williams([22, Theorem 5]) which can be proved directly by $\sigma(2m) = 3\sigma(m) - 2\sigma(m/2)$ and $\sigma_3(2m) = 9\sigma_3(m) - 8\sigma(m/2)$. If we use $\tilde{\sigma}(m) = \sigma_{1,1}(m;2) - \sigma_{1,0}(m;2)$ and $\tilde{\sigma}(m) = \sigma(m) - 2\sigma(m/2)$, then Corollary 3.12 (b) and Corollary 3.15 (b) are equivalent to a result in H. Hahn([21, Theorem 4.7]).

Theorem 3.18. (a) *If $N(\ge 3)$ is odd, then*

$$\sum_{k=1}^{\frac{N-1}{2}} \sigma_{1,0}(k;2)\sigma_{1,0}(N-2k;2) = 0.$$

(b) *If $N(\ge 4)$ is even, then*

$$\sum_{k=1}^{\frac{N}{2}-1} \sigma_{1,0}(k;2)\sigma_{1,0}(N-2k;2) = \frac{1}{24}\Big[2\sigma_3(N) - 10\sigma_3(\frac{N}{2}) + 6\sigma_{3,0}(\frac{N}{2};2)$$
$$+(2-3N)\sigma_{1,0}(N;2) + (2-6N)\sigma_{1,0}(\frac{N}{2};2)\Big].$$

In particular, if $\frac{N}{2}$ is an odd integer, then

$$\sum_{k=1}^{\frac{N}{2}-1} \sigma_{1,0}(k;2)\sigma_{1,0}(N-2k;2) = \frac{1}{12}\left[4\sigma_3(\frac{N}{2}) + (2-3N)\sigma_1(\frac{N}{2})\right]. \tag{21}$$

Proof. (a) Let N be an odd integer. Then we conclude that $N - 2k \equiv 1 \pmod 2$, $\sigma_{1,0}(N-2k;2) = 0$ and $\sum_{k=1}^{\frac{N-1}{2}} \sigma_{1,0}(k;2)\sigma_{1,0}(N-2k;2) = 0$.

(b) In [22, (4.4)], it is proved that

$$\sum_{m=1}^{\frac{N}{2}-1} \sigma_1(m)\sigma_1(N-2m) = \frac{1}{24}[2\sigma_3(N) + (1-3N)\sigma_1(N) + 8\sigma_3(N/2) + (1-6N)\sigma_1(N/2)]. \tag{22}$$

Let us consider the equation (22),

$$\begin{aligned} &\sum_{k=1}^{\frac{N}{2}-1} \sigma_1(k)\sigma_1(N-2k) \\ &= \sum_{k=1}^{\frac{N}{2}-1} [\sigma_{1,1}(k;2) + \sigma_{1,0}(k;2)][\sigma_{1,1}(N-2k;2) + \sigma_{1,0}(N-2k;2)] \\ &= \sum_{k=1}^{\frac{N}{2}-1} \sigma_{1,1}(k;2)\sigma_{1,1}(N-2k;2) + \sum_{k=1}^{\frac{N}{2}-1} \sigma_{1,1}(k;2)\sigma_{1,0}(N-2k;2) \\ &\quad + \sum_{k=1}^{\frac{N}{2}-1} \sigma_{1,0}(k;2)\sigma_{1,1}(N-2k;2) + \sum_{k=1}^{\frac{N}{2}-1} \sigma_{1,0}(k;2)\sigma_{1,0}(N-2k;2) \\ &= \frac{1}{24}[2\sigma_3(N) + (1-3N)\sigma_1(N) + 8\sigma_3(N/2) + (1-6N)\sigma_1(N/2)]. \end{aligned} \tag{23}$$

And, in [21, (4.14)] Hahn showed that

$$8\sum_{m=1}^{\frac{N}{2}-1} \tilde{\sigma_1}(m)\tilde{\sigma_1}(n-2m) = -2\tilde{\sigma_3}(n/2) + (n-1)\tilde{\sigma_1}(n) + (2n-1)\tilde{\sigma_1}(n/2). \tag{24}$$

Similarly, let us consider the equation (24),

$$\begin{aligned}
&\sum_{k=1}^{\frac{N}{2}-1} \tilde{\sigma_1}(k)\tilde{\sigma_1}(N-2k) \\
&= \sum_{k=1}^{\frac{N}{2}-1} [\sigma_{1,1}(k;2)-\sigma_{1,0}(k;2)][\sigma_{1,1}(N-2k;2)-\sigma_{1,0}(N-2k;2)] \\
&= \sum_{k=1}^{\frac{N}{2}-1} \sigma_{1,1}(k;2)\sigma_{1,1}(N-2k;2) - \sum_{k=1}^{\frac{N}{2}-1} \sigma_{1,1}(k;2)\sigma_{1,0}(N-2k;2) \\
&\quad - \sum_{k=1}^{\frac{N}{2}-1} \sigma_{1,0}(k;2)\sigma_{1,1}(N-2k;2) + \sum_{k=1}^{\frac{N}{2}-1} \sigma_{1,0}(k;2)\sigma_{1,0}(N-2k;2) \\
&= \frac{1}{8}[-2\tilde{\sigma_3}(N/2)+(N-1)\tilde{\sigma_1}(N)+(2N-1)\tilde{\sigma_1}(N/2)].
\end{aligned} \tag{25}$$

Then, we can calculate (23) plus (25);

$$\begin{aligned}
&\sum_{k=1}^{\frac{N}{2}-1} \sigma_{1,0}(k;2)\sigma_{1,0}(N-2k;2) \\
&= \frac{1}{24}[\sigma_3(N)+\sigma_3(N/2)+6\sigma_{3,0}(N/2;2)-\sigma_1(N)-\sigma_1(N/2) \\
&\quad +3(1-N)\sigma_{1,0}(N;2)+3(1-2N)\sigma_{1,0}(N/2;2)] \\
&\quad - \sum_{k=1}^{\frac{N}{2}-1} \sigma_{1,1}(k;2)\sigma_{1,1}(N-2k;2).
\end{aligned} \tag{26}$$

By using Theorem 3.18 (b) and (26), we deduce that

$$\begin{aligned}
&\sum_{k=1}^{\frac{N}{2}-1} \sigma_{1,0}(k;2)\sigma_{1,0}(N-2k;2) \\
&= \frac{1}{12}\sigma_3(N) - \frac{5}{12}\sigma_3(N/2) + \frac{1}{4}\sigma_{3,0}(N/2;2) \\
&\quad + (\frac{1}{12}-\frac{N}{8})\sigma_{1,0}(N;2) + (\frac{1}{12}-\frac{N}{4})\sigma_{1,0}(N/2;2).
\end{aligned} \tag{27}$$

Now suppose that $\frac{N}{2} \equiv 1 \pmod 2$. Then, we have

$$\sigma_{3,0}(\frac{N}{2};2) = \sigma_{1,0}(\frac{N}{2};2) = 0 \tag{28}$$

and

$$\begin{aligned}\sigma_{1,0}(N;2) &= \sigma_1(N;2) - \sigma_{1,1}(N;2) \\ &= \sigma_1(2)\sigma_1(\frac{N}{2}) - \sigma_{1,1}(\frac{N}{2};2) = 2\sigma_1(\frac{N}{2}).\end{aligned} \tag{29}$$

From (27), (28) and (29), we have the identity (21).

□

Theorem 3.19. (a) *If $N(\geq 2)$ is any integer, then*

$$\sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,1}(2N-2k;2) = \frac{1}{24}\left[\sigma_3(N) - \sigma_1(N) + 3\sigma_{3,1}(N;2) \right. \\ \left. -3(2N-1)\sigma_{1,1}(N;2)\right].$$

(b) *If $N(\geq 1)$ is any integer, then*

$$\sum_{k=1}^{N} \sigma_{1,0}(k;2)\sigma_{1,1}(2N+1-2k;2) = \frac{1}{24}\left[\sigma_3(2N+1) - (6N+1)\sigma_1(2N+1)\right].$$

Proof. (a) It is well-known that $\sigma_{1,1}(2N-2k;2) = \sigma_{1,1}(N-k;2)$.
Then, recalling (12), we have

$$\sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,1}(N-k;2) = \frac{1}{24}\left[\sigma_3(N) - \sigma_1(N) + 3\sigma_{3,1}(N;2) \right. \\ \left. -3(2N-1)\sigma_{1,1}(N;2)\right].$$

(b) In [22, (4.4)] and [32, (8)], it is proved that

$$\sum_{m<\frac{n}{2}} \sigma_1(m)\sigma_1(n-2m) = \frac{1}{24}\left[2\sigma_3(n) + (1-3n)\sigma_1(n) \right. \\ \left. +8\sigma_3(\frac{n}{2}) + (1-6n)\sigma_1(\frac{n}{2})\right]. \tag{30}$$

In (30), $n = 2N+1$ yields

$$\sum_{k=1}^{N} \sigma_1(k)\sigma_1(2N+1-2k) = \sum_{k=1}^{N}(\sigma_{1,1}(k;2) + \sigma_{1,0}(k;2))\sigma_{1,1}(2N+1-2k;2)$$
$$= \sum_{k=1}^{N} \sigma_{1,1}(k;2)\sigma_{1,1}(2N+1-2k;2) + \sum_{k=1}^{N} \sigma_{1,0}(k;2)\sigma_{1,1}(2N+1-2k;2)$$

and

$$\sum_{k=1}^{N} \sigma_{1,0}(k;2)\sigma_{1,1}(2N+1-2k;2)$$
$$= \frac{1}{24}\left[2\sigma_3(2N+1) + (-6N-2)\sigma_1(2N+1)\right]$$
$$- \frac{1}{24}\left[\sigma_3(2N+1) - \sigma_1(2N+1)\right]$$
$$= \frac{1}{24}\left[\sigma_3(2N+1) - (6N+1)\sigma_1(2N+1)\right].$$

□

Corollary 3.20. *Let $N \geq 2$ be a positive integer. Then,*

(a)

$$\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,1}(2N-2k;2) = \frac{1}{24}[11\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(N;2)].$$

(b)

$$\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,0}(2N-2k;2) = \frac{1}{24}[\sigma_3(2N) - \sigma_1(2N) + 3\sigma_{3,1}(N;2) - 3(4N-1)\sigma_{1,1}(N;2)].$$

(c)

$$\sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,1}(2N-2k;2) = \frac{1}{24}[\sigma_3(N) - \sigma_1(N) + 3\sigma_{3,1}(N;2) - 3(2N-1)\sigma_{1,1}(N;2)].$$

(d)

$$\sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,0}(2N-2k;2) = \frac{1}{24}[2\sigma_3(2N) - 10\sigma_3(N) + 6\sigma_{3,0}(N;2) + 2(1-3N)\sigma_{1,0}(2N;2) + 2(1-6N)\sigma_{1,0}(N;2)].$$

Proof. (a) It is obtained by $\sigma_{1,1}(k;2) = \sigma_{1,1}(2k;2)$ in Theorem 3.11(b).

(b) Putting $\sigma_{1,1}(k;2) = \sigma_{1,1}(2k;2)$ in Corollary 3.10(a), we obtain (b).

(c) It is obvious by Theorem 3.19(a).

(d) Letting $N \longrightarrow 2N$ in Theorem 3.18(b), we obtain (d).

□

From the above corollary, we get the following.

Corollary 3.21. *Let $N \geq 2$ is any integer.*

(a) *(See Melfi* [32], *(8), J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams* [22], *(4.4))*

$$\begin{aligned}\overline{D}_{1,2}(2N) &:= \sum_{k=1}^{N-1} \sigma_1(k)\sigma_1(2N-2k)\\ &= \frac{1}{24}[2\sigma_3(2N) + (1-6N)\sigma_1(2N) + 8\sigma_3(N) + (1-12N)\sigma_1(N)].\end{aligned}$$

(b)

$$\begin{aligned}\overline{E}_{1,2}(2N) &:= \sum_{k=1}^{N-1} \tilde{\sigma_1}(k)\tilde{\sigma_1}(2N-2k)\\ &= \frac{1}{8}\{-2[\sigma_3(N) - 2\sigma_{3,0}(N;2)] + (2N-1)[\sigma_1(2N) - 2\sigma_{1,0}(2N;2)]\\ &\quad + (4N-1)[\sigma_1(N) - 2\sigma_{1,0}(N;2)]\}.\end{aligned}$$

(c)

$$\begin{aligned}\overline{F}_{1,2}(2N) :&= \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,1}(2N-2k;2)\\ &= \frac{1}{24}[11\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(N;2)].\end{aligned}$$

(d)

$$\begin{aligned}\overline{G}_{1,2}(2N) &:= \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,0}(2N-2k;2)\\ &= \frac{1}{24}[2\sigma_3(2N) - 10\sigma_3(N) + 6\sigma_{3,0}(N;2) + 2(1-3N)\sigma_{1,0}(2N;2)\\ &\quad + 2(1-6N)\sigma_{1,0}(N;2)].\end{aligned}$$

(e)

$$\begin{aligned}\overline{H}_{1,2}(2N) :&= \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,1}(2N-2k;2)\\ &= \frac{1}{24}[\sigma_3(N) - \sigma_1(N) + 3\sigma_{3,1}(N;2) - 3(2N-1)\sigma_{1,1}(N;2)].\end{aligned}$$

(e')

$$\tilde{H}_{1,2}(2N) := \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,0}(2N-2k;2)$$
$$= \frac{1}{24}[\sigma_3(2N) - \sigma_1(2N) + 3\sigma_{3,1}(N;2) - 3(4N-1)\sigma_{1,1}(N;2)].$$

(f)

$$\overline{I}_{1,2}(2N) := \sum_{k=1}^{N-1} \sigma_1(k)\tilde{\sigma_1}(2N-2k)$$
$$= \frac{1}{24}[22\sigma_3(N) - 4\sigma_3(2N) + (6N-1)\sigma_1(2N) - \sigma_1(N)$$
$$- 6\sigma_{3,0}(N;2) + 2(6N-1)\sigma_{1,0}(N;2)].$$

(f')

$$\tilde{I}_{1,2}(2N) := \sum_{k=1}^{N-1} \tilde{\sigma_1}(k)\sigma_1(2N-2k)$$
$$= \frac{1}{24}[20\sigma_3(N) - 2\sigma_3(2N) - (1+6N)\sigma_1(N) - \sigma_1(2N)$$
$$- 6\sigma_{3,0}(N;2) - 2(1-3N)\sigma_{1,0}(2N;2) + 18N\sigma_{1,0}(N;2)].$$

(g)

$$\overline{J}_{1,2}(2N) := \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_1(2N-2k)$$
$$= \frac{1}{24}[11\sigma_3(N) - \sigma_1(2N) + 3\sigma_{3,1}(N;2) - (12N-1)\sigma_{1,1}(N;2)].$$

(g')

$$\tilde{J}_{1,2}(2N) := \sum_{k=1}^{N-1} \sigma_1(k)\sigma_{1,1}(2N-2k;2) = \sum_{k=1}^{N-1} \sigma_1(k)\sigma_{1,1}(N-k;2)$$
$$= \frac{1}{24}[12\sigma_3(N) - \sigma_3(2N) + 3\sigma_{3,1}(N;2) - \sigma_1(N) + (1-6N)\sigma_{1,1}(N;2)].$$

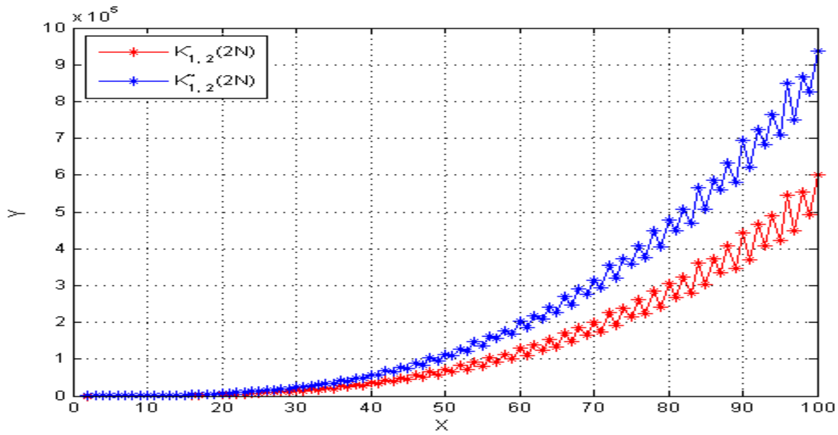

FIGURE 7. $\overline{K}_{1,2}(2N)$ and $\tilde{K}_{1,2}(2N)$ $(3 \le N \le 100)$

(h)

$$\begin{aligned}\overline{K}_{1,2}(2N) :&= \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_1(2N-2k)\\ &= \frac{1}{24}[2\sigma_3(2N) - 3\sigma_3(N) - 3\sigma_{3,1}(N;2) + (2-6N)\sigma_1(2N)\\ &\quad + (1-12N)\sigma_{1,0}(N;2)].\end{aligned}$$

(h')

$$\begin{aligned}\tilde{K}_{1,2}(2N) :&= \sum_{k=1}^{N-1} \sigma_1(k)\sigma_{1,0}(2N-2k;2)\\ &= \frac{1}{24}[3\sigma_{3,0}(2N;2) - 4\sigma_3(N) + (1-6N)\sigma_{1,0}(2N;2)\\ &\quad + 2(1-6N)\sigma_1(N)].\end{aligned}$$

(i)

$$\begin{aligned}\overline{L}_{1,2}(2N) :&= \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\tilde{\sigma_1}(2N-2k)\\ &= \frac{1}{24}[11\sigma_3(N) - 2\sigma_3(2N) + \sigma_1(2N) - 3\sigma_{3,1}(2N;2)\\ &\quad + (12N-5)\sigma_{1,1}(2N;2)].\end{aligned}$$

(i')

$$\tilde{L}_{1,2}(2N) := \sum_{k=1}^{N-1} \tilde{\sigma_1}(k)\sigma_{1,1}(2N-2k;2) = \sum_{k=1}^{N-1} \tilde{\sigma_1}(k)\sigma_{1,1}(N-k;2)$$
$$= \frac{1}{24}[10\sigma_3(N) - \sigma_3(2N) - 3\sigma_{3,1}(N;2) + \sigma_1(N) + (6N-5)\sigma_{1,1}(N;2)].$$

(j)

$$\overline{M}_{1,2}(2N) := \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\tilde{\sigma_1}(2N-2k)$$
$$= \frac{1}{24}\{14\sigma_3(N) - 2\sigma_3(2N) - 9\sigma_{3,0}(N;2) + (18N-5)\sigma_{1,0}(N;2) + 2(1-3N)[\sigma_1(N) - \sigma_{1,0}(2N;2)]\}.$$

(j')

$$\tilde{M}_{1,2}(2N) := \sum_{k=1}^{N-1} \tilde{\sigma_1}(k)\sigma_{1,0}(2N-2k;2)$$
$$= \frac{1}{24}[-\sigma_3(2N) + 13\sigma_3(N) - \sigma_1(2N) - 3(4N-1)\sigma_1(N) - 9\sigma_{3,0}(N;2) - 2(1-3N)\sigma_{1,0}(2N;2) + (24N-5)\sigma_{1,0}(N;2)].$$

In particular, if N is odd, then the above can be simplified to the following convolution formulas for the cases of $\overline{D}_{1,2}(2N) \sim \overline{M}_{1,2}(2N)$.

N: odd	convolution formulas	N: odd	convolution formulas
$\overline{D}_{1,2}(2N)$	$\frac{1}{12}[13\sigma_3(N) + (2-15N)\sigma_1(N)]$	$\overline{J}_{1,2}(2N)$	$\frac{1}{12}[7\sigma_3(N) - (6N+1)\sigma_1(N)]$
$\overline{E}_{1,2}(2N)$	$\frac{1}{4}[-\sigma_3(N) + N\sigma_1(N)]$	$\tilde{J}_{1,2}(2N)$	$\frac{1}{4}[\sigma_3(N) - N\sigma_1(N)]$
$\overline{F}_{1,2}(2N)$	$\frac{1}{12}[\sigma_3(N) - \sigma_1(N)]$	$\overline{K}_{1,2}(2N)$	$\frac{1}{4}[2\sigma_3(N) + (1-3N)\sigma_1(N)]$
$\overline{G}_{1,2}(2N)$	$\frac{1}{6}[2\sigma_3(N) + (1-3N)\sigma_1(N)]$	$\tilde{K}_{1,2}(2N)$	$\frac{1}{6}[5\sigma_3(N) + (1-6N)\sigma_1(N)]$
$\overline{H}_{1,2}(2N)$	$\frac{1}{12}[2\sigma_3(N) + (1-3N)\sigma_1(N)]$	$\overline{L}_{1,2}(2N)$	$\frac{1}{12}[-5\sigma_3(N) + (6N-1)\sigma_1(N)]$
$\tilde{H}_{1,2}(2N)$	$\frac{1}{2}[\sigma_3(N) - N\sigma_1(N)]$	$\tilde{L}_{1,2}(2N)$	$\frac{1}{12}[-\sigma_3(N) + (3N-2)\sigma_1(N)]$
$\overline{I}_{1,2}(2N)$	$\frac{1}{12}[-7\sigma_3(N) + (9N-2)\sigma_1(N)]$	$\overline{M}_{1,2}(2N)$	$\frac{1}{12}[-2\sigma_3(N) + (3N-1)\sigma_1(N)]$
$\tilde{I}_{1,2}(2N)$	$\frac{1}{12}[\sigma_3(N) + (3N-4)\sigma_1(N)]$	$\tilde{M}_{1,2}(2N)$	$\frac{1}{6}[\sigma_3(N) - \sigma_1(N)]$

<TABLE 11. Convolution formulas for $\overline{D}_{1,2}(2N) \sim \overline{M}_{1,2}(2N)(N : odd)$>

Example 3.22. *The first eleven values of $\overline{D}_{1,2}(2N) \sim \tilde{M}_{1,2}(2N)$ are given in the following table.*

N	2	3	4	5	6	7	8	9	10	11	12
$\overline{D}_{1,2}(2N)$	3	16	45	100	178	304	465	676	952	1280	1702
$\overline{E}_{1,2}(2N)$	-1	-4	-3	-24	10	-72	41	-160	120	-300	230
$\overline{F}_{1,2}(2N)$	1	2	9	10	30	28	73	62	136	110	254
$\overline{G}_{1,2}(2N)$	0	4	12	28	64	88	180	196	400	380	712
$\overline{H}_{1,2}(2N)$	0	2	2	14	8	44	22	98	48	190	76
$\tilde{H}_{1,2}(2N)$	2	8	22	48	76	144	190	320	368	600	660
$\overline{I}_{1,2}(2N)$	-1	-8	-23	-52	-102	-160	-275	-356	-584	-680	-1042
$\tilde{I}_{1,2}(2N)$	3	4	17	16	34	40	61	88	56	140	126
$\overline{J}_{1,2}(2N)$	3	10	31	58	106	172	263	382	504	710	914
$\tilde{J}_{1,2}(2N)$	1	4	11	24	38	72	95	160	184	300	330
$\overline{K}_{1,2}(2N)$	0	6	14	42	72	132	202	294	448	570	788
$\tilde{K}_{1,2}(2N)$	2	12	34	76	140	232	370	516	768	980	1372
$\overline{L}_{1,2}(2N)$	-1	-6	-13	-38	-46	-116	-117	-258	-232	-490	-406
$\tilde{L}_{1,2}(2N)$	1	0	7	-4	22	-16	51	-36	88	-80	178
$\overline{M}_{1,2}(2N)$	0	-2	-10	-14	-56	-44	-158	-98	-352	-190	-636
$\tilde{M}_{1,2}(2N)$	2	4	10	20	12	56	10	124	-32	220	-52

< TABLE 12. Examples for $\overline{D}_{1,2}(2N) \sim \tilde{M}_{1,2}(2N) (2 \leq N \leq 12)$ *>*

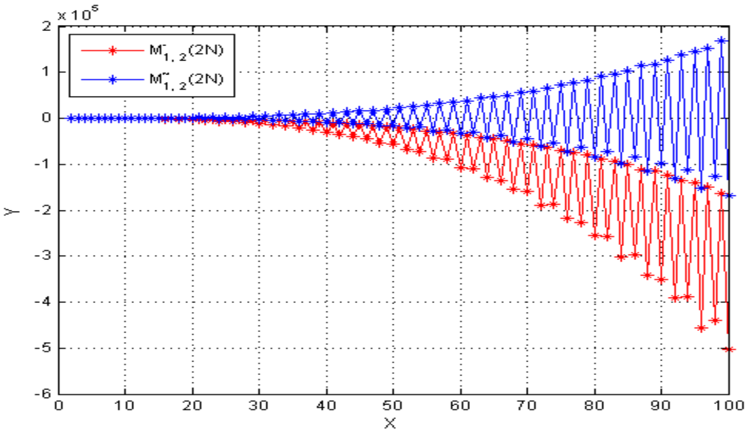

FIGURE 8. $\overline{M}_{1,2}(2N)$ and $\tilde{M}_{1,2}(2N)$ $(3 \leq N \leq 100)$

Corollary 3.23. *Let* $N \geq 1$ *be a positive integer. Then,*

(a)

$$\sum_{k=1}^{N} \sigma_{1,1}(k;2)\sigma_{1,1}(2N+1-2k;2) = \frac{1}{24}[\sigma_3(2N+1) - \sigma_1(2N+1)].$$

(b)

$$\sum_{k=1}^{N} \sigma_{1,1}(k;2)\sigma_{1,0}(2N+1-2k;2) = 0.$$

(c)

$$\sum_{k=1}^{N} \sigma_{1,0}(k;2)\sigma_{1,1}(2N+1-2k;2) = \frac{1}{24}[\sigma_3(2N+1) - (6N+1)\sigma_1(2N+1)].$$

(d)

$$\sum_{k=1}^{N} \sigma_{1,0}(k;2)\sigma_{1,0}(2N+1-2k;2) = 0.$$

Proof. (a) We can consider $\sigma_{1,1}(k;2) = \sigma_{1,1}(2k;2)$ in Theorem 3.11 (a).

(b) Since $\sigma_{1,0}(2N+1-2k;2) = 0$, we obtain (b).

(c) It is obvious by Theorem 3.19(b).

(d) By $\sigma_{1,0}(2N+1-2k;2) = 0$, it is obtained.

□

From the above Corollary, we get the following.

Corollary 3.24. *Let $N \geq 1$ is any integer. Then*

(a) *(See Melfi* [32], *(8), J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams* [22], *(4.4))*

$$\overline{D}_{1,2}(2N+1) := \sum_{k=1}^{N} \sigma_1(k)\sigma_1(2N+1-2k) = \frac{1}{24}[2\sigma_3(2N+1) - 2(3N+1)\sigma_1(2N+1)].$$

(b)

$$\overline{E}_{1,2}(2N+1) := \sum_{k=1}^{N} \tilde{\sigma_1}(k)\tilde{\sigma_1}(2N+1-2k) = \frac{N}{4}\sigma_1(2N+1).$$

(c)

$$\overline{F}_{1,2}(2N+1) := \sum_{k=1}^{N} \sigma_{1,1}(k;2)\sigma_{1,1}(2N+1-2k;2) = \frac{1}{24}(\sigma_3(2N+1) - \sigma_1(2N+1)).$$

(d)

$$\overline{G}_{1,2}(2N+1) := \sum_{k=1}^{N} \sigma_{1,0}(k;2)\sigma_{1,0}(2N+1-2k;2) = 0.$$

(e)

$$\overline{H}_{1,2}(2N+1) := \sum_{k=1}^{N} \sigma_{1,1}(k;2)\sigma_{1,0}(2N+1-2k;2) = 0.$$

(e')

$$\begin{aligned}\tilde{H}_{1,2}(2N+1) : &= \sum_{k=1}^{N} \sigma_{1,0}(k;2)\sigma_{1,1}(2N+1-2k;2) \\ &= \frac{1}{24}(\sigma_3(2N+1) - (6N+1)\sigma_1(2N+1)).\end{aligned}$$

(f)

$$\begin{aligned}\overline{I}_{1,2}(2N+1) : &= \sum_{k=1}^{N} \sigma_1(k)\tilde{\sigma_1}(2N+1-2k) = \sum_{k=1}^{N} \sigma_1(k)\sigma_1(2N+1-2k) \\ &= \frac{1}{24}[2\sigma_3(2N+1) - 2(3N+1)\sigma_1(2N+1)].\end{aligned}$$

(f')

$$\begin{aligned}\tilde{I}_{1,2}(2N+1) : &= \sum_{k=1}^{N} \tilde{\sigma_1}(k)\sigma_1(2N+1-2k) = \sum_{k=1}^{N} \tilde{\sigma_1}(k)\tilde{\sigma_1}(2N+1-2k) \\ &= \frac{N}{4}\sigma_1(2N+1).\end{aligned}$$

(g)

$$\begin{aligned}\overline{J}_{1,2}(2N+1) &:= \sum_{k=1}^{N} \sigma_{1,1}(k;2)\sigma_1(2N+1-2k) \\ &= \sum_{k=1}^{N} \sigma_{1,1}(k;2)\sigma_{1,1}(2N+1-2k;2) = \frac{1}{24}[\sigma_3(2N+1) - \sigma_1(2N+1)].\end{aligned}$$

(g')

$$\tilde{J}_{1,2}(2N+1) := \sum_{k=1}^{N} \sigma_1(k)\sigma_{1,1}(2N+1-2k;2)$$
$$= \sum_{k=1}^{N} \sigma_1(k)\tilde{\sigma_1}(2N+1-2k;2)$$
$$= \frac{1}{24}[2\sigma_3(2N+1) - 2(3N+1)\sigma_1(2N+1)].$$

(h)

$$\overline{K}_{1,2}(2N+1) := \sum_{k=1}^{N} \sigma_{1,0}(k;2)\sigma_1(2N+1-2k)$$
$$= \frac{1}{24}[\sigma_3(2N+1) - (6N+1)\sigma_1(2N+1)].$$

(h')

$$\tilde{K}_{1,2}(2N+1) := \sum_{k=1}^{N} \sigma_1(k)\sigma_{1,0}(2N+1-2k;2) = 0.$$

(i)

$$\overline{L}_{1,2}(2N+1) := \sum_{k=1}^{N} \sigma_{1,1}(k;2)\tilde{\sigma_1}(2N+1-2k)$$
$$= \sum_{k=1}^{N} \sigma_{1,1}(k;2)\sigma_{1,1}(2N+1-2k;2)$$
$$= \frac{1}{24}[\sigma_3(2N+1) - \sigma_1(2N+1)].$$

(i')

$$\tilde{L}_{1,2}(2N+1) := \sum_{k=1}^{N} \tilde{\sigma_1}(k)\sigma_{1,1}(2N+1-2k;2)$$
$$= \sum_{k=1}^{N} \tilde{\sigma_1}(k)\tilde{\sigma_1}(2N+1-2k)$$
$$= \frac{N}{4}\sigma_1(2N+1).$$

(j)

$$\begin{aligned}\overline{M}_{1,2}(2N+1) :&= \sum_{k=1}^{N} \sigma_{1,0}(k;2)\tilde{\sigma_1}(2N+1-2k) \\ &= \sum_{k=1}^{N} \sigma_{1,0}(k;2)\sigma_{1,1}(2N+1-2k;2) \\ &= \frac{1}{24}[\sigma_3(2N+1) - (6N+1)\sigma_1(2N+1)].\end{aligned}$$

(j')

$$\tilde{M}_{1,2}(2N+1) := \sum_{k=1}^{N} \tilde{\sigma_1}(k)\sigma_{1,0}(2N+1-2k;2) = 0.$$

With the integers u, a, b and c as defined in Theorem 1.1, the values of $\overline{D}_{1,2}(2N+1) \sim \tilde{M}_{1,2}(2N+1)$ are given as follows:

N	u	a	b	c	N	u	a	b	c
$\overline{D}_{1,2}(2N+1)$	24	2	-3	1	$\overline{J}_{1,2}(2N+1)$	24	1	0	-1
$\overline{E}_{1,2}(2N+1)$	8	0	1	-1	$\tilde{J}_{1,2}(2N+1)$	24	2	-3	1
$\overline{F}_{1,2}(2N+1)$	24	1	0	-1	$\overline{K}_{1,2}(2N+1)$	24	1	-3	2
$\overline{G}_{1,2}(2N+1)$	1	0	0	0	$\tilde{K}_{1,2}(2N+1)$	1	0	0	0
$\overline{H}_{1,2}(2N+1)$	1	0	0	0	$\overline{L}_{1,2}(2N+1)$	24	1	0	-1
$\tilde{H}_{1,2}(2N+1)$	24	1	-3	2	$\tilde{L}_{1,2}(2N+1)$	8	0	1	-1
$\overline{I}_{1,2}(2N+1)$	24	2	-3	1	$\overline{M}_{1,2}(2N+1)$	24	1	-3	2
$\tilde{I}_{1,2}(2N+1)$	8	0	1	-1	$\tilde{M}_{1,2}(2N+1)$	1	0	0	0

<TABLE 13. a, b, c for $\overline{D}_{1,2}(2N+1) \sim \tilde{M}_{1,2}(2N+1)$($N$: integer)>

Example 3.25. *The first thirteen values of $\overline{D}_{1,2}(2N+1) \sim \tilde{M}_{1,2}(2N+1)$ are given in the following table.*

N	1	2	3	4	5	6	7	8	9	10	11	12	13
$\overline{D}_{1,2}(2N+1)$	1	7	22	49	95	161	250	372	525	720	946	1217	1570
$\overline{E}_{1,2}(2N+1)$	1	3	6	13	15	21	42	36	45	80	66	93	130
$\overline{F}_{1,2}(2N+1)$	1	5	14	31	55	91	146	204	285	400	506	655	850
$\overline{G}_{1,2}(2N+1)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\overline{H}_{1,2}(2N+1)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\tilde{H}_{1,2}(2N+1)$	0	2	8	18	40	70	104	168	240	320	440	562	720
$\overline{I}_{1,2}(2N+1)$	1	7	22	49	95	161	250	372	525	720	946	1217	1570
$\tilde{I}_{1,2}(2N+1)$	1	3	6	13	15	21	42	36	45	80	66	93	130
$\overline{J}_{1,2}(2N+1)$	1	5	14	31	55	90	146	204	285	400	506	655	850
$\tilde{J}_{1,2}(2N+1)$	1	7	22	49	95	161	250	372	525	720	946	1217	1570
$\overline{K}_{1,2}(2N+1)$	0	2	8	18	40	70	104	168	240	320	440	562	720
$\tilde{K}_{1,2}(2N+1)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\overline{L}_{1,2}(2N+1)$	1	5	14	31	55	91	146	204	285	400	506	655	850
$\tilde{L}_{1,2}(2N+1)$	1	3	6	13	15	21	42	36	45	80	66	93	130
$\overline{M}_{1,2}(2N+1)$	0	2	8	18	40	70	104	168	240	320	440	562	720
$\tilde{M}_{1,2}(2N+1)$	0	0	0	0	0	0	0	0	0	0	0	0	0

< TABLE 14. Examples for $\overline{D}_{1,2}(2N+1) \sim \tilde{M}_{1,2}(2N+1)(1 \leq N \leq 13)$ >

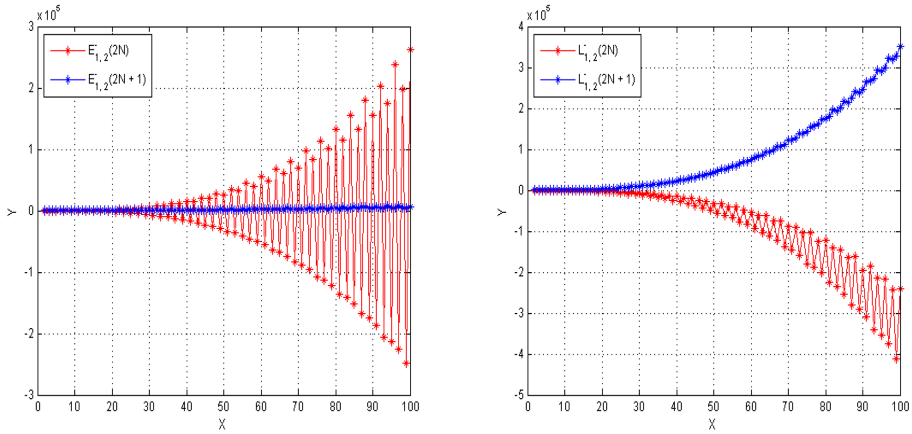

FIGURE 9. $\overline{E}_{1,2}(2N)$, $\tilde{E}_{1,2}(2N+1)$ and $\overline{L}_{1,2}(2N+1)$, $\tilde{L}_{1,2}(2N+1)$ $(3 \leq N \leq 100)$

Remark 3.26. Formulae $\overline{D}_{1,2}(2N+1)$ were obtained Melfi[32] and N. Cheng and K. S. Williams [10, (5.3)]. $\overline{E}_{1,2}(2N)$ and $\overline{E}_{1,2}(2N+1)$ are given in Hahn[21].

Consider values of $\overline{F}_{1,2}(2N+1)$ and $\tilde{I}_{1,2}(2N-1)$ in TABLE 14, FIGURE 10 and 11. From Corollary 3.24 (c), (f'), we deduce two curious properties of $\tilde{I}_{1,2}(2N-1)$ and $\overline{F}_{1,2}(2N+1)$. For example, we see that $\tilde{I}_{1,2}(3 = 2\times 1+1) = 1 = \frac{1}{2}(1\times 2)$, $\tilde{I}_{1,2}(5 = 2\times 2+1) = 1+2 = \frac{1}{2}(2\times 3)$, $\tilde{I}_{1,2}(7 = 2\times 3+1) = 1+2+3 = \frac{1}{2}(3\times 4)$, $\overline{F}_{1,2}(2\times 1+1) = 1^2$, $\overline{F}_{1,2}(2\times 2+1) = 1^2+2^2$, $\overline{F}_{1,2}(2\times 3+1) = 1^2+2^2+3^2$ and $\tilde{I}_{1,2}(7 = 2\times 3+1) = \tilde{I}_{1,2}(5 = 2\times 3-1)+3$ and $\overline{F}_{1,2}(7) = \overline{F}_{1,2}(5)+3^2$(see FIGURE 10, 11). The following corollary is immediate from Corollary 3.24 (c), (f').

Corollary 3.27. *Let* $S(N) := \sum_{k=1}^{N} k = \frac{1}{2}N(N+1)$ *and* $T(N) := \sum_{k=1}^{N} k^2 = \frac{1}{6}N(N+1)(2N+1)$. *Then we get the following.*

(a) *If* $p = 2N+1$ *is a prime, then* $\tilde{I}_{1,2}(2N+1) = S(N)$ *and* $\overline{F}_{1,2}(2N+1) = T(N)$.
(b) *If* $p-2 = 2N-1$ *and* $p = 2N+1$ *are twin primes,* $\tilde{I}_{1,2}(2N+1) - \tilde{I}_{1,2}(2N-1) = N$ *and* $\overline{F}_{1,2}(2N+1) - \overline{F}_{1,2}(2N-1) = N^2$.

Example 3.28. *The first eleven values of* $S(N)$, $\tilde{I}_{1,2}(2N+1)$, $T(N)$ *and* $\overline{F}_{1,2}(2N+1)$ *are given in the following table.*

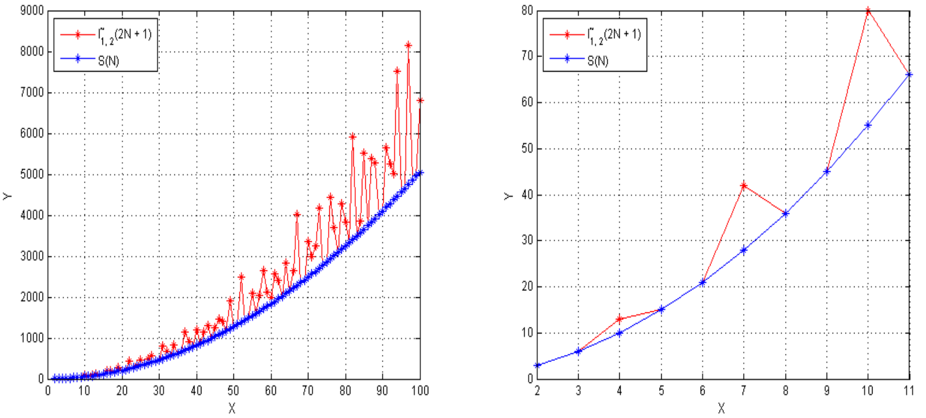

FIGURE 10. $\tilde{I}_{1,2}(2N+1)$ and $S(N)$

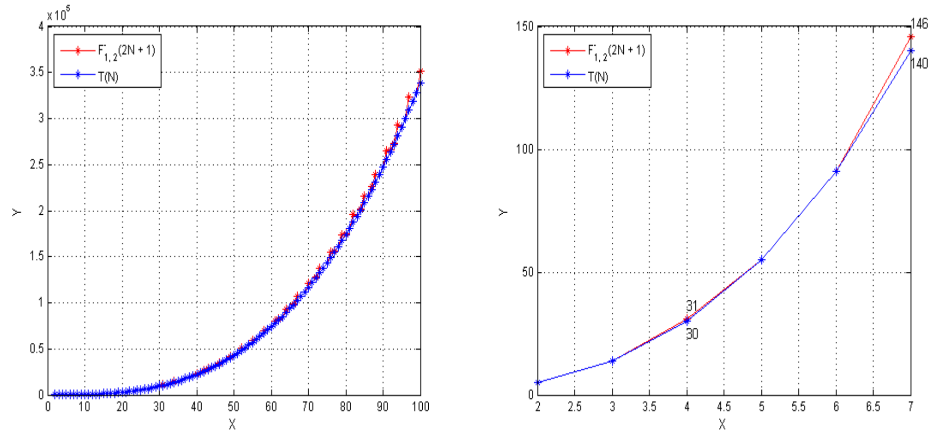

FIGURE 11. $\overline{F}_{1,2}(2N+1)$ and $T(N)$

N	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
$S(N)$	1	3	6	10	15	21	28	36	45	55	66
$\tilde{I}_{1,2}(2N+1)$	1	3	6	13	15	21	42	36	45	80	66
$T(N)$	1	5	14	30	55	91	140	204	285	385	506
$\overline{F}_{1,2}(2N+1)$	1	5	14	31	55	91	146	204	285	400	506

<*TABLE 14-1. Examples for* $S(N)$, $\tilde{I}_{1,2}(2N+1)$, $T(N)$, $\overline{F}_{1,2}(2N+1)$>

4. Some convolution sums of $f_{d,m}(4N)$

In this section, we discuss some new convolution sums that are derived from the existing ones. First, we will consider the values of $\overline{D}_{a,4}(4N)$ with $a = 1, 2, 4$ in Figure 12. Directly, we see that $\overline{D}_{1,4}(4N) < \overline{D}_{2,4}(4N) < \overline{D}_{4,4}(4N)$. Hence, in this section, we will find the exact values for $f_{a,4}(4N) \in \{\overline{D}_{a,4}(4N), ..., \tilde{M}_{a,4}(4N)\}$ with $a = 1, 2, 4$.

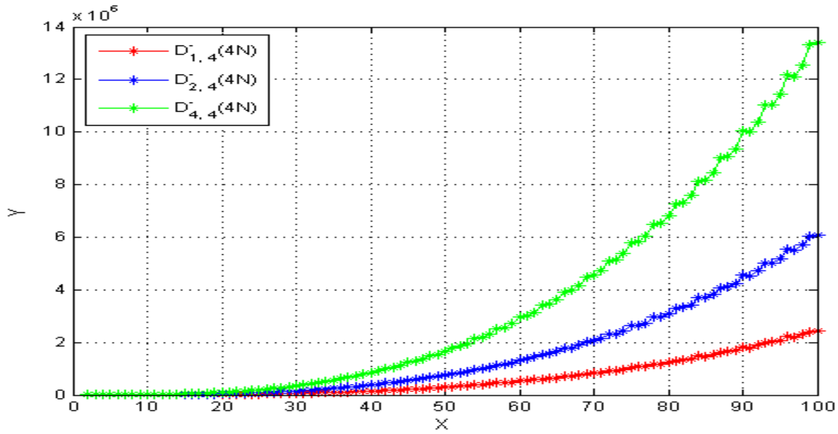

FIGURE 12. $\overline{D}_{a,4}(4N)$ with $a = 1, 2, 4$ $(3 \le N \le 100)$

Now it is easy to see that

$$\sigma_{1,1}(2l; 2) = \sigma_{1,1}(l; 2). \tag{31}$$

From (12) and (31), we have

$$\sum_{k=1}^{N-1} \sigma_{1,1}(4k; 2)\sigma_{1,1}(4(N-k); 2) = \frac{1}{24}[11\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(N; 2)]. \tag{32}$$

Consider

(33)
$$\begin{aligned}&\sum_{k=1}^{N-1}\sigma_{1,0}(4k;2)\sigma_{1,0}(4(N-k);2)\\&=\sum_{k=1}^{N-1}\sigma_{1,0}(2k;2)\sigma_{1,0}(4N-2k;2)-\sum_{k=1}^{N-1}\sigma_{1,0}(4k-2;2)\sigma_{1,0}(4N-(4k-2);2)\\&=\sum_{k=1}^{N-1}\sigma_{1,0}(2k;2)\sigma_{1,0}(4N-2k;2)-4\sum_{k=1}^{N-1}\sigma_{1}(2k-1;2)\sigma_{1}(2N-(2k-1);2)\end{aligned}$$

with $\sigma_{1,0}(2(2l-1);2)=2\sigma_1(2l-1)$.

From (11) and Corollary 3.8 (a), we deduce that

$$\begin{aligned}&\sum_{k=1}^{N-1}\sigma_{1,0}(4k;2)\sigma_{1,0}(4(N-k);2)\\&=\frac{1}{4}\sigma_{3,0}(4N;2)-\frac{5}{6}\sigma_3(2N)+\frac{1}{2}\sigma_3(N)+\left(\frac{1}{6}-2N\right)\sigma_{1,0}(4N;2).\end{aligned}\tag{34}$$

Similarly, we get

(35)
$$\begin{aligned}\sum_{k=1}^{N-1}\sigma_{1,1}(4k;2)\sigma_{1,0}(4(N-k);2)&=\frac{1}{24}[\sigma_3(4N)-\sigma_1(4N)\\&+3\{\sigma_{3,1}(N;2)-(8N-1)\sigma_{1,1}(N;2)\}-6\{\sigma_3(2N)-\sigma_3(N)\}].\end{aligned}$$

By (32), (34) and (35), we get the following theorem.

Theorem 4.1. *Let $N\geq 2$ be a positive integer. Then we get the following.*

(a)

$$\begin{aligned}\overline{D}_{4,4}(4N):&=\sum_{k=1}^{N-1}\sigma_1(4k)\sigma_1(4N-4k)\\&=\frac{1}{24}[35\sigma_3(N)-33\sigma_3(2N)+8\sigma_3(4N)+2(1-24N)\sigma_1(4N)].\end{aligned}$$

(b)

$$\overline{E}_{4,4}(4N) := \sum_{k=1}^{N-1} \tilde{\sigma_1}(4k)\tilde{\sigma_1}(4N-4k)$$
$$= \frac{1}{24}[11\sigma_3(N) - 9\sigma_3(2N) + 6(8N-1)\sigma_1(4N) - 8\sigma_3(4N)$$
$$+ 12(1-8N)\sigma_{1,0}(4N;2) + 12\sigma_{3,0}(4N;2)].$$

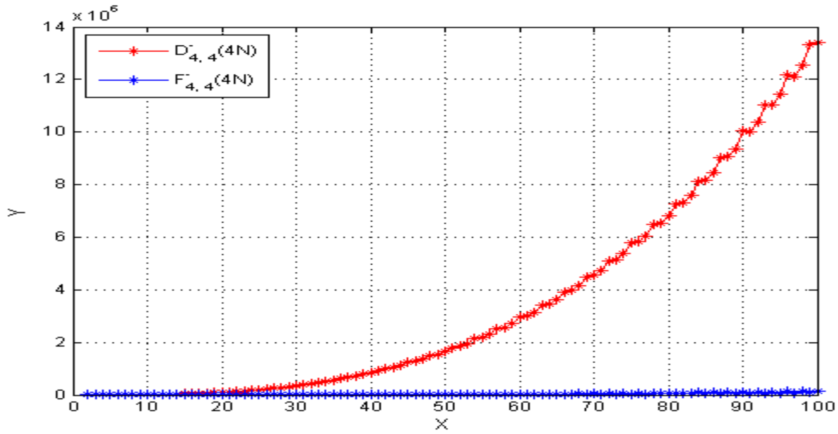

FIGURE 13. $\overline{D}_{4,4}(4N)$ and $\overline{F}_{4,4}(4N)$ $(3 \leq N \leq 100)$

(c)

$$\overline{F}_{4,4}(4N) := \sum_{k=1}^{N-1} \sigma_{1,1}(4k;2)\sigma_{1,1}(4N-4k;2) = \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,1}(N-k;2)$$
$$= \frac{1}{24}[11\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(N;2)].$$

(d)

$$\overline{G}_{4,4}(4N) := \sum_{k=1}^{N-1} \sigma_{1,0}(4k;2)\sigma_{1,0}(4N-4k;2)$$
$$= \frac{1}{4}\sigma_{3,0}(4N;2) - \frac{5}{6}\sigma_3(2N) + \frac{1}{2}\sigma_3(N) + \left(\frac{1}{6} - 2N\right)\sigma_{1,0}(4N;2).$$

(e)

$$\begin{aligned}\overline{H}_{4,4}(4N) :&= \sum_{k=1}^{N-1} \sigma_{1,0}(4k;2)\sigma_{1,1}(4N-4k;2) \\ &= \frac{1}{24}[\sigma_3(4N) - \sigma_1(4N) + 3\{\sigma_{3,1}(N;2) - (8N-1)\sigma_{1,1}(N;2)\} \\ &\quad - 6\{\sigma_3(2N) - \sigma_3(N)\}].\end{aligned}$$

(f)

$$\begin{aligned}\overline{I}_{4,4}(4N) :&= \sum_{k=1}^{N-1} \sigma_1(4k)\tilde{\sigma_1}(4N-4k) \\ &= \frac{1}{24}[-\sigma_3(N) + 19\sigma_3(2N) - 2\sigma_1(4N) + 2(24N-1)\sigma_{1,0}(4N;2) \\ &\quad - 6\sigma_{3,0}(4N;2)].\end{aligned}$$

(g)

$$\begin{aligned}\overline{J}_{4,4}(4N) :&= \sum_{k=1}^{N-1} \sigma_{1,1}(4k;2)\sigma_1(4N-4k) \\ &= \frac{1}{24}[17\sigma_3(N) - 7\sigma_3(2N) + \sigma_3(4N) - \sigma_1(4N) + 3\sigma_{3,1}(N;2) \\ &\quad + (1-24N)\sigma_{1,1}(N;2)].\end{aligned}$$

(h)

$$\begin{aligned}\overline{K}_{4,4}(4N) :&= \sum_{k=1}^{N-1} \sigma_{1,0}(4k;2)\sigma_1(4N-4k) \\ &= \frac{1}{24}[4\sigma_3(4N) + 3\sigma_{3,0}(4N;2) - 26\sigma_3(2N) + 18\sigma_3(N) \\ &\quad + 2(1-12N)\sigma_1(4N) + (1-24N)\sigma_{1,0}(4N;2)].\end{aligned}$$

(i)

$$\begin{aligned}\overline{L}_{4,4}(4N) :&= \sum_{k=1}^{N-1} \sigma_{1,1}(4k;2)\tilde{\sigma_1}(4N-4k) \\ &= \frac{1}{24}[5\{\sigma_3(N) + \sigma_3(2N)\} - \sigma_3(4N) + \sigma_1(4N) \\ &\quad + (24N-5)\sigma_{1,1}(N;2) - 3\sigma_{3,1}(N;2)].\end{aligned}$$

(j)

$$\begin{aligned}\overline{M}_{4,4}(4N) :&= \sum_{k=1}^{N-1} \sigma_{1,0}(4k;2)\tilde{\sigma_1}(4N-4k) \\ &= \frac{1}{24}[4\sigma_3(4N) - 9\sigma_{3,0}(4N;2) + 2(1-12N)\sigma_1(4N) \\ &\qquad + 14\sigma_3(2N) - 6\sigma_3(N) - (7-72N)\sigma_{1,0}(4N;2)]].\end{aligned}$$

In particular, if N is odd, then the above equation can be simplified to the following convolution formulas for the cases of $\overline{D}_{4,4}(4N) \sim \overline{M}_{4,4}(4N)$.

N: odd	convolution formulas	N: odd	convolution formulas
$\overline{D}_{4,4}(4N)$	$\frac{1}{12}[161\sigma_3(N) + (7-168N)\sigma_1(N)]$	$\overline{I}_{4,4}(4N)$	$\frac{1}{12}[-131\sigma_3(N) + (144N-13)\sigma_1(N)]$
$\overline{E}_{4,4}(4N)$	$\frac{5}{4}[7\sigma_3(N) + (1-8N)\sigma_1(N)]$	$\overline{J}_{4,4}(4N)$	$\frac{1}{4}[5\sigma_3(N) - (4N+1)\sigma_1(N)]$
$\overline{F}_{4,4}(4N)$	$\frac{1}{12}[\sigma_3(N) - \sigma_1(N)]$	$\overline{K}_{4,4}(4N)$	$\frac{1}{6}[73\sigma_3(N) + (5-78N)\sigma_1(N)]$
$\overline{G}_{4,4}(4N)$	$11\sigma_3(N) + (1-12N)\sigma_1(N)$	$\overline{L}_{4,4}(4N)$	$\frac{1}{12}[-13\sigma_3(N) + (1+12N)\sigma_1(N)]$
$\overline{H}_{4,4}(4N)$	$\frac{1}{6}[7\sigma_3(N) - (6N+1)\sigma_1(N)]$	$\overline{M}_{4,4}(4N)$	$\frac{1}{6}[-59\sigma_3(N) + (66N-7)\sigma_1(N)]$

<TABLE 15. Convolution formulas for $\overline{D}_{4,4}(4N) \sim \overline{M}_{4,4}(4N)(N:$ odd)>

Example 4.2. *We suggest TABLE 16 for $\overline{D}_{4,4}(4N) \sim \overline{M}_{4,4}(4N)$.*

N	2	3	4	5	6	7	8	9	10	11
$\overline{D}_{4,4}(4N)$	49	210	617	1274	2302	3836	5897	8526	11784	16030
$\overline{E}_{4,4}(4N)$	25	130	369	810	1454	2460	3793	5470	7752	10350
$\overline{F}_{4,4}(4N)$	1	2	9	10	30	28	73	62	136	110
$\overline{G}_{4,4}(4N)$	36	168	484	1032	1848	3120	4772	6936	9632	13080
$\overline{H}_{4,4}(4N)$	6	20	62	116	212	344	526	764	1008	1420
$\overline{I}_{4,4}(4N)$	−35	−166	−475	−1022	−1818	−3092	−4699	−6874	−9496	−12970
$\overline{J}_{4,4}(4N)$	7	22	71	126	242	372	599	826	1144	1530
$\overline{K}_{4,4}(4N)$	42	188	546	1148	2060	3464	5298	7700	10640	14500
$\overline{L}_{4,4}(4N)$	−5	−18	−53	−106	−182	−316	−453	−702	−872	−1310
$\overline{M}_{4,4}(4N)$	−30	−148	−422	−916	−1636	−2776	−4246	−6172	−8624	−11660

< TABLE 16. Examples for $\overline{D}_{4,4}(4N) \sim \overline{M}_{4,4}(4N)(2 \le N \le 11)$ >

Remark 4.3. Using $\sigma_3(4N) = 9\sigma_3(2N) - 8\sigma_3(N)$, Theorem 4.1 (a) becomes

(36)

$$\begin{aligned}\sum_{k=1}^{N-1} \sigma_1(4k)\sigma_1(4(N-k)) &= \frac{29}{192}\sigma_3(4N) + \frac{17}{64}\sigma_3(2N) + \left(\frac{1}{12} - 2N\right)\sigma_1(4N) \\ &= \frac{1}{12}\left[161\sigma_3(N) - 156\sigma_3(\frac{N}{2}) + (1-24N)\left(7\sigma(N) - 6\sigma(\frac{N}{2})\right)\right].\end{aligned}$$

Formulae (36) were derived by N. Cheng and K. S. Williams [10, (4.1)]. A similar result for (36) is in [22, Theorem 9].

It follows from (12), Theorem 4.1 (e), Corollay 3.10 (a) and Corollary 3.21 (d) that

$$\begin{aligned}(37)\qquad \sum_{k=1}^{N-1}\sigma_{1,1}(2k;2)\sigma_{1,1}(4N-4k;2) &= \sum_{k=1}^{N-1}\sigma_{1,1}(k;2)\sigma_{1,1}(N-k;2)\\ &= \frac{1}{24}[11\sigma_3(N)-\sigma_3(2N)-2\sigma_{1,1}(N;2)],\end{aligned}$$

(38)

$$\begin{aligned}\sum_{k=1}^{N-1}\sigma_{1,1}(2k;2)\sigma_{1,0}(4N-4k;2) &= \sum_{k=1}^{N-1}\sigma_{1,1}(4k;2)\sigma_{1,0}(4N-4k;2)\\ &= \frac{1}{24}[\sigma_3(4N)-\sigma_1(4N)+3\{\sigma_{3,1}(N;2)-(8N-1)\sigma_{1,1}(N;2)\}\\ &\quad -6\{\sigma_3(2N)-\sigma_3(N)\}],\end{aligned}$$

(39)

$$\begin{aligned}\sum_{k=1}^{N-1}\sigma_{1,0}(2k;2)\sigma_{1,1}(4N-4k;2) &= \sum_{k=1}^{N-1}\sigma_{1,0}(2k;2)\sigma_{1,1}(2N-2k;2)\\ &= \frac{1}{24}[\sigma_3(2N)-\sigma_1(2N)+3\sigma_{3,1}(N;2)-3(4N-1)\sigma_{1,1}(N;2)]\end{aligned}$$

and

(40)

$$\begin{aligned}\sum_{k=1}^{2N-1}\sigma_{1,0}(k;2)\sigma_{1,0}(4N-2k;2) &= \sum_{k=1}^{N-1}\sigma_{1,0}(2k;2)\sigma_{1,0}(4N-4k;2)\\ &= \frac{1}{24}[2\sigma_3(4N)-10\sigma_3(2N)+6\sigma_{3,0}(2N;2)+2(1-6N)\sigma_{1,0}(4N;2)\\ &\quad +2(1-12N)\sigma_{1,0}(2N;2)]\end{aligned}$$

with $\sigma_{1,0}(2k-1;2)=0$ and $\sigma_{1,1}(2k;2)=\sigma_{1,1}(k;2)=0$.

From (37)$\sim$(40) we obtain the following identities involving divisor functions.

Corollary 4.4. *Let $N \geq 2$ be a positive integer. Then,*

(a)

$$\overline{D}_{2,4}(4N) := \sum_{k=1}^{N-1} \sigma_1(2k)\sigma_1(4N-4k)$$
$$= \frac{1}{24}[17\sigma_3(N) - 10\sigma_3(2N) + 3\sigma_3(4N) + (1-12N)\sigma_1(4N) + (1-24N)\sigma_1(2N)].$$

(b)

$$\overline{E}_{2,4}(4N) := \sum_{k=1}^{N-1} \tilde{\sigma_1}(2k)\tilde{\sigma_1}(4N-4k)$$
$$= \frac{1}{24}[5\sigma_3(N) - 12\sigma_3(2N) + \sigma_3(4N) + \sigma_1(4N) + 12\sigma_{3,0}(2N;2) + (36N-7)\sigma_1(2N) + 2(1-6N)\{5\sigma_{1,0}(2N;2) + \sigma_{1,0}(4N;2)\}].$$

(c)

$$\overline{F}_{2,4}(4N) := \sum_{k=1}^{N-1} \sigma_{1,1}(2k;2)\sigma_{1,1}(4N-4k;2)$$
$$= \frac{1}{24}[11\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(N;2)].$$

(d)

$$\overline{G}_{2,4}(4N) := \sum_{k=1}^{N-1} \sigma_{1,0}(2k;2)\sigma_{1,0}(4N-4k;2)$$
$$= \frac{1}{24}[2\sigma_3(4N) - 10\sigma_3(2N) + 6\sigma_{3,0}(2N;2) + 2(1-6N)\sigma_{1,0}(4N;2) + 2(1-12N)\sigma_{1,0}(2N;2)].$$

(e)

$$\overline{H}_{2,4}(4N) := \sum_{k=1}^{N-1} \sigma_{1,0}(2k;2)\sigma_{1,1}(4N-4k;2)$$
$$= \frac{1}{24}[\sigma_3(2N) - \sigma_1(2N) + 3\sigma_{3,1}(N;2) - 3(4N-1)\sigma_{1,1}(N;2)].$$

(e')

$$\begin{aligned}\tilde{H}_{2,4}(4N) &:= \sum_{k=1}^{N-1} \sigma_{1,1}(2k;2)\sigma_{1,0}(4N-4k;2)\\ &= \frac{1}{24}[\sigma_3(4N)-\sigma_1(4N)+3\{\sigma_{3,1}(N;2)-(8N-1)\sigma_{1,1}(N;2)\}\\ &\quad -6\{\sigma_3(2N)-\sigma_3(N)\}].\end{aligned}$$

(f)

$$\begin{aligned}\overline{I}_{2,4}(4N) &:= \sum_{k=1}^{N-1} \sigma_1(2k)\tilde{\sigma_1}(4N-4k)\\ &= \frac{1}{24}[5\sigma_3(N)+16\sigma_3(2N)-3\sigma_3(4N)+\sigma_1(4N)+3(4N-1)\sigma_1(2N)\\ &\quad -6\sigma_{3,0}(2N;2)-2(1-6N)\sigma_{1,0}(4N;2)+12N\sigma_{1,0}(2N;2)].\end{aligned}$$

(f')

$$\begin{aligned}\tilde{I}_{2,4}(4N) &:= \sum_{k=1}^{N-1} \sigma_1\tilde{(2k)}\sigma_1(4N-4k)\\ &= \frac{1}{24}[-\sigma_3(4N)+2\sigma_3(2N)+17\sigma_3(N)-\sigma_1(4N)+\sigma_1(2N)\\ &\quad -6\sigma_{3,0}(2N;2)-2(6N+1)\sigma_{1,1}(N;2)+2(6N-1)\sigma_{1,0}(4N;2)\\ &\quad +2(12N-1)\sigma_{1,0}(2N;2)].\end{aligned}$$

(g)

$$\begin{aligned}\overline{J}_{2,4}(4N) &:= \sum_{k=1}^{N-1} \sigma_{1,1}(2k;2)\sigma_1(4N-4k) = \sum_{k=1}^{N-1} \sigma_{1,1}(4k;2)\sigma_1(4N-4k)\\ &= \frac{1}{24}[17\sigma_3(N)-7\sigma_3(2N)+\sigma_3(4N)-\sigma_1(4N)+3\sigma_{3,1}(N;2)\\ &\quad +(1-24N)\sigma_{1,1}(N;2)].\end{aligned}$$

(g')

$$\begin{aligned}\tilde{J}_{2,4}(4N) &:= \sum_{k=1}^{N-1} \sigma_1(2k)\sigma_{1,1}(4N-4k;2) = \sum_{k=1}^{N-1} \sigma_1(2k)\sigma_{1,1}(2N-2k;2)\\ &= J_2(2N)\\ &= \frac{1}{24}[11\sigma_3(N)-\sigma_1(2N)+3\sigma_{3,1}(2N;2)-(12N-1)\sigma_{1,1}(2N;2)].\end{aligned}$$

(h)

$$\overline{K}_{2,4}(4N) := \sum_{k=1}^{N-1} \sigma_{1,0}(2k;2)\sigma_1(4N-4k)$$
$$= \frac{1}{24}[2\sigma_3(4N) + 3\sigma_{3,0}(2N;2) - 6\sigma_3(2N) + 2(1-6N)\sigma_1(4N)$$
$$+ \sigma_{1,1}(N;2) - \sigma_1(2N) + (2-24N)\sigma_{1,0}(2N;2)].$$

(h')

$$\tilde{K}_{2,4}(4N) := \sum_{k=1}^{N-1} \sigma_1(2k)\sigma_{1,0}(4N-4k;2)$$
$$= \frac{1}{12}[3\sigma_3(N) + 7\sigma_3(2N) + 2(1-12N)\sigma_1(2N)].$$

(i)

$$\overline{L}_{2,4}(4N) := \sum_{k=1}^{N-1} \sigma_{1,1}(2k;2)\tilde{\sigma_1}(4N-4k)$$
$$= \frac{1}{24}[5\{\sigma_3(N) + \sigma_3(2N)\} - \sigma_3(4N) + \sigma_1(4N)$$
$$+ (24N-5)\sigma_{1,1}(N;2) - 3\sigma_{3,1}(N;2)].$$

(i')

$$\tilde{L}_{2,4}(4N) := \sum_{k=1}^{N-1} \tilde{\sigma_1}(2k)\sigma_{1,1}(4N-4k;2)$$
$$= \frac{1}{24}[11\sigma_3(N) - 2\sigma_3(2N) + \sigma_1(2N) - 3\sigma_{3,1}(2N;2)$$
$$+ (12N-5)\sigma_{1,1}(2N;2)].$$

(j)

$$\overline{M}_{2,4}(4N) := \sum_{k=1}^{N-1} \sigma_{1,0}(2k;2)\tilde{\sigma_1}(4N-4k)$$
$$= \frac{1}{24}[11\sigma_3(2N) - 2\sigma_3(4N) - \sigma_1(2N) + 3\sigma_{3,1}(N;2) - 6\sigma_{3,0}(2N;2)$$
$$- 3(4N-1)\sigma_{1,1}(N;2) - 2(1-6N)\sigma_{1,0}(4N;2)$$
$$- 2(1-12N)\sigma_{1,0}(2N;2)].$$

(j')

$$\begin{aligned}\tilde{M}_{2,4}(4N) &:= \sum_{k=1}^{N-1} \tilde{\sigma_1}(2k)\sigma_{1,0}(4k-4k;2)\\ &= \frac{1}{12}[19\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(2N;2) - 6\sigma_{3,0}(2N;2)\\ &\quad - 4(1-6N)\sigma_{1,0}(2N;2)].\end{aligned}$$

In particular, if N is odd, then the above equation can be simplified to the following convolution formulas for the cases of $\overline{D}_{2,4}(4N) \sim \tilde{M}_{2,4}(4N)$.

N: odd	*convolution formulas*	*N: odd*	*convolution formulas*
$\overline{D}_{2,4}(4N)$	$\frac{1}{12}[73\sigma_3(N)+(5-78N)\sigma_1(N)]$	$\overline{J}_{2,4}(4N)$	$\frac{1}{12}[15\sigma_3(N)-(12N+3)\sigma_1(N)]$
$\overline{E}_{2,4}(4N)$	$\frac{1}{12}[33\sigma_3(N)+(9-42N)\sigma_1(N)]$	$\tilde{J}_{2,4}(4N)$	$\frac{1}{12}[7\sigma_3(N)-(6N+1)\sigma_1(N)]$
$\overline{F}_{2,4}(4N)$	$\frac{1}{12}[\sigma_3(N)-\sigma_1(N)]$	$\overline{K}_{2,4}(4N)$	$\frac{1}{6}[29\sigma_3(N)+(4-33N)\sigma_1(N)]$
$\overline{G}_{2,4}(4N)$	$\frac{1}{3}[13\sigma_3(N)+(2-15N)\sigma_1(N)]$	$\tilde{K}_{2,4}(4N)$	$\frac{1}{2}[11\sigma_3(N)+(1-12N)\sigma_1(N)]$
$\overline{H}_{2,4}(4N)$	$\frac{1}{2}[\sigma_3(N)-N\sigma_1(N)]$	$\overline{L}_{2,4}(4N)$	$\frac{1}{12}[-13\sigma_3(N)+(12N+1)\sigma_1(N)]$
$\tilde{H}_{2,4}(4N)$	$\frac{1}{6}[7\sigma_3(N)-(6N+1)\sigma_1(N)]$	$\tilde{L}_{2,4}(4N)$	$\frac{1}{12}[-5\sigma_3(N)+(6N-1)\sigma_1(N)]$
$\overline{I}_{2,4}(4N)$	$\frac{1}{12}[-59\sigma_3(N)+(66N-7)\sigma_1(N)]$	$\overline{M}_{2,4}(4N)$	$\frac{1}{12}[-23\sigma_3(N)+(27N-4)\sigma_1(N)]$
$\tilde{I}_{2,4}(4N)$	$\frac{1}{12}[-43\sigma_3(N)+(54N-11)\sigma_1(N)]$	$\tilde{M}_{2,4}(4N)$	$\frac{1}{6}[-19\sigma_3(N)+(24N-5)\sigma_1(N)]$

<TABLE 17. Convolution formulas for $\overline{D}_{2,4}(4N) \sim \overline{M}_{2,4}(4N)(N:$ odd)>

Example 4.5. *The first ten values of $\overline{D}_{2,4}(4N) \sim \tilde{M}_{2,4}(4N)$ are given in the following table.*

N	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
$\overline{D}_{2,4}(4N)$	21	94	273	574	1030	1732	2649	3850	5320	7250
$\overline{E}_{2,4}(4N)$	5	38	105	246	454	756	1217	1682	2568	3210
$\overline{F}_{2,4}(4N)$	1	2	9	10	30	28	73	62	136	110
$\overline{G}_{2,4}(4N)$	12	64	180	400	712	1216	1860	2704	3808	5120
$\overline{H}_{2,4}(4N)$	2	8	22	48	76	144	190	320	368	600
$\tilde{H}_{2,4}(4N)$	6	20	62	116	212	344	526	764	1008	1420
$\overline{I}_{2,4}(4N)$	−15	−74	−211	−458	−818	−1388	−2123	−3086	−4312	−5830
$\tilde{I}_{2,4}(4N)$	−7	−50	−131	−322	−546	−988	−1451	−2198	−3032	−4190
$\overline{J}_{2,4}(4N)$	7	22	71	126	242	372	599	826	1144	1530
$\tilde{J}_{2,4}(4N)$	3	10	31	58	106	172	263	382	504	710
$\overline{K}_{2,4}(4N)$	14	72	202	448	788	1360	2050	3024	4176	5720
$\tilde{K}_{2,4}(4N)$	18	84	242	516	924	1560	2386	3468	4816	6540
$\overline{L}_{2,4}(4N)$	−5	−18	−53	−106	−182	−316	−453	−702	−872	−1310
$\tilde{L}_{2,4}(4N)$	−1	−6	−13	−38	−46	−116	−117	−258	−232	−490
$\overline{M}_{2,4}(4N)$	−10	−56	−158	−352	−636	−1072	−1670	−2384	−3440	−4520
$\tilde{M}_{2,4}(4N)$	−6	−44	−118	−284	−500	−872	−1334	−1940	−2800	−3700

< TABLE 18. Examples for $\overline{D}_{2,4}(4N) \sim \tilde{M}_{2,4}(4N)(2 \le N \le 11)$ >

It follows from (12), Theorem 4.1 (e), Corollay 3.10 (a), Theorem 3.1 (b) and Corollary 3.21 (h) that

$$\begin{aligned}(41)\quad &\sum_{k=1}^{N-1}\sigma_{1,1}(k;2)\sigma_{1,1}(4N-4k;2)=\sum_{k=1}^{N-1}\sigma_{1,1}(k;2)\sigma_{1,1}(N-k;2)\\ &=\frac{1}{24}[11\sigma_3(N)-\sigma_3(2N)-2\sigma_{1,1}(N;2)],\end{aligned}$$

$$\begin{aligned}(42)\quad &\sum_{k=1}^{N-1}\sigma_{1,1}(k;2)\sigma_{1,0}(4N-4k;2)=\sum_{k=1}^{N-1}\sigma_{1,1}(4k;2)\sigma_{1,0}(4N-4k;2)\\ &=\frac{1}{24}[\sigma_3(4N)-\sigma_1(4N)+3\{\sigma_{3,1}(N;2)-(8N-1)\sigma_{1,1}(N;2)\}\\ &\quad-6\{\sigma_3(2N)-\sigma_3(N)\}],\end{aligned}$$

$$\begin{aligned}(43)\quad &\sum_{k=1}^{N-1}\sigma_{1,0}(k;2)\sigma_{1,1}(4N-4k;2)=\sum_{k=1}^{N-1}\sigma_{1,0}(k;2)\sigma_{1,1}(N-k;2)\\ &=\frac{1}{24}[\sigma_3(N)-\sigma_1(N)+3\sigma_{3,1}(N;2)-3(2N-1)\sigma_{1,1}(N;2)]\end{aligned}$$

and

$$\begin{aligned}(44)\quad &\sum_{k=1}^{N-1}\sigma_{1,0}(k;2)\sigma_{1,0}(4N-4k;2)=2\sum_{k=1}^{N-1}\sigma_{1,0}(k;2)\sigma_1(2N-2k)\\ &=\frac{1}{12}[2\sigma_3(2N)-3\sigma_3(N)-3\sigma_{3,1}(N;2)+(2-6N)\sigma_1(2N)\\ &\quad+(1-12N)\sigma_{1,0}(N;2)].\end{aligned}$$

From (41)∼(44) we obtain the following corollary.

Corollary 4.6. *Let $N\geq 2$ be a positive integer. Then, we obtain the next result.*

(a) *See* ([32], *(11)), (*[22], *Theorem 4).*

$$\begin{aligned}\overline{D}_{1,4}(4N):&=\sum_{k=1}^{N-1}\sigma_1(k)\sigma_1(4N-4k)\\ &=\frac{1}{24}[4\sigma_3(N)+6\sigma_3(2N)+(1-12N)\sigma_1(2N)\\ &\quad+3(1-8N)\sigma_1(N)+2(1-3N)\sigma_{1,1}(N;2)].\end{aligned}$$

(b)

$$\overline{E}_{1,4}(4N) := \sum_{k=1}^{N-1} \tilde{\sigma_1}(k)\tilde{\sigma_1}(4N-4k)$$
$$= \frac{1}{24}[6\sigma_3(N) + (7-12N)\sigma_1(2N) + (1-24N)\sigma_1(N) - 12\sigma_{3,1}(N;2) + 2(27N-5)\sigma_{1,1}(N;2)].$$

(c)

$$\overline{F}_{1,4}(4N) := \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,1}(4N-4k;2)$$
$$= \frac{1}{24}[11\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(N;2)].$$

(d)

$$\overline{G}_{1,4}(4N) := \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,0}(4N-4k;2)$$
$$= 2\sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_1(2N-2k)$$
$$= \frac{1}{12}[2\sigma_3(2N) - 3\sigma_3(N) - 3\sigma_{3,1}(N;2) + 2(1-3N)\sigma_1(2N) + (1-12N)\sigma_{1,0}(N;2)].$$

(e)

$$\overline{H}_{1,4}(4N) := \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,1}(4N-4k;2)$$
$$= \frac{1}{24}[\sigma_3(N) - \sigma_1(N) + 3\sigma_{3,1}(N;2) - 3(2N-1)\sigma_{1,1}(N;2)].$$

(e')

$$\tilde{H}_{1,4}(4N) := \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,0}(4N-4k;2)$$
$$= \sum_{k=1}^{N-1} \sigma_{1,1}(4k;2)\sigma_{1,0}(4N-4k;2) = \frac{1}{24}[3[\sigma_3(2N) - \sigma_1(2N)] - 2\{\sigma_3(N) - \sigma_1(N)\} + 3\{\sigma_{3,1}(N;2) - (8N-1)\sigma_{1,1}(N;2)\}].$$

(f)

$$\overline{I}_{1,4}(4N) := \sum_{k=1}^{N-1} \sigma_1(k)\tilde{\sigma_1}(4N-4k)$$
$$= \frac{1}{24}[20\sigma_3(N) - 8\sigma_3(2N) + (12N-1)\sigma_1(2N) + (18N-5)\sigma_1(N) + 6\sigma_{3,1}(N;2) + 6N\sigma_{1,0}(N;2)].$$

(f')

$$\tilde{I}_{1,4}(4N) := \sum_{k=1}^{N-1} \tilde{\sigma_1}(k)\sigma_1(4N-4k)$$
$$= \frac{1}{24}[14\sigma_3(N) - 2\sigma_3(2N) + (12N-7)\sigma_1(2N) + (1+24N)\sigma_1(N) + 6\sigma_{3,1}(N;2) - 42N\sigma_{1,1}(N;2)].$$

(g)

$$\overline{J}_{1,4}(4N) := \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_1(4N-4k) = \sum_{k=1}^{N-1} \sigma_{1,1}(4k;2)\sigma_1(4N-4k)$$
$$= \frac{1}{24}[9\sigma_3(N) + 2\sigma_3(2N) - \sigma_1(4N) + 3\sigma_{3,1}(N;2) + (1-24N)\sigma_{1,1}(N;2)].$$

(g')

$$\tilde{J}_{1,4}(4N) := \sum_{k=1}^{N-1} \sigma_1(k)\sigma_{1,1}(4N-4k;2) = \sum_{k=1}^{N-1} \sigma_1(k)\sigma_{1,1}(N-k;2)$$
$$= \frac{1}{24}\left[12\sigma_3(N) - \sigma_3(2N) + 3\sigma_{3,1}(N;2) - \sigma_1(N) + (1-6N)\sigma_{1,1}(N;2)\right].$$

(h)

$$\overline{K}_{1,4}(4N) := \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_1(4N-4k)$$
$$= \frac{1}{24}[-5\sigma_3(N) + 4\sigma_3(2N) + 4(1-3N)\sigma_1(2N) + (1-24N)\sigma_1(N) - 3\sigma_{3,1}(N;2) + (1+18N)\sigma_{1,1}(N;2)].$$

(h')

$$\begin{aligned}\tilde{K}_{1,4}(4N) := &\sum_{k=1}^{N-1}\sigma_1(k)\sigma_{1,0}(4N-4k;2)\\ &= \frac{1}{24}[-8\sigma_3(N)+7\sigma_3(2N)+(1-12N)\sigma_1(2N)\\ &\quad +4(1-6N)\sigma_1(N)-3\sigma_{3,1}(N;2)+\sigma_{1,1}(N;2)].\end{aligned}$$

(i)

$$\begin{aligned}\overline{L}_{1,4}(4N) := &\sum_{k=1}^{N-1}\sigma_{1,1}(k;2)\tilde{\sigma_1}(4N-4k) = \sum_{k=1}^{N-1}\sigma_{1,1}(2k;2)\tilde{\sigma_1}(4N-4k)\\ &= \frac{1}{24}[13\sigma_3(N)-4\sigma_3(2N)+\sigma_1(4N)-3\sigma_{3,1}(N;2)\\ &\quad +(24N-5)\sigma_{1,1}(N;2)].\end{aligned}$$

(i')

$$\begin{aligned}\tilde{L}_{1,4}(4N) := &\sum_{k=1}^{N-1}\tilde{\sigma_1}(k)\sigma_{1,1}(4N-4k;2)\\ &= \frac{1}{24}[10\sigma_3(N)-\sigma_3(2N)+\sigma_1(N)-3\sigma_{3,1}(N;2)\\ &\quad +(6N-5)\sigma_{1,1}(N;2)].\end{aligned}$$

(j)

$$\begin{aligned}\overline{M}_{1,4}(4N) := &\sum_{k=1}^{N-1}\sigma_{1,0}(k;2)\tilde{\sigma_1}(4N-4k)\\ &= \frac{1}{24}[7\sigma_3(N)-4\sigma_3(2N)-4(1-3N)\sigma_1(2N)\\ &\quad -3(1-8N)\sigma_1(N)+9\sigma_{3,1}(N;2)+5(1-6N)\sigma_{1,1}(N;2)].\end{aligned}$$

(j')

$$\begin{aligned}\tilde{M}_{1,4}(4N) := &\sum_{k=1}^{N-1}\tilde{\sigma_1}(k)\sigma_{1,0}(4N-4k;2)\\ &= \frac{1}{24}[4\sigma_3(N)-\sigma_3(2N)-(7-12N)\sigma_1(2N)+24N\sigma_1(N)\\ &\quad +9\sigma_{3,1}(N;2)+(5-48N)\sigma_{1,1}(N;2)].\end{aligned}$$

In particular, if N is odd, then the above can be simplified to the following convolution formulas for the cases of $\overline{D}_{1,4}(4N) \sim \tilde{M}_{1,4}(4N)$.

N: odd	convolution formulas	N: odd	convolution formulas
$\overline{D}_{1,4}(4N)$	$\frac{1}{12}[29\sigma_3(N) + (4-33N)\sigma_1(N)]$	$\overline{J}_{1,4}(4N)$	$\frac{1}{4}[5\sigma_3(N) - (4N+1)\sigma_1(N)]$
$\overline{E}_{1,4}(4N)$	$\frac{1}{4}[-\sigma_3(N) + (2-N)\sigma_1(N)]$	$\tilde{J}_{1,4}(4N)$	$\frac{1}{4}[\sigma_3(N) - N\sigma_1(N)]$
$\overline{F}_{1,4}(4N)$	$\frac{1}{12}[\sigma_3(N) - \sigma_1(N)]$	$\overline{K}_{1,4}(4N)$	$\frac{1}{12}[14\sigma_3(N) + (7-21N)\sigma_1(N)]$
$\overline{G}_{1,4}(4N)$	$\frac{1}{2}[2\sigma_3(N) + (1-3N)\sigma_1(N)]$	$\tilde{K}_{1,4}(4N)$	$\frac{1}{6}[13\sigma_3(N) + (2-15N)\sigma_1(N)]$
$\overline{H}_{1,4}(4N)$	$\frac{1}{12}[2\sigma_3(N) + (1-3N)\sigma_1(N)]$	$\overline{L}_{1,4}(4N)$	$\frac{1}{12}[-13\sigma_3(N) + (12N+1)\sigma_1(N)]$
$\tilde{H}_{1,4}(4N)$	$\frac{1}{6}[7\sigma_3(N) - (6N+1)\sigma_1(N)]$	$\tilde{L}_{1,4}(4N)$	$\frac{1}{12}[-\sigma_3(N) + (3N-2)\sigma_1(N)]$
$\overline{I}_{1,4}(4N)$	$\frac{1}{12}[-23\sigma_3(N) + (27N-4)\sigma_1(N)]$	$\overline{M}_{1,4}(4N)$	$\frac{1}{12}[-10\sigma_3(N) + 5(3N-1)\sigma_1(N)]$
$\tilde{I}_{1,4}(4N)$	$\frac{1}{12}[\sigma_3(N) + (9N-10)\sigma_1(N)]$	$\tilde{M}_{1,4}(4N)$	$\frac{1}{6}[\sigma_3(N) + (3N-4)\sigma_1(N)]$

<TABLE 19. Convolution formulas for $\overline{D}_{1,4}(4N) \sim \tilde{M}_{1,4}(4N)(N:$ odd) >

Example 4.7. *The values of $\overline{D}_{1,4}(4N) \sim \tilde{M}_{1,4}(4N)$ for $N = 2, 3, \cdots, 11$ are given in the following table.*

N	2	3	4	5	6	7	8	9	10	11
$\overline{D}_{1,4}(4N)$	7	36	101	224	394	680	1025	1512	2088	2860
$\overline{E}_{1,4}(4N)$	−5	−8	−27	−36	−46	−96	−71	−212	−24	−360
$\overline{F}_{1,4}(4N)$	1	2	9	10	30	28	73	62	136	110
$\overline{G}_{1,4}(4N)$	0	12	28	84	144	264	404	588	896	1140
$\overline{H}_{1,4}(4N)$	0	2	2	14	8	44	22	98	48	190
$\tilde{H}_{1,4}(4N)$	6	20	62	116	212	344	526	764	1008	1420
$\overline{I}_{1,4}(4N)$	−5	−28	−79	−176	−318	−536	−835	−1192	−1720	−2260
$\tilde{I}_{1,4}(4N)$	7	8	41	28	90	64	173	140	200	200
$\overline{J}_{1,4}(4N)$	7	22	71	126	242	372	599	826	1144	1530
$\tilde{J}_{1,4}(4N)$	1	4	11	24	38	72	95	160	184	300
$\overline{K}_{1,4}(4N)$	0	14	30	98	152	308	426	686	944	1330
$\tilde{K}_{1,4}(4N)$	6	32	90	200	356	608	930	1352	1904	2560
$\overline{L}_{1,4}(4N)$	−5	−18	−53	−106	−182	−316	−453	−702	−872	−1310
$\tilde{L}_{1,4}(4N)$	1	0	7	−4	22	−16	51	−36	88	−80
$\overline{M}_{1,4}(4N)$	0	−10	−26	−70	−136	−220	−382	−490	−848	−950
$\tilde{M}_{1,4}(4N)$	12	16	68	64	136	160	244	352	224	560

< TABLE 20. Examples for $\overline{D}_{1,4}(4N) \sim \tilde{M}_{1,4}(4N)(2 \leq N \leq 11)$ >

Corollary 4.8. *Let N be an odd integer.*

(a) $7|\overline{D}_{4,4}(4N)$.
(b) $7|\overline{K}_{1,4}(4N)$.
(c) $5|\overline{M}_{1,4}(4N)$.

Proof. It is immediate from Table 15 and 19. □

Using these results in Section 3 and 4, we will prove our main theorem.

Proof. (Proof of Theorem 1.1)

(a) By TABLE 5, 7, 9, 11, 13, 15, 17 and 19, we deduce the desired result.

(b) If p is an odd prime number, then $\sigma_3(p) = p^3+1$ and $\sigma_1(p) = p+1$. Since $a+b+c = 0$, we obtain

$$\begin{aligned}\frac{1}{u}[a\sigma_3(p) + bp\sigma_1(p) + c\sigma_1(p)] &= \frac{1}{u}[a(p^3+1) + bp(p+1) + c(p+1)\\ &= \frac{1}{u}[a(p^3+1) + bp(p+1) + (-a-b)(p+1)]\\ &= \frac{1}{u}(ap+b)(p^2-1).\end{aligned}$$

(c) From (b), we obtain $\frac{1}{u}(ap+b)(p^2-1) = \frac{1}{u}(a(2L+1)+b)(2L)(2L+2) = \frac{1}{u}[\frac{24aL(L+1)(2L+1)}{6} + \frac{8bL(L+1)}{2}]$.

□

Corollary 4.9. *Let N be an odd positive integer and $s, l \in \mathbb{Z}^+ \cup \{0\}$. If $f_{2^s,2^l} \in \{\overline{D}_{2^s,2^l}, ..., \tilde{M}_{2^s,2^l}\}$ and $s \leq l$, then there exist $a, b, c, u \in \mathbb{Z}$ satisfying*

$$f_{2^s,2^l}(2^l N) = \frac{1}{u}[a\sigma_3(N) + bN\sigma_1(N) + c\sigma_1(N)]$$

with $a+b+c = 0$ and $(u, a, b, c) = 1$.

Proof. Obviously, $\sigma_{1,1}(2^m N; 2) = \sigma_{1,1}(N; 2)$ and $\sigma_{1,0}(2^m N; 2) = 2\sigma_1(2^{m-1}N)$. Hence

$$\begin{aligned}\sigma_{1,0}(2^m N; 2) &= 2\sigma_1(2^{m-1}N)\\ &= 2\{\sigma_{1,1}(2^{m-1}N; 2) + \sigma_{1,0}(2^{m-1}N; 2)\}\\ &= 2\sigma_{1,1}(N; 2) + 2\sigma_1(2^{m-2}N)\\ &= 2\sigma_{1,1}(N; 2) + 2\{\sigma_{1,1}(2^{m-2}N; 2) + \sigma_{1,0}(2^{m-2}N; 2)\}\\ &= ...\\ &= \alpha\sigma_{1,1}(N; 2) + \beta\sigma_1(N)\end{aligned}$$

where $\alpha, \beta \in \mathbb{Z}$. Then there exist $A, B, C \in \mathbb{Z}$ satisfying

$$
\begin{aligned}
f_{2^s,2^l}(2^l N) &= A \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,1}(N-k;2) \\
&\quad + B \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,1}(N-k;2) + C \sum_{k=1}^{N-1} \sigma_{1,0}(k;2)\sigma_{1,0}(N-k;2) \\
&= \frac{A}{12}[\sigma_3(N) - \sigma_1(N)] + \frac{B}{12}[2\sigma_3(N) + (1-3N)\sigma_1(N)] + \frac{C}{12}[0]
\end{aligned}
$$

and $\frac{A}{12}(1-1) + \frac{B}{12}(2-3+1) + \frac{C}{12}(0) = 0$, by Table 5. □

Example 4.10. (a)

$$
\begin{aligned}
\overline{D}_{2^n,2^n}(2^n N) &:= \sum_{k=1}^{N-1} \sigma_1(2^n k)\sigma_1(2^n N - 2^n k) \\
&= \frac{1}{24}[(45 \cdot 4^n - 46 \cdot 2^n + 11)\sigma_3(N) - (3 \cdot 4^n - 4 \cdot 2^n + 1)\sigma_3(2N) \\
&\quad + 6(4^n - 2^n)\sigma_{3,1}(N;2) + (-12N \cdot 4^n + 2 \cdot 2^n)\sigma_1(N) \\
&\quad + (-12N \cdot 4^n + (12N+2)2^n - 2)\sigma_{1,1}(N;2)].
\end{aligned}
$$

Finally, we will give a counter example for Theorem 1.1. From (12) we obtain

$$
\begin{aligned}
&\sum_{k=1}^{2N-1} \sigma_{1,1}(k;2)\sigma_{1,1}(2N-k;2) \\
&= 2\sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,1}(2N-k;2) + \{\sigma_{1,1}(N;2)\}^2 \\
&= \frac{1}{24}[11\sigma_3(2N) - \sigma_3(4N) - 2\sigma_{1,1}(N;2)].
\end{aligned}
$$

Thus, we get a counter example in Question 1.2 as follows.

Example 4.11. *If $N > 1$ is any integer, then we obtain*

$$
\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{1,1}(4k;2)\sigma_{1,1}(4N-2k;2) \\
&= \frac{1}{48}[11\sigma_3(2N) - \sigma_3(4N) - 2\sigma_{1,1}(N;2)\{1 + 12\sigma_{1,1}(N;2)\}].
\end{aligned}
$$

In particular, if N is odd, then

$$\sum_{k=1}^{N-1}\sigma_{1,1}(4k;2)\sigma_{1,1}(4N-2k;2)=\frac{1}{24}[23\sigma_3(N)-22\{\sigma_1(N)\}^2-\sigma_1(N)].$$

If we change a condition of Question 1.2, i.e., replace $N\sigma_1(N)$ with $\{\sigma_1(N)\}^2$, we obtain $a+b+c=0$.

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