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AN IMPROVEMENT OF THE HÖRMANDER-MIKHLIN MULTIPLIER CONDITIONS

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Abstract. We give an L^p Fourier multiplier condition which implies the Hörmander-Mikhlin multiplier theorem.

1. Introduction and statement of results

The Fourier transform of a Schwartz function f on $\mathbb{R}^d$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} x\cdot\xi} f(x)dx$$

and its inverse Fourier transform by

$$\mathcal{F}^{-1}[f](x) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi \mathrm{i} x\cdot\xi} f(\xi)d\xi.$$

For each $\alpha \in \mathbb{R}$, the Sobolev space $W^\alpha(\mathbb{R}^d)$ is defined as the space of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|u\|_{W^\alpha} = \left(\int_{\mathbb{R}^d} (1+|x|)^{2\alpha} |\widehat{u}(x)|^2 dx\right)^{1/2} < \infty.$$

Let $m \in L^\infty(\mathbb{R}^d)$, then for $f \in \mathcal{S}(\mathbb{R}^d)$ the Fourier multiplier operator T_m is defined by

$$\widehat{T_m f} = m\widehat{f}.$$

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Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ be a radial function such that

$$\operatorname{supp}(\widehat{\varphi}) \subset \{\xi : 1/2 \leq |\xi| \leq 2\}, \quad \sum_{s\in\mathbb{Z}} \widehat{\varphi}(\xi/2^s) = 1 \quad \text{for all } \xi \neq 0.$$

The Hörmander-Mikhlin multiplier theorem [6] states that if $m \in L^\infty(\mathbb{R}^d)$ and

$$\sup_{s\in\mathbb{Z}} \|m(2^s\cdot)\widehat{\varphi}(\cdot)\|_{W^\alpha(\mathbb{R}^d)} < \infty \quad \text{for some } \alpha > d/2, \tag{1.1}$$

then T_m is bounded on $L^p(\mathbb{R}^d)$ for all $1 < p < \infty$. Let $H_s = \mathcal{F}^{-1}[m(2^s\cdot)\widehat{\varphi}(\cdot)]$ then by Plancherel's theorem the condition (1.1) is equivalent to

$$\sup_{s\in\mathbb{Z}} \left(\|H_s\|_{L^2[(1+|x|)^{2\alpha}dx]}\right) := \sup_{s\in\mathbb{Z}} \left(\int_{\mathbb{R}^d} |H_s(x)|^2(1+|x|)^{2\alpha}dx\right)^{\frac{1}{2}} < \infty, \tag{1.2}$$

for some $\alpha > d/2$. In this paper we give an L^p Fourier multiplier condition which implies the Hörmander-Mikhlin multiplier theorem.

Theorem 1. *Let $d \geq 1$. If $\epsilon > 0$ then for $1 < p < \infty$*

$$\|T_m f\|_p \leq C_\epsilon \sup_{s\in\mathbb{Z}} \left(\|H_s\|_{L^1[(1+|x|)^\epsilon dx]}\right) \|f\|_p. \tag{1.3}$$

By Hölder's inequality, if $\alpha > \frac{d}{2} + \epsilon$ then

$$\|H_s\|_{L^1[(1+|x|)^\epsilon dx]} \leq C_{\alpha,\epsilon}\|H_s\|_{L^2[(1+|x|)^{2\alpha}dx]}. \tag{1.4}$$

Therefore by (1.1), (1.2) and (1.4), Theorem 1 implies the Hörmander-Mikhlin multiplier theorem.

2. Reductions

Notation. *For each $k \in \mathbb{Z}$, $\mathcal{D}_k(\mathbb{R}^d)$ denotes the collection of dyadic cubes $Q \subset \mathbb{R}^d$ with side-length 2^k. Let $\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$. For each cube $Q \subset \mathbb{R}^d$ with side-length 2^k, we write $L(Q) = k$. For each cube Q and $r > 0$, rQ denotes the cube having r times side-length of Q with the same center as Q. The d-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^d$ will be denoted by $|E|$ or meas(E).*

Let φ be a radial Schwartz function such that $\widehat{\varphi}$ is supported in $\{\xi : 1/2 < |\xi| < 2\}$ and satisfies

$$\sum_{s\in\mathbb{Z}}[\widehat{\varphi}(2^{-s}\xi)]^2 = 1 \quad \text{for all } \xi \neq 0.$$

As in [7], let ψ be a radial Schwartz function with compact support in $\{|x| < 1/10\}$ and $\widehat{\psi} > 0$ on $\{\xi : 1/4 < |\xi| < 4\}$. Let $\eta = \mathcal{F}^{-1}[\widehat{\varphi}(\widehat{\psi})^{-1}]$. For each $s \in \mathbb{Z}$, let

$$H_s = \mathcal{F}^{-1}[m(2^s\cdot)\widehat{\varphi}(\cdot)].$$

Now let $K_s = 2^{sd}H_s(2^s\cdot)$, $\psi_s = 2^{sd}\psi(2^s\cdot)$ and $\eta_s = 2^{sd}\eta(2^s\cdot)$ then

$$T_m f = \sum_{s\in\mathbb{Z}} \psi_s * K_s * (\eta_s * f). \tag{2.1}$$

Theorem 1 will be proved in Section 3 by using atomic decompositions constructed from Peetre's maximal square function (cf. [8], [10] and [9]) using ideas from work by Chang and Fefferman [1].

3. Atomic decompositions and proof of Theorem 1

As in [7] we use atomic decompositions constructed from a non-tangential Peetre type maximal square function (cf. [8–10]).

Atomic decompositions

Consider the non-tangential version of Peetre's maximal square function

$$Sf(x) = \Big(\sum_s \sup_{|y|\leq 10d\cdot 2^{-s}} |\eta_s * f(x+y)|^2\Big)^{1/2}.$$

Then $\|Sf\|_p \leq C_p\|f\|_p$ for $1 < p < \infty$. So by (2.1), the proof of the L^p boundedness of T_m reduces to

$$\Big\|\sum_s \psi_s * K_s * (\eta_s * f)\Big\|_p \leq C_\epsilon \sup_s \big(\|H_s\|_{L^1[(1+|x|)^\epsilon dx]}\big)\, \|Sf\|_p.$$

For each integer j, we introduce the set

$$\Omega_j = \{x : Sf(x) > 2^j\}.$$

Let $\mathcal{Q}_j^s$ be the set of all dyadic cubes of side-length 2^{-s} which have the property that $|Q \cap \Omega_j| \geq |Q|/2$ but $|Q \cap \Omega_{j+1}| < |Q|/2$. Note that for fixed s, the sets

$$\bigcup_{Q \in \mathcal{Q}_j^s} Q, \quad j \in \mathbb{Z},$$

are disjoint. We also set

$$\Omega_j^* = \{x : M\chi_{\Omega_j}(x) > 100^{-d}\}$$

where M is the Hardy-Littlewood maximal operator. Ω_j^* is an open set containing Ω_j and $|\Omega_j^*| \leq C_d|\Omega_j|$.

Let $\mathcal{W}_j$ be the set of all dyadic cubes W such that the 20-fold dilate of W is contained in Ω_j^* and W is maximal with respect to this property. Clearly, the interiors of these cubes are disjoint, and we shall refer to them as Whitney cubes for Ω_j^*. Observe that $\{10W : W \in \mathcal{W}_j\}$ have bounded overlap.

Note that each $Q \in \mathcal{Q}_j^s$ is contained in a unique $W \in \mathcal{W}_j$. For each $W \in \mathcal{W}_j$, set

$$A_{s,W,j} = \sum_{\substack{Q \in \mathcal{Q}_j^s \\ Q \subset W}} (\eta_s * f)\chi_Q;$$

note that only terms with $L(W) + s \geq 0$ occur. Since any dyadic cube W can be a Whitney cube for several Ω_j^*, we also define "cumulative atoms",

$$A_{s,W} = \sum_{j : W \in \mathcal{W}_j} A_{s,W,j}.$$

Note that

$$\eta_s * f = \sum_{W \in \bigcup_j \mathcal{W}_j} A_{s,W} = \sum_j \sum_{W \in \mathcal{W}_j} A_{s,W,j}.$$

Standard facts about these atoms are summarized in the following lemma that is taken from [7].

Lemma 3.1. *For each $j \in \mathbb{Z}$, the following inequalities hold.*

(1)

$$\sum_{W \in \mathcal{W}_j} \sum_s \|A_{s,W,j}\|_2^2 \leq C2^{2j}\text{meas}(\Omega_j). \tag{3.1}$$

(2) There is a constant C_d such that for every assignment $W \mapsto s(W) \in \mathbb{Z}$, defined on $\mathcal{W}_j$, and for $0 \le p \le 2$,

$$\sum_{W \in \mathcal{W}_j} \operatorname{meas}(W) \|A_{s(W),W,j}\|_\infty^p \le C_d 2^{pj} \operatorname{meas}(\Omega_j). \tag{3.2}$$

With this notation we need to show the inequality

$$\Big\| \sum_{s,j} \sum_{\ell \ge 0} \sum_{\substack{W \in \mathcal{W}_j \\ L(W)=l-s}} \psi_s * K_s * A_{s,W,j} \Big\|_p \le C_\epsilon \sup_s \left(\|H_s\|_{L^1[(1+|x|)^\epsilon dx]} \right) \|Sf\|_p, \tag{3.3}$$

for any $\epsilon > 0$ and $1 < p \le 2$. For each integer l in this sum, we split K_s into short range and long range pieces, $K_{s,\ell}^{sh}$ and $K_{s,\ell}^{lg}$. To define them, let

$$\begin{aligned} K_s(x) = 2^{sd} H_s(2^s x) &= 2^{sd} H_s(2^s x) \chi_{(|2^s x| \le 2^\ell)} + 2^{sd} H_s(2^s x) \chi_{(|2^s x| > 2^\ell)} \\ &:= K_{s,\ell}^{sh}(x) + K_{s,\ell}^{lg}(x), \end{aligned}$$

where χ denotes the characteristic function.

Lemma 3.2. *We have the following.*

(1) For $1 \le p \le \infty$

$$\|K_{s,\ell}^{sh} * f\|_p \le C \|H_s\|_1 \|f\|_p. \tag{3.4}$$

(2) If $\epsilon > 0$ then for $1 \le p \le \infty$

$$\|K_{s,\ell}^{lg} * f\|_p \le C 2^{-\epsilon\ell} \left(\|H_s\|_{L^1[(1+|x|)^\epsilon dx]} \right) \|f\|_p. \tag{3.5}$$

Proof. For (3.4), by scaling

$$\|K_{s,\ell}^{sh} * f\|_p \le C \|K_{s,\ell}^{sh}\|_1 \|f\|_p \le C \|H_s \chi_{(|x| \le 2^\ell)}\|_1 \|f\|_p.$$

For (3.5), by scaling

$$\|K_{s,\ell}^{lg} * f\|_p \le C \|K_{s,\ell}^{lg}\|_1 \|f\|_p \le C \|H_s \chi_{(|x| > 2^\ell)}\|_1 \|f\|_p.$$

And so if $\epsilon > 0$ then

$$\|K_{s,\ell}^{lg} * f\|_p \le C 2^{-\epsilon\ell} \left(\|H_s\|_{L^1[(1+|x|)^\epsilon dx]} \right) \|f\|_p.$$

□

Lemma 3.3 (Lemma 2.2 in [7]). *Let $0 < p_0 < p_1 < \infty$. Let $\{F_j\}_{j\in\mathbb{Z}}$ be a sequence of measurable functions on a measure space $\{\Omega, \mu\}$, and let $\{s_j\}$ be a sequence of nonnegative numbers. Assume that, for all j, the inequality*

$$\|F_j\|_{p_v}^{p_v} \leq M^{p_v} 2^{jp_v} s_j$$

holds for $v = 0$ and $v = 1$. Then for every $p \in (p_0, p_1)$, there is a constant $C = C(p_0, p_1, p)$ such that

$$\|\sum_j F_j\|_p^p \leq C^p M^p \sum_j 2^{jp} s_j.$$

For the short range estimate, it suffices to show that

$$\Big\| \sum_{s,j} \sum_{\ell \geq 0} \sum_{\substack{W\in\mathcal{W}_j \\ L(W)=\ell-s}} \psi_s * K_{s,\ell}^{sh} * A_{s,W,j} \Big\|_\tau^\tau \leq C^\tau \sup_{s\in\mathbb{Z}} (\|H_s\|_1)^\tau \|Sf\|_\tau^\tau \tag{3.6}$$

for $1 \leq \tau \leq 2$. If we have (3.6) then $\tau = p$ will give the desired result. To prove (3.6), by Lemma 3.3, it suffices to show that for fixed j, and for $1 \leq \tau \leq 2$,

$$\Big\| \sum_{s} \sum_{\ell \geq 0} \sum_{\substack{W\in\mathcal{W}_j \\ L(W)=\ell-s}} \psi_s * K_{s,\ell}^{sh} * A_{s,W,j} \Big\|_\tau^\tau \leq C^\tau \sup_{s\in\mathbb{Z}} (\|H_s\|_1)^\tau \operatorname{meas}(\Omega_j). \tag{3.7}$$

If we show (3.7), then by Lemma 3.3 the left hand side of (3.6) is controlled by

$$C^\tau \sup_{s\in\mathbb{Z}} (\|H_s\|_1)^\tau \sum_j 2^{j\tau} \operatorname{meas}(\Omega_j) \leq C^\tau \sup_{s\in\mathbb{Z}} (\|H_s\|_1)^\tau \|Sf\|_\tau^\tau.$$

Now we prove (3.7). For $\tau = 2$, note that

$$\Big\| \sum_{s} \sum_{\ell \geq 0} \sum_{\substack{W\in\mathcal{W}_j \\ L(W)=\ell-s}} \psi_s * K_{s,\ell}^{sh} * A_{s,W,j} \Big\|_2^2 = \Big\| \sum_{W\in\mathcal{W}_j} \sum_{s} \psi_s * K_{s,L(W)+s}^{sh} * A_{s,W,j} \Big\|_2^2.$$

For each fixed j, since $\sum_s \psi_s * K^{sh}_{s,L(W)+s} * A_{s,W,j}$ is supported in W^* and W^*, $W \in \mathcal{W}_j$, have bounded overlap, by (3.4) and (3.1),

$$\begin{aligned}\text{The left hand side of (3.7)} &\le C^2 \sum_{W\in\mathcal{W}_j} \Big\| \sum_s \psi_s * K^{sh}_{s,L(W)+s} * A_{s,W,j}\Big\|_2^2 \\ &\le C^2 \sum_{W\in\mathcal{W}_j} \sum_s \Big\| K^{sh}_{s,L(W)+s} * A_{s,W,j}\Big\|_2^2 \\ &\le C^2 \sum_{W\in\mathcal{W}_j} \sum_s \sup_{s\in\mathbb{Z}} \left(\|H_s\|_1\right)^2 \|A_{s,W,j}\|_2^2 \\ &\le C^2 \sup_{s\in\mathbb{Z}} \left(\|H_s\|_1\right)^2 2^{2j}\text{meas}(\Omega_j).\end{aligned}$$

Inequality (3.7) for $\tau < 2$ follows from (3.7) for $\tau = 2$ by Hölder's inequality. Here we use that the relevant expressions are supported in Ω_j^* and $|\Omega_j^*| \le C_d|\Omega_j|$.

For the long range estimate, that is,

(3.8)
$$\Big\| \sum_{s,j} \sum_{\ell\ge 0} \sum_{\substack{W\in\mathcal{W}_j \\ L(W)=\ell-s}} \psi_s * K^{lg}_{s,l} * A_{s,W,j}\Big\|_p \le C_\epsilon \sup_s \left(\|H_s\|_{L^1[(1+|x|)^\epsilon dx]}\right) \|Sf\|_p,$$

for $\epsilon > 0$ and $1 < p \le 2$, we use

$$\| \sum_s \psi_s * h_s\|_\tau^\tau \le C\big(\sum_s \|h_s\|_\tau^\tau\big)^{1/\tau}, \quad 1 \le \tau \le 2. \tag{3.9}$$

And so by Minkowski's inequality and (3.9), the left hand side of (3.8) is dominated by

$$\sum_{\ell\ge 0} \Big(\sum_s \Big\| \sum_j \sum_{\substack{W\in\mathcal{W}_j \\ L(W)=\ell-s}} K^{lg}_{s,\ell} * A_{s,W,j}\Big\|_p^p\Big)^{1/p}. \tag{3.10}$$

For fixed ℓ, s, by (3.5), if $\epsilon > 0$ then

(3.11)
$$\Big\| \sum_j \sum_{\substack{W\in\mathcal{W}_j \\ L(W)=\ell-s}} K^{lg}_{s,\ell} * A_{s,W,j}\Big\|_p \le C2^{-\epsilon\ell} \|H_s\|_{L^1[(1+|x|)^\epsilon dx]} \Big\| \sum_j \sum_{\substack{W\in\mathcal{W}_j \\ L(W)=\ell-s}} A_{s,W,j}\Big\|_p.$$

For fixed ℓ, s, the functions $\sum_{\substack{W\in\mathcal{W}_j\\ L(W)=\ell-s}} A_{s,W,j}$, $j\in\mathbb{Z}$, live on disjoint sets (since the sets $\bigcup_{Q\in\mathcal{Q}_j^s} Q$, $j\in\mathbb{Z}$, are disjoint). Hence

$$\Big\|\sum_j \sum_{\substack{W\in\mathcal{W}_j\\ L(W)=\ell-s}} A_{s,W,j}\Big\|_p^p = \sum_j \Big\| \sum_{\substack{W\in\mathcal{W}_j\\ L(W)=\ell-s}} A_{s,W,j}\Big\|_p^p = \sum_j \sum_{\substack{W\in\mathcal{W}_j\\ L(W)=\ell-s}} \|A_{s,W,j}\|_p^p.$$

Therefore by (3.2)

(3.12)

$$\begin{aligned}\sum_s \Big\|\sum_j \sum_{\substack{W\in\mathcal{W}_j\\ L(W)=\ell-s}} A_{s,W,j}\Big\|_p^p &\le \sum_j\sum_s \sum_{\substack{W\in\mathcal{W}_j\\ L(W)=\ell-s}} \|A_{s,W,j}\|_p^p\\ &\le \sum_j \sum_{W\in\mathcal{W}_j} \|A_{\ell-L(W),W,j}\|_p^p\\ &\le \sum_j \sum_{W\in\mathcal{W}_j} \mathrm{meas}(W)\|A_{\ell-L(W),W,j}\|_\infty^p\\ &\le C\sum_j 2^{jp}\mathrm{meas}(\Omega_j) \le C\|Sf\|_p^p \le C\|f\|_p^p.\end{aligned}$$

By (3.11) and (3.12), it follows that the expression (3.10) is

$$\le \sum_{\ell>0} C2^{-\ell\epsilon} \sup_s \left(\|H_s\|_{L^1[(1+|x|)^\epsilon dx]}\right)\|f\|_p \le C_\epsilon \sup_s \left(\|H_s\|_{L^1[(1+|x|)^\epsilon dx]}\right)\|f\|_p,$$

if $\epsilon>0$ and $1<p\le 2$. And this completes the proof.

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