Honam Mathematical J. **35** (2013), No. 3, pp. 395–406
http://dx.doi.org/10.5831/HMJ.2013.35.3.395

CLASSIFICATION OF GENERALIZED PAPER FOLDING SEQUENCES

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Abstract. Generalized paper folding sequences X_p^n and $(X_pY_q)^n$ where $X, Y \in \{R, L, U, D\}$, and $n, p, q \in \mathbb{N}$ with $p, q \geq 2$ are classified in this paper. We show that all generalized paper folding sequences X_p^n are classified into one type if we classify generalized paper folding sequences along with the numbers of downwards and upwards. In addition, we investigate the numbers of downwards and upwards in $(X_pY_q)^n$ and prove that all generalized paper folding sequences $(X_pY_q)^n$ are classified into two types.

1. Introduction and Preliminaries

Recently, paper folding sequences have been investigated extensively by many researchers[1-8]. Davis and Knuth [4] introduced a paper folding sequence and they used 0 for a crease that makes the paper upward and 1 for a crease that makes the paper downward. Bates, Bunder and Tognetti [2] investigated the structure of mirroring and interleaving in the paper folding sequence and Bercoff [3] showed the effective construction of 2-uniform tag systems related to paper folding sequences. Lee, Kim and Choi [7] explained the trace of generalized paper folding sequences using $(0, 1)$ codes and $(0, 1)$ matrices but they didn't obtain the exact numbers of upwards and downwards of generalized paper folding sequences.

In this paper, we adapt the notations in [8]. We use R when we fold a sheet of paper left over right, L when we fold a sheet of paper right over left, U when we fold a sheet of paper bottom over top and D when we fold a sheet of paper top over bottom. When we fold a sheet of paper

Received May 22, 2013. Accepted May 31, 2013.
2010 Mathematics Subject Classification. 97A20, 68R99.
Key words and phrases. classification, paper folding sequences, upward, downward.
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left over right and rotate it 180^0 counterclockwise, the creases are the same as those of the paper folding right over left.

Let $p, q, n \in \mathbb{N}$ with $p, q \geq 2$. If we fold a sheet of paper in p, left over right, n times, we get a generalized paper folding sequence and denote it by R_p^n. We define L_p^n, U_p^n and D_p^n similarly.

If we fold a sheet of paper in p left over right and then fold the result in q left over right, we get a paper folding sequence and denote it by R_pR_q. If we iterate R_pR_q process n times, then we get a generalized paper folding sequence and denote it by $(R_pR_q)^n$. Generalized paper folding sequences $(X_pY_q)^n$ where $X, Y \in \{R, L, U, D\}$ are defined similarly.

The letters X and Y may be different at different occurrences.

Example 1.1. *Some examples of generalized paper folding sequences are given as follows :*

(1) $(R_2L_3)^2$: (0 0 0 1 1 0 0 0 1 1 1 0 0 0 0 1 1 1 0 0 1 1 1 1 0 0 0 1 1 1 0 0 1 1 1)

(2) $(R_2U_3)^2$:
$$\begin{pmatrix}
 & 0 & & 1 & & 1 & \\
0 & & 1 & & 0 & & 1 \\
 & 0 & & 1 & & 1 & \\
0 & & 1 & & 0 & & 1 \\
 & 0 & & 1 & & 1 & \\
0 & & 0 & & 1 & & 1 \\
 & 1 & & 1 & & 0 & \\
1 & & 0 & & 1 & & 0 \\
 & 1 & & 1 & & 0 & \\
1 & & 0 & & 1 & & 0 \\
 & 1 & & 1 & & 0 & \\
0 & & 0 & & 1 & & 1 \\
 & 0 & & 1 & & 1 & \\
0 & & 1 & & 0 & & 1 \\
 & 0 & & 1 & & 1 & \\
0 & & 1 & & 0 & & 1 \\
 & 0 & & 1 & & 1 &
\end{pmatrix}$$

For a paper folding sequence X, we define X^c the paper folding sequence obtained by reversing the order and swapping 0s and 1s in X.

We define $|X|_0$, $|X|_1$ and $|X|$ the number of all 0s in X, all 1s in X and the number of all 0s and 1s in X, respectively. Lemma 1.2 (see [8]) is obtained by the definitions of $|X|, |X|_0, |X|_1$ and X^c.

Lemma 1.2. *Let X be a paper folding sequence. Then we have*
(1) $|X| = |X|_0 + |X|_1$.
(2) $|X^c|_0 = |X|_1$.
(3) $|X^c|_1 = |X|_0$.
(4) $|X^c| = |X|$.

For the classification of generalized paper folding sequences, we define the following.

Definition 1.3. *Let X and Y be paper folding sequences. If $|X|_0 = |Y|_0$ and $|X|_1 = |Y|_1$, then we say that X and Y are the same type and denote it by $X \equiv Y$.*

For generalized paper folding sequences X_p^n where $X \in \{R, L, U, D\}$, we can easily classify them.

Theorem 1.4. *Let $p, n \in \mathbb{N}$ with $p \geq 2$. Then*

$$R_p^n \equiv U_p^n \equiv L_p^n \equiv D_p^n. \tag{1.1}$$

Proof. If R_p^n is rotated $90^0, 180^0$ and 270^0 counterclockwise, we obtain U_p^n, L_p^n and D_p^n, respectively. This implies that the numbers of 0s and 1s in R_p^n, U_p^n, L_p^n and D_p^n are the same. Thus $R_p^n \equiv U_p^n \equiv L_p^n \equiv D_p^n$. □

Theorem 1.4 implies that R_p^n, U_p^n, L_p^n and D_p^n are classified into one type.

2. Classification of $(X_pY_q)^n$ where $X, Y \in \{R, L, U, D\}$

Let $p, q, n \in \mathbb{N}$ with $p, q \geq 2$ and let $X, Y \in \{R, L, U, D\}$. Then there are 16 generalized paper folding sequences $(X_pY_q)^n$ as follows :

$$\begin{aligned}
&(R_pR_q)^n, (R_pL_q)^n, (R_pU_q)^n, (R_pD_q)^n,\\
&(L_pR_q)^n, (L_pL_q)^n, (L_pU_q)^n, (L_pD_q)^n,\\
&(U_pR_q)^n, (U_pL_q)^n, (U_pU_q)^n, (U_pD_q)^n,\\
&(D_pR_q)^n, (D_pL_q)^n, (D_pU_q)^n, (D_pD_q)^n.
\end{aligned} \tag{2.1}$$

First, we show that these 16 generalized paper folding sequences are classified into four types.

Theorem 2.1. *Let $p, q, n \in \mathbb{N}$ with $p, q \geq 2$. Then*
(1) $(R_pR_q)^n \equiv (U_pU_q)^n \equiv (L_pL_q)^n \equiv (D_pD_q)^n$.
(2) $(R_pL_q)^n \equiv (U_pD_q)^n \equiv (L_pR_q)^n \equiv (D_pU_q)^n$.
(3) $(R_pU_q)^n \equiv (U_pL_q)^n \equiv (L_pD_q)^n \equiv (D_pR_q)^n$.
(4) $(R_pD_q)^n \equiv (U_pR_q)^n \equiv (L_pU_q)^n \equiv (D_pL_q)^n$.

Proof. If $(R_pR_q)^n$ is rotated $90^0, 180^0$ and 270^0 counterclockwise, we obtain $(U_pU_q)^n, (L_pL_q)^n$ and $(D_pD_q)^n$, respectively. Since a rotation of a paper folding sequence doesn't change the numbers of 0s and 1s, we have $(R_pR_q)^n \equiv (U_pU_q)^n \equiv (L_pL_q)^n \equiv (D_pD_q)^n$. Thus we prove (1). The proofs of (2), (3) and (4) are same as those of (1). □

The next theorem is a modification of Theorem 2.1 in [8]. Note that there are $p-1$ 1s in R_p, and all 1s in R_p are not divided at all in R_pX if X is a paper folding sequence related only to R or L. Thus we have the following theorem.

Theorem 2.2. *If X is a paper folding sequence related only to R or L, then*

$$R_pX = \begin{cases} (X^c 1 X 1 X^c 1 X 1 \cdots 1 X^c 1 X) & \text{if } p \text{ is even} \\ (X 1 X^c 1 X 1 X^c 1 \cdots 1 X^c 1 X) & \text{if } p \text{ is odd,} \end{cases} \tag{2.2}$$

and

$$L_pX = \begin{cases} (X 1 X^c 1 X 1 X^c 1 \cdots 1 X 1 X^c) & \text{if } p \text{ is even} \\ (X 1 X^c 1 X 1 X^c 1 \cdots 1 X^c 1 X) & \text{if } p \text{ is odd.} \end{cases} \tag{2.3}$$

In (2.2) and (2.3), X and X^c appear $\frac{p}{2}$ times and $\frac{p}{2}$ times, respectively, when p is even. In addition, X and X^c appear $\frac{p+1}{2}$ times and $\frac{p-1}{2}$ times, respectively, when p is odd.
From Theorem 2.2, we get the following result.

Lemma 2.3. *If X and Y are paper folding sequences related only to R or L, we have*

$$R_pX \equiv L_pX \tag{2.4}$$

and

$$R_pX \equiv R_pY \;\text{ if }\; X \equiv Y. \tag{2.5}$$

Proof. By Theorem 2.2, we have

$$R_pX = (X\,1\,X^c \cdots X^c\,1\,X) = L_pX \tag{2.6}$$

when p is odd. Since X and X^c appear $\frac{p+1}{2}$ times and $\frac{p-1}{2}$ times, respectively, and 1 appears $p-1$ times in R_pX and L_pX, we have $R_pX \equiv L_pX$ when p is odd. In addition,

$$R_pX = (X^c\,1\,X \cdots X^c\,1\,X) \text{ and } L_pX = (X\,1\,X^c \cdots X\,1\,X^c) \tag{2.7}$$

when p is even. Since X and X^c appear $\frac{p}{2}$ times and $\frac{p}{2}$ times, respectively, 1 appears $p-1$ times in R_pX and L_pX, we have $R_pX \equiv L_pX$ when p is even. Thus we prove (2.4).
Assume that $X \equiv Y$. Then $|X|_0 = |Y|_0$ and $|X|_1 = |Y|_1$.
If p is even, we get

$$\begin{aligned}
|R_pX|_0 &= |X^c\,1\,X \cdots X^c\,1\,X|_0 \\
&= |X^c|_0 + |1|_0 + |X|_0 + \cdots + |X^c|_0 + |1|_0 + |X|_0 \\
&= |X|_1 + |1|_0 + |X|_0 + \cdots + |X|_1 + |1|_0 + |X|_0 \\
&= |Y|_1 + |1|_0 + |Y|_0 + \cdots + |Y|_1 + |1|_0 + |Y|_0 \\
&= |Y^c|_0 + |1|_0 + |Y|_0 + \cdots + |Y^c|_0 + |1|_0 + |Y|_0 \\
&= |Y^c\,1\,Y \cdots Y^c\,1\,Y|_0 \\
&= |R_pY|_0,
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
|R_pX|_1 &= |X^c\,1\,X \cdots X^c\,1\,X|_1 \\
&= |X^c|_1 + |1|_1 + |X|_1 + \cdots + |X^c|_1 + |1|_1 + |X|_1 \\
&= |X|_0 + |1|_1 + |X|_1 + \cdots + |X|_0 + |1|_1 + |X|_1 \\
&= |Y|_0 + |1|_1 + |Y|_1 + \cdots + |Y|_0 + |1|_1 + |Y|_1 \\
&= |Y^c|_1 + |1|_1 + |Y|_1 + \cdots + |Y^c|_1 + |1|_1 + |Y|_1 \\
&= |Y^c\,1\,Y \cdots Y^c\,1\,Y|_1 \\
&= |R_pY|_1.
\end{aligned} \tag{2.9}$$

By (2.8) and (2.9), we have $R_pX \equiv R_pY$ when p is even.
Similarly, we get $R_pX \equiv R_pY$ when p is odd. Thus we prove (2.5). □

As a result of Lemma 2.3, we have

$$R_pX \equiv L_pY \text{ if } X \equiv Y, \tag{2.10}$$

where X and Y are paper folding sequences related only to R or L.
From Lemma 2.3, we obtain the following result.

Theorem 2.4. *Let $p, q, n \in \mathbb{N}$ with $p, q \geq 2$. Then*

$$(R_pR_q)^n \equiv (R_pL_q)^n. \tag{2.11}$$

Proof. We use the mathematical induction.
When $n = 1$, $R_pR_q \equiv R_pL_q$ by (2.5) since $R_q \equiv L_q$.
Assume that $(R_pR_q)^n \equiv (R_pL_q)^n$ for $n = k$.
We set $Z = (R_pR_q)^k$ and $W = (R_pL_q)^k$. By the induction hypothesis, we have $Z \equiv W$. Hence we get $R_qZ \equiv L_qW$ by (2.10) since Z and W are paper folding sequences related only to R or L. Thus

$$R_p(R_qZ) \equiv R_p(L_qW) \tag{2.12}$$

by (2.5). Since $(R_pR_q)^{k+1} = R_p(R_qZ)$ and $(R_pL_q)^{k+1} = R_p(L_qW)$, we have, by (2.12),

$$(R_pR_q)^{k+1} \equiv (R_pL_q)^{k+1}. \tag{2.13}$$

Therefore $(R_pR_q)^n \equiv (R_pL_q)^n$ for all $n \in \mathbb{N}$. □

By Theorem 2.1 and Theorem 2.4, we obtain the following.

Theorem 2.5. *Let $p, q, n \in \mathbb{N}$ with $p, q \geq 2$. Then*

$$\begin{aligned} & (R_pR_q)^n \equiv (U_pU_q)^n \equiv (L_pL_q)^n \equiv (D_pD_q)^n \\ \equiv\ & (R_pL_q)^n \equiv (U_pD_q)^n \equiv (L_pR_q)^n \equiv (D_pU_q)^n. \end{aligned} \tag{2.14}$$

Let $p \in \mathbb{N}$ with $p \geq 2$. If X is a paper folding sequence related only to R or L, then each 1 in R_p is not divided by X in R_pX. But, if Y is a paper folding sequence including U or D, each 1 in R_p is divided into some 1s in R_pY and so we denote it by 1_Y. Then $|1_Y|_1 > 1 = |1|_1$ and $|1_Y|_0 = 0 = |1|_0$. Now we obtain the following result from Theorem 2.2.

Theorem 2.6. *Let $p \in \mathbb{N}$ with $p \geq 2$.*
(1) If X is a paper folding sequence related only to R or L, then

$$R_pX = \begin{cases} (X^c\,1\,X\,1\,X^c\,1\,X\,\cdots\,1\,X^c\,1\,X) & \text{if } p \text{ is even} \\ (X\,1\,X^c\,1\,X\,1\,X^c\,\cdots\,1\,X^c\,1\,X) & \text{if } p \text{ is odd.} \end{cases} \tag{2.15}$$

(2) If Y is a paper folding sequence including U or D, then

$$R_pY = \begin{cases} (Y^c\,1_Y\,Y\,1_Y\,Y^c\,1_Y\,Y\,\cdots\,1_Y\,Y^c\,1_Y\,Y) & \text{if } p \text{ is even} \\ (Y\,1_Y\,Y^c\,1_Y\,Y\,1_Y\,Y^c\,\cdots\,1_Y\,Y^c\,1_Y\,Y) & \text{if } p \text{ is odd.} \end{cases} \tag{2.16}$$

Note that 1 of 1_Y in (2.16) is a downward in R_p. We need to prove the following lemmas in order to obtain the conclusion. The proof of Lemma 2.7 is similar to that of Lemma 2.3.

Lemma 2.7. *Let $p \in \mathbb{N}$ with $p \geq 2$ and let X be a paper folding sequence including U or D. Then we have*

$$R_pX \equiv L_pX. \tag{2.17}$$

Proof. Note that

$$R_pX = (X\,1_X\,X^c \cdots X^c\,1_X\,X) = L_pX \tag{2.18}$$

when p is odd. Since X and X^c appear $\frac{p+1}{2}$ times and $\frac{p-1}{2}$ times, respectively, and 1_X appears $p-1$ times in R_pX and L_pX, we have $R_pX \equiv L_pX$ when p is odd. In addition,

$$R_pX = (X^c\,1_X\,X \cdots 1_X\,X) \text{ and } L_pX = (X\,1_X\,X^c \cdots 1_X\,X^c) \tag{2.19}$$

when p is even. Since X and X^c appear $\frac{p}{2}$ times and $\frac{p}{2}$ times, respectively, 1_X appears $p-1$ times in R_pX and L_pX, we have $R_pX \equiv L_pX$ when p is even. Thus we prove (2.17). □

Lemma 2.8. *Let $p \in \mathbb{N}$ with $p \geq 2$ and let X and Y be paper folding sequences including U or D. Then we have*

$$R_pX \equiv R_pY \text{ if } X \equiv Y \text{ and } 1_X \equiv 1_Y, \tag{2.20}$$

where 1 is a downward in R_p.

Proof. By the hypothesis, we have $|X|_0 = |Y|_0$, $|X|_1 = |Y|_1$, $|1_X|_0 = |1_Y|_0$ and $|1_X|_1 = |1_Y|_1$. If p is even, we get

$$\begin{aligned} |R_pX|_0 &= |X^c\,1_X\,X \cdots X^c\,1_X\,X|_0 \\ &= |X^c|_0 + |1_X|_0 + |X|_0 + \cdots + |X^c|_0 + |1_X|_0 + |X|_0 \\ &= |X|_1 + |1_X|_0 + |X|_0 + \cdots + |X|_1 + |1_X|_0 + |X|_0 \\ &= |Y|_1 + |1_Y|_0 + |Y|_0 + \cdots + |Y|_1 + |1_Y|_0 + |Y|_0 \\ &= |Y^c|_0 + |1_Y|_0 + |Y|_0 + \cdots + |Y^c|_0 + |1_Y|_0 + |Y|_0 \\ &= |Y^c\,1_Y\,Y \cdots Y^c\,1_Y\,Y|_0 \\ &= |R_pY|_0, \end{aligned} \tag{2.21}$$

and

$$
\begin{aligned}
|R_pX|_1 &= |X^c 1_X X \cdots X^c 1_X X|_1 \\
&= |X^c|_1 + |1_X|_1 + |X|_1 + \cdots + |X^c|_1 + |1_X|_1 + |X|_1 \\
&= |X|_0 + |1_X|_1 + |X|_1 + \cdots + |X|_0 + |1_X|_1 + |X|_1 \\
&= |Y|_0 + |1_Y|_1 + |Y|_1 + \cdots + |Y|_0 + |1_Y|_1 + |Y|_1 \qquad (2.22)\\
&= |Y^c|_1 + |1_Y|_1 + |Y|_1 + \cdots + |Y^c|_1 + |1_Y|_1 + |Y|_1 \\
&= |Y^c 1_Y Y \cdots Y^c 1_Y Y|_1 \\
&= |R_pY|_1,
\end{aligned}
$$

By (2.21) and (2.22), we have $R_pX \equiv R_pY$ when p is even.
Similarly, we get $R_pX \equiv R_pY$ when p is odd. Thus we prove (2.20). □

As a result of Lemma 2.7 and Lemma 2.8, we have

$$R_pX \equiv L_pY \text{ if } X \equiv Y \text{ and } 1_X \equiv 1_Y, \tag{2.23}$$

where X and Y are paper folding sequences including U or D, and 1 is a downward in R_p or L_p.

From Lemma 2.7, Lemma 2.8 and (2.23), we have the following.

Corollary 2.9. *Let $q \in \mathbb{N}$ with $q \geq 2$ and let X and Y be paper folding sequences including R or L. Then we have*

$$U_qX \equiv D_qY \text{ if } X \equiv Y \text{ and } 1_X \equiv 1_Y, \tag{2.24}$$

where 1 is a downward in U_q or D_q.

We prove the next theorem using Lemma 2.8 and Corollary 2.9.

Theorem 2.10. *Let $p, q, n \in \mathbb{N}$ with $p, q \geq 2$. Then*

$$(R_pU_q)^n \equiv (R_pD_q)^n. \tag{2.25}$$

Proof. We use the mathematical induction.
When $n = 1$, $R_pU_q \equiv R_pD_q$ by Lemma 2.8, since $U_q \equiv D_q$ and $1_{U_q} \equiv 1_{D_q}$ where 1 is a downward in R_p.
Assume that $(R_pU_q)^n \equiv (R_pD_q)^n$ for $n = k$.
We set $Z = (R_pU_q)^k$ and $W = (R_pD_q)^k$. By the induction hypothesis, we have $Z \equiv W$. Note that 1 in U_q or D_q is divided by p $1s$ by R_p although it is not divided by another U_q or D_q. Thus, for any $k \in \mathbb{N}$,

$$|1_Z|_1 = |1_{(R_pU_q)^k}|_1 = p^k = |1_{(R_pD_q)^k}|_1 = |1_W|_1 \tag{2.26}$$

and

$$|1_Z|_0 = 0 = |1_W|_0, \tag{2.27}$$

and so $1_Z \equiv 1_W$ by (2.26) and (2.27) when 1 is a downward in U_q or D_q. Since Z and W are paper folding sequences including R or L and $1_Z \equiv 1_W$, we get $U_qZ \equiv D_qW$ by Corollary 2.9. Similarly, if 1 is a downward in R_p, we have

$$|1_{U_qZ}|_1 = q^{k+1} = |1_{D_qW}|_1 \text{ and } |1_{U_qZ}|_0 = 0 = |1_{D_qW}|_0 \tag{2.28}$$

and hence $1_{U_qZ} \equiv 1_{D_qW}$. By Lemma 2.8, we get

$$R_p(U_qZ) \equiv R_p(D_qW). \tag{2.29}$$

Since $(R_pU_q)^{k+1} = R_p(U_qZ)$ and $(R_pD_q)^{k+1} = R_p(D_qW)$, we have, by (2.29),

$$(R_pU_q)^{k+1} \equiv (R_pD_q)^{k+1}. \tag{2.30}$$

Therefore $(R_pU_q)^n \equiv (R_pD_q)^n$ for all $n \in \mathbb{N}$. □

By Lemma 2.8, Corollary 2.9 and Theorem 2.10, we obtain the following result.

Theorem 2.11. *Let $p, q, n \in \mathbb{N}$ with $p, q \geq 2$. Then*

$$\begin{aligned} &(R_pU_q)^n \equiv (U_pL_q)^n \equiv (L_pD_q)^n \equiv (D_pR_q)^n \\ \equiv\ &(R_pD_q)^n \equiv (U_pR_q)^n \equiv (L_pU_q)^n \equiv (D_pL_q)^n. \end{aligned} \tag{2.31}$$

Theorem 2.5 and Theorem 2.11 show that 16 generalized paper folding sequences $(X_pY_q)^n$ where $X, Y \in \{R, L, U, D\}$ can be classified into two types at most. Now, we prove that 16 generalized paper folding sequences can't be classified into one type.

Theorem 2.12. *Let $p, q \in \mathbb{N}$ with $p, q \geq 2$. Then*

$$(R_pR_q)^n \not\equiv (R_pU_q)^n \tag{2.32}$$

for any $n \in \mathbb{N}$.

Proof. We prove $|(R_pR_q)^n| < |(R_pU_q)^n|$ for all $n \in \mathbb{N}$ by the mathematical induction.
Note that $|R_q^c| = |R_q|$, $|U_q^c| = |U_q|$ and $|R_q| = |U_q|$. In addition, $|1| = 1 < q = |1_{U_q}|$ if 1 is a downward in R_p. By Theorem 2.2 and Theorem 2.6, we have

$$R_pR_q = (R_q^c\, 1\, R_q \cdots 1\, R_q) \text{ and } R_pU_q = (U_q^c\, 1_{U_q}\, U_q \cdots 1_{U_q}\, U_q) \tag{2.33}$$

if p is even, and

$$(2.34)\quad R_pR_q = (R_q\,1\,R_q^c \cdots 1\,R_q) \text{ and } R_pU_q = (U_q\,1_{U_q}\,U_q^c \cdots 1_{U_q}\,U_q)$$

if p is odd. Hence

$$\begin{aligned} |R_pR_q| &= |R_q^c| + |1| + |R_q| + |1| + \cdots + |1| + |R_q^c| + |1| + |R_q| \\ &= |U_q^c| + |1| + |U_q| + |1| + \cdots + |1| + |U_q^c| + |1| + |U_q| \\ &< |U_q^c| + |1_{U_q}| + |U_q| + |1_{U_q}| + \cdots + |U_q^c| + |1_{U_q}| + |U_q| \qquad (2.35) \\ &= |R_pU_q| \end{aligned}$$

if p is even, and

$$\begin{aligned} |R_pR_q| &= |R_q| + |1| + |R_q^c| + |1| + \cdots + |1| + |R_q^c| + |1| + |R_q| \\ &= |U_q| + |1| + |U_q^c| + |1| + \cdots + |1| + |U_q^c| + |1| + |U_q| \\ &< |U_q| + |1_{U_q}| + |U_q^c| + |1_{U_q}| + \cdots + |U_q^c| + |1_{U_q}| + |U_q| \qquad (2.36) \\ &= |R_pU_q| \end{aligned}$$

if p is odd. Note that 1 and 1 of 1_{U_q} in (2.35) and (2.36) are downwards in R_p. Thus $|R_pR_q| < |R_pU_q|$.
Assume that $|(R_pR_q)^n| < |(R_pU_q)^n|$ for $n = k$.
We set $Z = (R_pR_q)^k$ and $W = (R_pU_q)^k$. Then Z is a paper folding sequence related only to R, and W is a paper folding sequence including R and U. In addition,

$$(2.37)\qquad (R_pR_q)^{k+1} = R_p(R_q(R_pR_q)^k) = R_p(R_qZ)$$

and

$$(2.38)\qquad (R_pU_q)^{k+1} = R_p(U_q(R_pU_q)^k) = R_p(U_qW).$$

By Lemma 1.2 and the induction hypothesis, we have $|Z^c| = |Z| < |W| = |W^c|$. Therefore

$$\begin{aligned} |R_qZ| &= |Z^c| + |1| + |Z| + |1| + \cdots + |1| + |Z| \\ &< |W^c| + |1| + |W| + |1| + \cdots + |1| + |W| \\ &< |W^c| + |1_W| + |W| + |1_W| + \cdots + |1_W| + |W| \qquad (2.39) \\ &= |U_qW| \end{aligned}$$

if q is even, and

$$
\begin{aligned}
|R_qZ| &= |Z| + |1| + |Z^c| + |1| + \cdots + |1| + |Z| \\
&< |W| + |1| + |W^c| + |1| + \cdots + |1| + |W| \\
&< |W| + |1_W| + |W^c| + |1_W| + \cdots + |1_W| + |W| \qquad (2.40)\\
&= |U_qW|
\end{aligned}
$$

if q is odd. Note that 1 in (2.39) and (2.40) is a downward in R_q but 1 of 1_W in (2.39) and (2.40) is a downward in U_q.
Similarly, we have

$$
\begin{aligned}
|R_p(R_qZ)| &= |(R_qZ)^c| + |1| + |R_qZ| + |1| + \cdots + |1| + |R_qZ| \\
&< |(U_qW)^c| + |1| + |U_qW| + |1| + \cdots + |1| + |U_qW| \\
&< |(U_qW)^c| + |1_{U_qW}| + |U_qW| + |1_{U_qW}| + \cdots + |U_qW| \qquad (2.41)\\
&= |R_p(U_qW)|
\end{aligned}
$$

if p is even, and

$$
\begin{aligned}
|R_p(R_qZ)| &= |R_qZ| + |1| + |(R_qZ)^c| + |1| + \cdots + |1| + |R_qZ| \\
&< |U_qW| + |1| + |(U_qW)^c| + |1| + \cdots + |1| + |U_qW| \\
&< |U_qW| + |1_{U_qW}| + |(U_qW)^c| + |1_{U_qW}| + \cdots + |U_qW| \qquad (2.42)\\
&= |R_p(U_qW)|
\end{aligned}
$$

if p is odd. Note that 1 and 1 of 1_{U_qW} in (2.41) and (2.42) are downwards in R_p. By (2.41) and (2.42), we have

$$
|(R_pR_q)^{k+1}| = |R_p(R_qZ)| < |R_p(U_qW)| = |(R_pU_q)^{k+1}|. \qquad (2.43)
$$

Therefore $|(R_pR_q)^n| < |(R_pU_q)^n|$ for all $n \in \mathbb{N}$ and this implies that $(R_pR_q)^n \not\equiv (R_pU_q)^n$ for all $n \in \mathbb{N}$. □

As we mentioned before, Theorem 2.12 shows that 16 generalized paper folding sequences $(X_pY_q)^n$ where $X, Y \in \{R, L, U, D\}$ are classified into two types exactly.

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