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A NOTE ON FULLY IDEMPOTENT S-ACTS

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Abstract. In this article we introduce the notion of fully idempotent acts and by this we give a characterization of commutative monoids over which all cyclic right acts are injective.

1. Introduction and Preliminaries

Throughout this paper, S will denote a monoid. A *right S-act* A_S (or A) is a non-empty set A together with a function $\mu : A\times S \longrightarrow A$, called the *action* of S, which is denoted by $\mu(a,s) = as$ such that $a.1 = a$ and $a(st) = (as)t$ for each $a \in A$ and $s,t \in S$. A *subact* B of an S-act A is a subset of A such that $bs \in B$, for all $b \in B$, $s \in S$. A subact of the S-act S is said to be a *right ideal* of the monoid S. A right S-act A is called *simple* if it does not contain a proper subact.

For a right S-act A, an element θ in A is called a *zero* of A if $\theta s = \theta$ for each $s \in S$. Moreover, $\Theta = \{\theta\}$ is the one-element right S-act. A right S-act A is called Θ-*simple* if it contains no subacts other than A and one-element subact.

Let A and B be two right S-acts. A mapping $f : A \longrightarrow B$ is called an *S-homomorphism* if $f(as) = f(a)s$ for all $a \in A, s \in S$. The set all of S-homomorphims from A to B is denoted by $Hom(A,B)$. If $f : A \longrightarrow B$ is an S-homomorphism, then f is called a *monomorphism* (an *epimorphism*) if f is left (right) cancellable, i.e., if C is a right S-act and $h,k : C \longrightarrow A$ ($h,k : B \longrightarrow C$) are S-homomorphisms, then the equality $fk = fh$ ($kf = hf$) implies that $k = h$. If for S-homomorphisms $f : A \longrightarrow B$ and $g : B \longrightarrow A$, $fg = 1_{B_S}$, then f is called a *retraction* and B is called a *retract* of A.

A subact B of an S-act A is called *essential* in A if any S-homomorphism $g : A \longrightarrow C$ such that $g|_B$ is a monomorphism is itself a monomorphism.

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In this case A is called an *essential extension* of B. It is well-known that an S-act A is said to be *injective* if for every monomorphism $h : B \longrightarrow C$ and every homomorphism $f : B \longrightarrow A$, there exists a homomorphism $g : C \longrightarrow A$ such that $gh = f$. Moreover, an extension A of an S-act B is called an *injective envelope* of B if it is an essential extension of B which is also injective. Note that injective envelopes of S-acts always exist (see Corollary 3.1.23 of [1]).

Recall that a monoid S is said to be *right self-injective* if the right S-act S_S is injective (see Definition 4.1.7 of [1])

A right S-act P is said to be *projective* if for any epimorphism $f : A \longrightarrow B$ and any homomorphism $g : P \longrightarrow B$, there exists a homomorphism $h : P \longrightarrow A$ such that $g=fh$. For more information about semigroups and acts see [1].

In this paper we introduce the notion of fully idempotent acts. This leads to the study of the relation between these kinds of acts and other classes of acts, such as injective and projective acts. A subact B of a right S-act A is called *idempotent* if $B = \bigcup_{f\in Hom(A,B)} f(B)$. Also, the right S-act A is called *fully idempotent* if every subact of A is an idempotent. We prove that the right S-act S_S is fully idempotent if and only if for every right ideal I of S, $I^2 = I$. Also, by the notion of fully idempotent acts we give a characterization of commutative monoids over which every cyclic right act is injective.

2. Fully idempotent acts

Definition 2.1. Let A be a right S-act and B be a subact of A. We say that B is an *idempotent* subact of A if $B = \bigcup_{f\in Hom(A,B)} f(B)$. Also, the right S-act A is called *fully idempotent* if every subact of A is an idempotent.

We show that the right S-act S_S is injective if and only if S_S is an idempotent subact of its injective envelope. Also, it is shown that over a commutative monoid S, all right S-acts are fully idempotent if and only if all cyclic right S-acts are injective. We need the following lemma, which is used frequently in the sequel.

Lemma 2.2. *Suppose S is a monoid and A is a right S-act. Then the following statements are equivalent:*

(i) *A is fully idempotent.*

(ii) *For every $a \in A$, aS is an idempotent subact of A.*

(iii) *For every $a \in A$, there exits a homomorphism $\varphi : A \longrightarrow aS$ such that $a = \varphi(at)$ for some $t \in S$.*

Proof. (i) $\Longrightarrow$ (ii) It is clear.

(ii) $\Longrightarrow$ (iii) Since for every $a \in A$, aS is an idempotent subact of A, the result follows by Definition 2.1.

(iii) $\Longrightarrow$ (i) If B is a subact of A, then $B = \bigcup_{b\in B} bS$ and by assumption $B = \bigcup_{f\in Hom(A,B)} f(B)$. Thus B is an idempotent subact of A and A is fully idempotent. □

The following lemma includes some general properties of fully idempotent right S-acts.

Lemma 2.3. *The following statements hold over a monoid S.*

(i) *If S contains a left zero and $\{A_i : i \in I\}$ is a family of fully idempotent right S-acts, then $\coprod_{i\in I} A_i$ is fully idempotent.*

(ii) *Every subact of a fully idempotent right S-act is fully idempotent.*

(iii) *A retract of a fully idempotent right S-act is fully idempotent.*

Proof. (i) The proof is easy and will be omitted.

(ii) Suppose B is a subact of a fully idempotent S-act A. Since A is fully idempotent, by Lemma 2.2, for every $b \in B$, there exists a homomorphism $\varphi : A \longrightarrow bS$ such that $a = \varphi(bt)$ for some $t \in S$. Now if we consider the restriction of φ to B, i.e. $\varphi|_B$, then B would be fully idempotent by Lemma 2.2.

(iii) Note that a retract of a right S-act A is isomorphic to a subact of A and thus the result follows by part (ii). □

By the following example we show that fully idempotent acts are not preserved under product and coproduct.

Example 2.4. (i) Let $S = (\mathbb{N}, .)$ and P be the collection of prime numbers. For every $p \in P$, $(\mathbb{Z}_p, +)$, the group of congruence classes modulo p, is a right S-act with multiplication $\overline{a}n = \overline{an}$ for every $\overline{a} \in \mathbb{Z}_p$ and every $n \in \mathbb{N}$. Clearly for every $p \in P$, $(\mathbb{Z}_p, +)$ is a Θ-simple right S-act and so by Lemma 2.2, it is a fully idempotent right S-act. Now, if $A = \Pi_{p\in P}\mathbb{Z}_p$ is fully idempotent, then by Lemma 2.2, there exists an epimorphism $f : A \longrightarrow (\overline{1}, \overline{1}, ...)S$. This implies that $f(\overline{0}, \overline{0}, ...) = (\overline{1}, \overline{1}, ...)m$ for some $m \in \mathbb{N}$ and hence $(\overline{mn}, \overline{mn}, ..) = (\overline{m}, \overline{m}, ..)$ for each $n \in \mathbb{N}$. This means that $mn \cong m$ $(mod\ p)$ for every $p \in P$ and every $n \in \mathbb{N}$, which is a contradiction.

(ii) Let S be a regular monoid which has no left zero element and Θ be the one-element right S-act. Then S_S and Θ are fully idempotent right S-acts, but $S \sqcup \Theta$ is not fully idempotent. Otherwise, by Lemma 2.2, there exists a homomorphism $f : S \sqcup \Theta \longrightarrow S$, which implies that S has a left zero element.

Recall that a monoid S is called $left\ reversible$ if any two right ideals of S have non-empty intersection (see Definition 1.3.18 of [1]).

Proposition 2.5. *Suppose S is a monoid with a right zero and A is a projective right S-act. Then A is injective if and only if A is an idempotent subact of its injective envelope.*

Proof. The necessity is clear. Conversely, suppose $A = \coprod_{i\in I} a_i S$ and $E(A)$ is the injective envelope of A. Then by assumption for every $i \in I$, $a_i = f_i(a_j s_i)$ for some homomorphism $f_i : E_i = E(A) \longrightarrow A$ and some $s_i \in S$. Define $f : \coprod_{i\in I} E_i \longrightarrow A$ by $f(x_i) = f_i(x_i)$ for every $x_i \in E_i, i \in I$. Clearly f is an epimorphism and by projectivity of A, there exists a homomorphism $h : A \longrightarrow \coprod_{i\in I} E_i$ such that $f\ o\ h{=}1_A$. Thus A is a retract of $\coprod_{i\in I} E_i$. Since S is left reversible, by Proposition 3.1.13 of [1], $\coprod_{i\in I} E_i$ is injective and so A is injective. □

By the following theorem we give a new characterization of right self-injective monoids by the notion of fully idempotent acts.

Theorem 2.6. *Suppose S is a monoid with a zero and $E(S)$ is the injective envelope of S_S. The following statements are equivalent:*
(i) *S_S is injective.*
(ii) *S_S is an idempotent subact of $E(S)$.*
Moreover, if S is commutative, then the above statements are equivalent to:
(iii) *$E(S)$ is projective.*

Proof. (i) $\Longleftrightarrow$ (ii) It is clear by the previous proposition.

(iii) $\Longrightarrow$ (i) By Theorem 3.17.8 of [1], $E(S) \cong \coprod_{i\in I} e_i S$, where for every $i \in I$, e_i is an idempotent element of S. Since $S \subseteq E(S)$ and S is commutative ,we conclude that for some $i \in I$, $e_i = 1$. Thus if $\pi : E(S) \longrightarrow S$ is the canonical projection, then clearly π is an epimorphism and so S_S is an idempotent subact of $E(S)$. Now the result follows by Proposition 2.5.

(i) $\Longrightarrow$ (iii) Since the right S-act S_S is injective, $S = E(S)$ and hence $E(S)$ is projective. □

Recall that a right S-act A is regular if every cyclic right subact of A is projective (see Corollary 3.19.3 of [1]).

Proposition 2.7. *Suppose S is a commutative monoid and A is a fully idempotent right S-act. Then every cyclic subact of A is a retract of A. In particular every projective fully idempotent right S-act is regular.*

Proof. By Lemma 2.2, for every $a \in A$, there exists $f : A \longrightarrow aS$ such that $a = f(a)s$ for some $s \in S$. Define $g : aS \longrightarrow A$ by $g(a) = as$. Clearly, g is a well-defined homomorphism and $f\ o\ g$=1. Thus aS is a retract of A. Also, this implies that every projective fully idempotent right S-act is regular. □

Now we give a classification of monoids by the fully idempotent acts.

Lemma 2.8. *Over a monoid S the following statements are equivalent:*
(i) *S_S is fully idempotent.*
(ii) *For every right ideal I of S, $I^2 = I$.*
(iii) *For every principal right ideal I of S, $I^2 = I$.*
(iv) *For every $s \in S$, there exist $x, t \in S$ such that $s = sxst$.*

Proof. (i) $\Longrightarrow$ (ii) Suppose I is a right ideal of S and $i \in I$. By Lemma 2.2, $i = f(it)$ where $f : S_S \longrightarrow iS$ is an S-homomorphism and $t \in S$. Thus $i = f(it) = f(1)(it) \in I^2$ because $f(1), it \in I$. Hence $I \subseteq I^2$ and so $I = I^2$.

(ii) $\Longrightarrow$ (iii) It is clear.

(iii) $\Longrightarrow$ (iv) For every $s \in S, (sS)^2 = sS$ and so $s = sxst$ for some $s, t \in S$.

(iv) $\Longrightarrow$ (i) Suppose $s \in S$. By assumption, $s = sxst$ for some $s, t \in S$. Define $\varphi : S \longrightarrow sS$ by $\varphi(z) = sxz$ for every $z \in S$. Clearly φ is a well-defined homomorphism and $\varphi(st) = sxst = s$. Hence by Lemma 2.2, S_S is fully idempotent □

Theorem 2.9. *Over a commutative monoid S the following statements are equivalent:*
(i) *All free right S-acts are fully idempotent.*
(ii) *All finitely generated free right S-acts are fully idempotent.*
(iii) *S_S is fully idempotent.*
(iv) *All cyclic right S-acts are fully idempotent.*
(v) *S is a regular monoid.*

Proof. (i) $\Longrightarrow$ (ii) It is clear.

(ii) $\Longrightarrow$ (iii) Since the right S-act S_S is free, the result follows.

(iii) $\Longrightarrow$ (i) Suppose F is a free right S-act with the basis $\{a_i\}_{i \in I}$ and suppose $a = a_i s$ is an element of F for some $i \in I$ and $s \in S$. Define $f : F \longrightarrow aS$ by $f(a_j t) = a_i st$ for every $j \in I$ and every $t \in S$. Clearly f

is well-defined and $f(a_i) = a_i s$. By Lemma 2.8, $s = s^2 x$ for some $x \in S$. Thus $f(a_i sx) = a_i s^2 x = a_i s$ and so $a = f(a)x$. Now by Lemma 2.2, F is fully idempotent.

(iii) $\Longrightarrow$ (iv) Suppose $A = aS$ is a cyclic right S-act and $B = asS$ is a cyclic subact of A for some $s \in S$. By Lemma 2.8, $s = sxst$ for some $x, t \in S$. Define $g : aS \longrightarrow asS$ by $g(a) = asx$. Clearly g is well-defined and $g(a)st = a(sxst) = as$. Thus by Lemma 2.2, A is fully idempotent.

(iv) $\Longrightarrow$ (iii) It is clear.

(iii) $\Longleftrightarrow$ (v) It is obvious by commutativity of S. □

Corollary 2.10. *Over a commutative monoid S with a zero the following statements are equivalent:*

(i) *All finitely generated right S-acts which satisfy condition (P) are fully idempotent.*

(ii) *All finitely generated strongly flat right S-acts are fully idempotent.*

(iii) *All projective right S-acts are fully idempotent.*

(iv) *All finitely generated projective right S-acts are fully idempotent.*

(v) *S is a regular monoid.*

Proof. (i) $\Longrightarrow$ (ii) It is clear.

(ii) $\Longrightarrow$ (v) Since S_S is strongly flat, S_S is fully idempotent and so it is regular by the previous theorem.

(v) $\Longrightarrow$ (i) Suppose A is a finitely generated right S-act which satisfies condition (P). By Proposition 3.13.14 of [1], $A = \coprod_{i=1}^{n} a_i S$ for some $n \in \mathbb{N}$. If aS is a cyclic subact of A, then $a = a_j s$ for some $j \in \{1, 2, ..., n\}$ and some $s \in S$. By Lemma 2.8, $s = sxst$ for some $x, t \in S$. Define $f : A \longrightarrow a_j sS = aS$ by

$$f(a_i) = \begin{cases} a_j sx; & i = j, \\ \theta; & i \neq j, \end{cases} \tag{2.1}$$

where θ is a fixed zero element of aS. Clearly f is a well-defined homomorphism and $f(a_j st) = a_j sxst = a_j s$. Hence $a = f(at)$ and by Lemma 2.2, A is fully idempotent.

(iii) $\Longrightarrow$ (iv) It is clear.

(iv) $\Longrightarrow$ (v) By the previous theorem is obvious.

(v) $\Longrightarrow$ (iii) By a similar proof of (v) $\Longrightarrow$ (i) is clear. □

Note that by Theorem 2.9 and Corollary 2.10, we can find many examples of fully idempotent acts over a monoid S.

Lemma 2.11. *Suppose A is a right S-act and B, C are two subacts of A such that $C \subseteq B$. If C is an idempotent subact of A, then C is an idempotent subact of B.*

Proof. Since C is an idempotent subact of A, by Definition 2.1, $C = \bigcup_{f \in Hom(A,C)} f(C)$. Let $\bar{f} = f \mid_B$ for every S-homomorphism $f : A \longrightarrow C$. Thus $C = \bigcup_{f \in Hom(A,C)} \bar{f}(C)$. Now, we can easily see that C is an idempotent subact of B. □

By Lemma 2.11, we have the following result.

Corollary 2.12. *Over a monoid S, all right S-acts are fully idempotent if and only if all injective right S-acts are fully idempotent.*

In [2], Zhang et al, gave a characterization of monoids over which every cyclic right act is injective. The next theorem gives a new characterization of these monoids by the fully idempotent acts.

Theorem 2.13. *Suppose S is a commutative monoid. Then the following statements are equivalent:*
(i) *All right S-acts are fully idempotent.*
(ii) *All injective right S-acts are fully idempotent.*
(iii) *All cyclic right S-acts are injective.*
(iv) *S_S is fully idempotent and factors of fully idempotent right S-acts are fully idempotent. Moreover, if S contains a zero then the above statements are equivalent to:*
(v) *All principally weakly flat right S-acts are fully idempotent.*
(vi) *All weakly flat right S-acts are fully idempotent.*
(vii) *All flat right S-acts are fully idempotent.*

Proof. (i) $\Longrightarrow$ (ii) It is clear.
(ii) $\Longrightarrow$ (iii) holds by Proposition 2.7.
(iii) $\Longrightarrow$ (i) If A is a right S-act and aS is a cyclic subact of A, then by assumption there exists a homomorphism $f : A \longrightarrow aS$ such that $f(a) = a$ and so A is fully idempotent by Lemma 2.2.
(iv) $\Longrightarrow$ (i) Note that every right S-act is a homomorphic image of a free right S-act. Now, the result follows by Theorem 2.9.
(i) $\Longrightarrow$ (iv) It is clear.
(i) $\Longrightarrow$ (v), (v) $\Longrightarrow$ (vi) and (vi) $\Longrightarrow$ (vii) are clear.
(vii)$\Longrightarrow$(i) By assumption and corollary 2.10, we deduce that S is a regular monoid. It is well known that over a commutative regular monoid S all acts are flat and so, by assumption, all right S-acts are fully idempotent. □

Remark. In part (ii) of the previous theorem, we can replace injectivity with every property which is weaker than injectivity. Also, this theorem shows that factors of fully idempotent acts are not necessarily fully idempotent.

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