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COMPACT TOEPLITZ OPERATORS

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Abstract. In this paper we prove that if Toeplitz operators T_u^α with symbols in RW satisfy $||uk_z^\alpha||_{s,\alpha} \to 0$ as $z \to \partial\mathbb{D}$ then T_u^α is compact and also prove that if T_u^α is compact then the Berezin transform of T_u^α equals to zero on $\partial\mathbb{D}$.

1. Introduction

Let dA denote the normalized area measure on the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. For any real number α with $\alpha > -1$, let $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ which is a probability measure on $\mathbb{D}$. For $p \geq 1$, the weighted Bergman space L_a^p is a closed subspace of $L^p(\mathbb{D}, dA_\alpha) \cap H(\mathbb{D})$, where $H(\mathbb{D})$ is the space of analytic functions on $\mathbb{D}$. For $z \in \mathbb{D}$, the weighted reproducing kernel is the function $K_z^\alpha \in L_a^2$ such that

$$f(z) \quad = \quad < f, K_z^\alpha >_\alpha$$

for every $f \in L_a^2$, where the norm $||\cdot||_{p,\alpha}$ and the inner product $< \, , \, >_\alpha$ are taken in the space $L^p(\mathbb{D}, dA_\alpha)$ and $L^2(\mathbb{D}, dA_\alpha)$, respectively. Since $K_z^\alpha(w) = \dfrac{1}{(1-\overline{z}w)^{2+\alpha}}$ and $||K_z^\alpha||_{2,\alpha} = (1-|z|^2)^{1+\frac{\alpha}{2}}$, the normalized weighted Bergman reproducing kernel k_z^α is the function

$$k_z^\alpha(w) = \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}}.$$

For a linear operator S on L_a^2, the Berezin transform of S is the function $\widetilde{S}$ on $\mathbb{D}$ defined by

$$\widetilde{S}(z) \quad = \quad < Sk_z^\alpha, k_z^\alpha >_\alpha.$$

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For $u \in L^1(\mathbb{D}, dA)$, the Toeplitz operator T_u^α with symbol u is the operator on L_a^2 defined by $T_u^\alpha(f) = P_\alpha(uf)$, where P_α is the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ onto L_a^2. Since $L^\infty(\mathbb{D}, dA)$ is dense in $L^1(\mathbb{D}, dA)$, the Toeplitz operator T_u^α with symbol u in $L^1(\mathbb{D}, dA)$ is densely defined on L_a^2 and the Berezin transform of u is defined to be the Berezin transform of T_u^α.

A common intuition is that for operators on the Bergman space "closely associated with function theory", compactness is equivalent to having vanishing Berezin transform on $\partial\mathbb{D}$. Our main result shows that this intuition is partially true if the operator is a weighted Toeplitz operator with symbol in RW, where $RW = \{f \in L^1(\mathbb{D}, dA) : ||f||_{RW} = \sup_{z\in\mathbb{D}} ||fK_z^\alpha||_{s,\alpha} < +\infty$ for some $s \in (2, \infty)\}$.

Throughout the paper we use p' to denote the conjugate of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$, for $1 < p < \infty$.

In this paper, we will show the following theorems.

Theorem 1.1. Suppose $2 < p < +\infty$ and $u \in RW$, that is, $\sup_{z\in\mathbb{D}} ||uk_z^\alpha||_{s,\alpha} < +\infty$ for some $s > 2$. If $p(2+\alpha) < s$ and $||uk_z^\alpha||_{s,\alpha} \to 0$ as $z \to \partial\mathbb{D}$ then T_u^α is a Hilbert-Schmidt operator and hence compact.

Corollary 1.2. Suppose $2 < p < +\infty$ and $u \in RW$, that is, $\sup_{z\in\mathbb{D}} ||uk_z^\alpha||_{s,\alpha} < +\infty$ for some $s > 2$. If $\dfrac{p(2+\alpha)}{l} < s$ for some l with $1 < l < p'$ and $||uk_z^\alpha||_{s,\alpha} \to 0$ as $z \to \partial\mathbb{D}$ then T_u^α is a compact operator.

2. Some estimates

To obtain our main theorem, we need to introduce some notations. For $z \in \mathbb{D}$, let φ_z be a Möbius transform of $\mathbb{D}$ onto $\mathbb{D}$ defined by

$$\varphi_z(w) = \frac{z - w}{1 - \overline{z}w}.$$

It is easy to show that $\varphi_z \circ \varphi_z$ is the identity function on $\mathbb{D}$.

For $z \in \mathbb{D}$, let $U_z^\alpha : L_a^2 \to L_a^2$ be defined by

$$U_z^\alpha f = (f \circ \varphi_z)(\varphi_z')^{1+\frac{\alpha}{2}}.$$

Since $(\varphi_z'(\varphi_z(w)))^{1+\frac{\alpha}{2}}(\varphi_z(w))^{1+\frac{\alpha}{2}} = 1$, a simple computation shows that $U_z^\alpha \circ U_z^\alpha$ is the identity function on L_a^2 and U_z^α is an isometry. Thus U_z^α is a self-adjoint unitary operator.

For an operator S on L_a^2, we define the conjugation operator S_z by $S_z = U_z^\alpha S U_z^\alpha$.

We consider the problem to determine when a Toeplitz operator is bounded and compact on L_a^2.

Suppose S is a finite sum of operators of the form $T_{u_1} \cdots T_{u_n}$, where each $u_j \in L^\infty(\mathbb{D}, dA)$. Axler and Zheng([2]) proved that the following are equivalent :

(i) S is compact ;

(ii) $||Sk_z||_2 \to 0$ as $z \to \partial\mathbb{D}$, where the norm $||\cdot||_2$ is taken in the space $L^2(\mathbb{D}, dA)$ and $k_z(w) = \dfrac{1-|z|^2}{(1-\overline{z}w)^2}$;

(iii) $\widetilde{S}(z) \to 0$ as $z \to \partial\mathbb{D}$;

(iv) $||S_z 1||_2 \to 0$ as $z \to \partial\mathbb{D}$.

Miao and Zheng ([3]) proved that whenever S is a finite sum of operators of the form $T_{f_1} \cdots T_{f_n}$, where each $f_j \in BT$, S is compact if and only if $\widetilde{S}(z) \to 0$ as $z \to \partial\mathbb{D}$.

Since dA_α is a probability measure on $\mathbb{D}$, for any $f \in RW$, $\sup_{z\in\mathbb{D}} ||fk_z^\alpha||_{2,\alpha} \leq ||f||_{RW}$ and $\sup_{z\in\mathbb{D}} |\widetilde{T_f^\alpha}(z)| \leq ||f||_{RW}$. Let μ be a finite positive Borel measure on $\mathbb{D}$ and let $1 \leq p < +\infty$. We say that μ is a Carleson measure for L_a^p if the inclusion map from L_a^p to $L^p(\mathbb{D}, d\mu)$ is bounded and hence the closed Graph Theorem shows that L_a^p is contained in $L^p(\mathbb{D}, d\mu)$

if and only if μ is a Carleson measure for L_a^p

if and only if $\sup \left\{ \frac{\int_\mathbb{D} |f|^p d\mu}{\int_\mathbb{D} |f|^p dA_\alpha} : f \in L_a^p \right\} < +\infty$

if and only if $\sup \widetilde{\mu}(z) = \sup \int_\mathbb{D} |k_z^\alpha(w)|^2 d\mu(w) < +\infty$ (see Zhu [5]). By the above observation, for each $f \in RW$, $|f|dA_\alpha$ is a Carleson measure on $\mathbb{D}$ and hence T_f^α is a bounded linear operator, in fact, T_f^α is bounded on L_a^p for $1 < p < +\infty$ ([4]).

Suppose $2 < s < +\infty$ and k is a positive integer. For $0 \leq x \leq 1$, we define

$$f(x) = \begin{cases} 2^{\frac{k}{s}} & \text{if } \frac{1}{2^k} - (\frac{1}{2^{k+1}})^2 \leq x \leq \frac{1}{2^k}; \\ 0 & \text{otherwise.} \end{cases}$$

So we get a radial function on $\mathbb{D}$, that is, for every $z \in \mathbb{D}$, $f(z) = f(|z|)$. Since $|k_z^\alpha(w)| \leq 2^{2+\alpha}$ for all $|w| \leq \dfrac{1}{2}$, $\int_\mathbb{D} |f(w)k_z^\alpha(w)|^s dA_\alpha(w) \leq 2^{(2+\alpha)s} \int_0^{\frac{1}{2}} |f(t)|^s dt = 2^{(2+\alpha)s} \sum_{k=1}^\infty \dfrac{2^k}{(2^{k+1})^2} < +\infty$ and hence $L^\infty(\mathbb{D}, dA)$ is a proper subset of RW. Since T_f has an infinite dimensional range, T_f is not compact. From $| < T_f k_z^\alpha, k_z^\alpha > | = | < f k_z^\alpha, k_z^\alpha > | = (1-|z|^2)^{1+\frac{\alpha}{2}} | < f k_z^\alpha, K_z^\alpha > | \leq (1-|z|^2)^{1+\frac{\alpha}{2}} ||f k_z^\alpha||_{s,\alpha} ||K_z^\alpha||_{s',\alpha}$, it is easy to show that $\widetilde{T}_f(z) \to 0$ as $z \to \partial\mathbb{D}$ because $f(z) = 0$ whenever $|z| > \frac{1}{2}$.

3. Compactness

This section contains some upper bounds that will be used in the proof of the main theorem. We begin by stating a simple lemma which is a special case of *Lemma* 4.2.2 in [5].

Lemma 3.1. Suppose $a < \alpha + 1$. If $a + b - \alpha < 2$ then

$$\sup_{z \in \mathbb{D}} \int_{\mathbb{D}} \frac{dA_\alpha(w)}{(1-|w|^2)^a |1-\bar{z}w|^b} < +\infty.$$

Lemma 3.2. Suppose $0 < a < 1$ and there exists s such that $2 < s < +\infty$ and $\sup_{z \in \mathbb{D}} ||u k_z^\alpha||_{s,\alpha} < +\infty$, where $u \in L^1(\mathbb{D}, dA)$. If $\frac{2+\alpha}{a} < s$ then there exists t in $(\frac{2+\alpha}{a}, s)$ and there exists a constant C such that

$$\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|w|^2)^a} dA_\alpha(w) \leq \frac{C||(T_u^\alpha)_z 1||_{t,\alpha}}{(1-|z|^2)^a} \leq \frac{C||u||_{RW}}{(1-|z|^2)^a}$$

for all $z \in \mathbb{D}$ and

$$\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|z|^2)^a} dA_\alpha(z) \leq \frac{C||(T_{\bar{u}}^\alpha)_z 1||_{t,\alpha}}{(1-|w|^2)^a} \leq \frac{C||u||_{RW}}{(1-|w|^2)^a}$$

for all $w \in \mathbb{D}$.

Proof. Since $\dfrac{2+\alpha}{a} < s$, there exists t in $\left(\dfrac{2+\alpha}{a}, s\right)$ such that $1 < t' < \dfrac{2+\alpha}{2-a+\alpha}$ and hence $(2-a+\alpha)t' - \alpha < 2$. Take any z in $\mathbb{D}$. Put $w = \varphi_z(\lambda)$. Since $T_u^\alpha K_z^\alpha = \dfrac{T_u^\alpha U_z^\alpha 1}{(|z|^2-1)^{1+\frac{\alpha}{2}}} = \dfrac{(T_u^\alpha)_z 1 \circ \varphi_z (\varphi_z')^{1+\frac{\alpha}{2}}}{(|z|^2-1)^{1+\frac{\alpha}{2}}}$,

$$\begin{aligned}\int_{\mathbb{D}} \frac{|T_u^\alpha K_z^\alpha(w)|}{(1-|w|^2)^a} &= \frac{1}{(1-|z|^2)^{1+\frac{\alpha}{2}}} \int_{\mathbb{D}} \frac{|(T_u^\alpha)_z 1(\varphi_z(w))||\varphi_z'(w)|^{1+\frac{\alpha}{2}}}{(1-|w|^2)^a} dA_\alpha(w) \\ &= \frac{1}{(1-|z|^2)^a} \int_{\mathbb{D}} \frac{|(T_u^\alpha)_z 1(\lambda)|(1-|w|^2)^a}{(1-|\lambda|^2)^a} dA_\alpha(\lambda) \\ &\leq \frac{||(T_u^\alpha)_z 1||_{t,\alpha}}{(1-|z|^2)^a} \Big(\int_{\mathbb{D}} \frac{dA_\alpha(\lambda)}{(1-|\lambda|^2)^{at'} |1-\overline{z}\lambda|^{(2-2a+\alpha)t'}} \Big)^{\frac{1}{t'}}.\end{aligned}$$

By Lemma 3.1, $(2-a+\alpha)t' - \alpha < 2$ implies that the above integral is finite.
Put $C^{t'} = \int_{\mathbb{D}} \frac{dA_\alpha(\lambda)}{(1-|\lambda|^2)^{at'} |1-\overline{z}\lambda|^{(2-2a+\alpha)t'}}$. Since $t < s$ and dA_α is a probability measure on $\mathbb{D}$, $||(T_u^\alpha)_z 1||_{t,\alpha} = ||U_z^\alpha T_u^\alpha U_z^\alpha 1||_{t,\alpha} \leq ||uk_z^\alpha||_{t,\alpha}$ $\leq ||uk_z^\alpha||_{s,\alpha}$ and hence $||(T_u^\alpha)_z 1||_{t,\alpha} \leq ||u||_{RW}$. Since $T_u^\alpha K_z^\alpha(w) = < T_u^\alpha K_z^\alpha, K_w^\alpha > = \overline{(T_u^\alpha)^* K_w(z)} = \overline{T_{\overline{u}}^\alpha K_w^\alpha(z)}$,

$$\begin{aligned}\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|\overline{z}|^2)^a} dA_\alpha(z) &= \int_{\mathbb{D}} \frac{|(T_{\overline{u}}^\alpha K_w^\alpha)(z)|}{(1-|z|^2)^a} dA_\alpha(z) \\ &\leq \frac{C||(T_{\overline{u}}^\alpha)_z 1||_{t,\alpha}}{(1-|w|^2)^a} \leq \frac{C||u||_{RW}}{(1-|w|^2)^a}.\end{aligned}$$

Thus one has the results.

Corollary 3.3. Suppose $u \in RW$, where $\sup_z ||uk_z^\alpha||_{s,\alpha} < +\infty$ for some $s > 2$. If $2+\alpha < s$ then there exists a in $(0,1)$ and there exists t such that $\frac{2+\alpha}{a} < s$ and $\frac{2+\alpha}{a} < t < s$. Moreover there exists a constant C such that

$$\int_{\mathbb{D}} \frac{|T_u^\alpha K_z^\alpha(w)|}{(1-|w|^2)^a} dA_\alpha(w) \leq \frac{C||(T_u^\alpha)_z 1||_{t,\alpha}}{(1-|z|^2)^a} \leq \frac{C||uk_z^\alpha||_{s,\alpha}}{(1-|z|^2)^a}$$

for all $z \in \mathbb{D}$ and

$$\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|z|^2)^a} dA_\alpha(z) \leq \frac{C||(T_{\overline{u}}^\alpha)_z 1||_{t,\alpha}}{(1-|w|^2)^a} \leq \frac{C||uk_z^\alpha||_{s,\alpha}}{(1-|w|^2)^a}$$

for all $w \in \mathbb{D}$.

Proof. Since $\lim_{a\to 1^-} \frac{2+\alpha}{a} = 2+\alpha$ and $\lim_{a\to 1^-} \frac{2+\alpha}{2-a+\alpha} = \frac{2+\alpha}{1+\alpha}$, it follows from Lemma 3.2.

Proposition 3.4. Suppose $u \in RW$ and T_u^α is compact. Then $||T_u^\alpha k_z^\alpha||_{2,\alpha} \to 0$ as $z \to \partial\mathbb{D}$ and hence $\widetilde{T_u^\alpha}(z) \to 0$ as $z \to \partial\mathbb{D}$.

Proof. Since for each $f \in L_a^2$, $< f, k_z^\alpha > = (1-|z|^2)^{1+\frac{\alpha}{2}} f(z)$ and H^∞ is dense in L_a^2, $k_z^\alpha \to 0$ weakly in L_a^2 as $z \to \partial\mathbb{D}$ and hence $||T_u^\alpha k_z^\alpha||_{2,\alpha} \to 0$ as $z \to \partial\mathbb{D}$.

To prove Theorem 1.1, we will need Schur's theorem ([5]) so that we will show that T_u^α is a compact operator under some condition.

Theorem 3.5. Suppose K is a nonnegative measurable function on $X \times X$, T is the integral operator with kernel K and $1 < p < +\infty$. If there exist positive constats C_1 and C_2 and a positive measurable function h on X such that

$$\int_X K(x,y)h(y)^{p'} d\mu(y) \le C_1 h(x)^{p'}$$

for almost every x in X and

$$\int_X K(x,y)h(x)^{p} d\mu(x) \le C_2 h(y)^{p}$$

for almost every y in X, then T is a bounded linear operator on $L^p(X, d\mu)$ with norm less than or equal to $C_1^{\frac{1}{p'}} C_2^{\frac{1}{p}}$.

Proof of Theorem 1.1. Suppose $f \in L_a^2$ and $w \in \mathbb{D}$. Then

$$\begin{aligned} (T_u^\alpha f)(w) &= < T_u^\alpha f, K_w^\alpha >_\alpha \\ &= \int_{\mathbb{D}} f(z) T_u^\alpha K_z^\alpha(w) dA_\alpha(z). \end{aligned}$$

For $0 < r < 1$, we define

$$\begin{aligned} (T_{u,r}^\alpha f)(w) &= \int_{r\mathbb{D}} f(z) T_u^\alpha K_z^\alpha(w) dA_\alpha(z) \\ &= \int_{\mathbb{D}} f(z) T_u^\alpha K_z^\alpha \chi_{r\mathbb{D}}(z) dA_\alpha(z). \end{aligned}$$

Thus T_u^α and $T_{u,r}^\alpha$ are the integral operators with kernel $T_u^\alpha K_z^\alpha(w)$ and $T_u^\alpha K_z^\alpha(w)\chi_{r\mathbb{D}}(z)$, respectively.

For each $r \in (0,1)$, we get

$$\begin{aligned}\int_{\mathbb{D}}\int_{\mathbb{D}} |T_u^\alpha K_z^\alpha(w)\chi_{r\mathbb{D}}(z)|^2 dA(w)dA(z) &= \int_{r\mathbb{D}}\int_{\mathbb{D}} |T_u^\alpha K_z^\alpha(w)|^2 dA(w)dA(z)\\ &= \int_{r\mathbb{D}} ||T_u^\alpha K_z^\alpha||_{2,\alpha}^2 dA(z)\\ &= ||T_u^\alpha||^2 \int_{r\mathbb{D}} ||K_z^\alpha||_{2,\alpha}^2 dA(z)\\ &\leq \frac{||T_u^\alpha||^2}{(1-r^2)^{2+\alpha}} < \infty.\end{aligned}$$

Thus each $T_{u,r}^\alpha$ is a Hilbert-Schmidt operator. Moreover, $T_u^\alpha - T_{u,r}^\alpha$ is the integral operator with kernel $T_u^\alpha K_z^\alpha(w)\chi_{\mathbb{D}\setminus r\mathbb{D}}$. Let $h(\lambda) = (1-|\lambda|^2)^{-\frac{1}{pp'}}$ be a positive measurable function on $\mathbb{D}$. Since $2 < p < +\infty$, $1 < p' < 2$ and hence $p'(2+\alpha) < s$. By Lemma 3.2, there exists t in $(p(2+\alpha), s)$ and there exists a constant C such that

$$\int_{\mathbb{D}} |(T_u^\alpha K_z^\alpha)(w)| h(w)^{p'} dA_\alpha(w) \leq C||(T_u^\alpha)_z 1||_{t,\alpha} h(z)^{p'}$$

and

$$\int_{\mathbb{D}} |(T_u^\alpha K_z^\alpha)(w)| h(z)^p dA_\alpha(z) \leq C||u||_{RW} h(w)^p.$$

Let $C_1 = C\sup\{||uk_z^\alpha||_{s,\alpha} : r \leq |z| < 1\}$. Then

$$\int_{\mathbb{D}} |(T_u^\alpha K_z^\alpha)(w)\chi_{\mathbb{D}\setminus r\mathbb{D}}(z)| h(w)^{p'} dA_\alpha(w) \leq C_1 h(z)^{p'}$$

for all $z \in \mathbb{D}$ and

$$\int_{\mathbb{D}} |(T_u^\alpha K_z^\alpha)(w)\chi_{\mathbb{D}\setminus r\mathbb{D}}(z)| h(z)^p dA_\alpha(z) \leq C||u||_{RW} h(w)^p$$

for all $w \in \mathbb{D}$. By Schur's theorem, $||T_u^\alpha - T_{u,r}^\alpha|| \leq C_1^{\frac{1}{p'}} (C||u||_{RW})^{\frac{1}{p}}$. Since $||(T_u^\alpha)_z 1||_{t,\alpha} \leq ||(T_u^\alpha)_z 1||_{s,\alpha}$ and $||uk_z^\alpha||_{s,\alpha} \to 0$ as $z \to \partial\mathbb{D}$, $\lim\limits_{r\to 1^-} C_1 = 0$ and hence $\lim\limits_{r\to 1^-} C_1{}^{\frac{1}{p'}} (C||u||_{RW})^{\frac{1}{p}} = 0$.

Since $||T_u^\alpha - T_{u,r}^\alpha|| \to 0$ as $r \to 1^-$ and the collection of Hilbert-Schmidt operators on L_a^2 is a Hilbert space, T_u^α is also a Hilbert-Schmidt operator. This implies that T_u^α is a compact operator.

Proof of Corollary 1.2. In the proof of Thoerem 1.1, let $h(\lambda) = (1-|\lambda|^2)^{-\frac{l}{pp'}}$. Since $\frac{l}{p'} < 1$, it follows from Theorem 1.1.

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