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TWO CHARACTERIZATION THEOREMS FOR LIGHTLIKE HYPERSURFACES OF A SEMI-RIEMANNIAN SPACE FORM

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Abstract. We study lightlike hypersurfaces M of a semi-Riemannian space form $\widetilde{M}(c)$ with a semi-symmetric non-metric connection whose structure vector field is tangent to M. Our main result is two characterization theorems for such a lightlike hypersurface.

1. Introduction

The theory of lightlike submanifolds is used in mathematical physics, in particular, in general relativity since lightlike submanifolds produce models of different types of horizons [10, 19]. Lightlike submanifolds are also studied in the theory of electromagnetism [4]. As for any semi-Riemannian manifold there is a natural existence of lightlike subspaces, Duggal and Bejancu published their work [4] on the general theory of lightlike submanifolds to fill a gap in the study of submanifolds. Since then there has been very active study on lightlike geometry of submanifolds (see up-to date results in two books [5, 9]).

Ageshe and Chafle [1] introduced the notion of a semi-symmetric non-metric connection on a Riemannian manifold. Although now we have lightlike version of a large variety of Riemannian submanifolds, the theory of lightlike submanifolds of semi-Riemannian manifolds with semi-symmetric non-metric connections has been few known. Yasar et al. [20] and Jin [11] $\sim$ [15] studied lightlike submanifolds of semi-Riemannian manifolds admitting semi-symmetric non-metric connections.

Călin proved the following result [2]: For any lightlike submanifolds M of indefinite almost contact manifolds $\widetilde{M}$, *if the structure vector field* ζ *of* $\widetilde{M}$ *is tangent to* M*, then it belongs to* $S(TM)$. After Călin's

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work, many earlier works [7, 8, 16], which have been written on lightlike submanifolds of indefinite almost contact manifolds or lightlike submanifolds of semi-Riemannian manifolds admitting semi-symmetric non-metric connections, obtained their results by using the Călin's result.

In this paper, first we prove that the afore cited Călin's result is not true for any lightlike hypersurfaces M of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric non-metric connection (see Theorem 3.2 and its corollary). Next several authors [18] have agreed the assertion that two screen conformalities, which are called *screen conformal* and *screen quasi-conformal*, of M are dependent to each other. We prove that such two screen conformalities are independent (see Theorem 3.2 and Theorem 3.3). In addition to these main results, we prove a classification theorem for Einstein lightlike hypersurfaces of a Lorentzian space form admitting a semi-symmetric non-metric connection.

2. Semi-symmetric non-metric connection

Let $(\widetilde{M}, \widetilde{g})$ be a semi-Riemannian manifold. A connection $\widetilde{\nabla}$ on $\widetilde{M}$ is called a *semi-symmetric non-metric connection* [1] if, for any vector fields X, Y and Z on $\widetilde{M}$, $\widetilde{\nabla}$ and its torsion tensor $\widetilde{T}$ satisfy

$$(\widetilde{\nabla}_X\widetilde{g})(Y,Z) = -\pi(Y)\widetilde{g}(X,Z) - \pi(Z)\widetilde{g}(X,Y), \tag{2.1}$$

$$\widetilde{T}(X,Y) = \pi(Y)X - \pi(X)Y, \tag{2.2}$$

where π is a 1-form associated with a non-vanishing smooth vector field ζ, which is called the *structure vector field*, by

$$\pi(X) = \widetilde{g}(X,\zeta). \tag{2.3}$$

Let (M, g) be a lightlike hypersurface of $\widetilde{M}$. Then the normal bundle $TM^{\perp}$ of M is a subbundle of the tangent bundle TM of M and coincides the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of M. Therefore there exists a complementary non-degenerate vector bundle $S(TM)$ of $Rad(TM)$ in TM, which is called a *screen distribution* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \tag{2.4}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. It is well-known [4] that, for any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ in $S(TM)^{\perp}$ satisfying

$$\widetilde{g}(\xi, N) = 1, \quad \widetilde{g}(N, N) = \widetilde{g}(N, X) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to $S(TM)$ respectively. Then the tangent bundle $T\widetilde{M}$ of $\widetilde{M}$ is given by

$$T\widetilde{M} = TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM). \tag{2.5}$$

In the entire discussion of this article we shall assume that ζ to be unit spacelike vector field of M. Therefore ζ is tangent to M. In the sequel, we take X, Y, $Z \in \Gamma(TM)$ unless otherwise specified.

Let P be the projection morphism of TM on $S(TM)$. The local Gauss and Weingartan formulas for M and $S(TM)$ are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.6}$$

$$\widetilde{\nabla}_X N = -A_N X + \tau(X)N; \tag{2.7}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.8}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.9}$$

where ∇ and ∇^* are the induced linear connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively, and τ is a 1-form on TM.

From (2.1), (2.2) and (2.6), we have

$$(\nabla_X g)(Y, Z) = -\pi(Y)g(X, Z) - \pi(Z)g(X, Y) + B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \tag{2.10}$$

$$T(X, Y) = \pi(Y)X - \pi(X)Y \tag{2.11}$$

and B is symmetric on TM, where T is the torsion tensor with respect to the induced connection ∇ of M and η is a 1-form on TM such that

$$\eta(X) = \widetilde{g}(X, N).$$

From the fact $B(X, Y) = \widetilde{g}(\widetilde{\nabla}_X Y, \xi)$, we know that B is independent of the choice of a screen distribution $S(TM)$. The above two local second fundamental forms are related to their shape operators by

$$g(A^*_\xi X, Y) = B(X, Y), \qquad \widetilde{g}(A^*_\xi X, N) = 0, \tag{2.12}$$

$$g(A_N X, PY) = C(X, PY) - fg(X, PY) - \eta(X)\pi(PY), \quad \widetilde{g}(A_N X, N) = -f\eta(X), \tag{2.13}$$

where f is the smooth function given by $f=\pi(N)$. By (2.12), we show that A^*_ξ is a $S(TM)$-valued self-adjoint operator and

$$B(X,\xi)=0, \qquad A^*_\xi\xi=0. \tag{2.14}$$

Denote by $\widetilde{R}$, R and R^* the curvature tensors of the semi-symmetric non-metric connection $\widetilde{\nabla}$ on $\widetilde{M}$, the induced connection ∇ on M and the induced connection ∇^* on $S(TM)$ respectively. Using the Gauss-Weingarten formulas for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$\begin{aligned}\tilde{g}(\widetilde{R}(X,Y)Z,\,PW) &= g(R(X,Y)Z,\,PW)\\ &\quad + B(X,Z)g(A_N Y,PW) - B(Y,Z)g(A_N X,PW),\end{aligned} \tag{2.15}$$

$$\begin{aligned}\tilde{g}(\widetilde{R}(X,Y)Z,\,\xi) &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z)\\ &\quad + B(Y,Z)\{\tau(X)-\pi(X)\} - B(X,Z)\{\tau(Y)-\pi(Y)\},\end{aligned} \tag{2.16}$$

$$\begin{aligned}\tilde{g}(\widetilde{R}(X,Y)Z,\,N) &= \tilde{g}(R(X,Y)Z,\,N)\\ &\quad + f\{B(Y,Z)\eta(X) - B(X,Z)\eta(Y)\},\end{aligned} \tag{2.17}$$

$$\begin{aligned}\tilde{g}(\widetilde{R}(X,Y)\xi,\,N) &= B(X,A_N Y) - B(Y,A_N X) - 2d\tau(X,Y)\\ &= C(Y,A^*_\xi X) - C(X,A^*_\xi Y) - 2d\tau(X,Y),\end{aligned} \tag{2.18}$$

$$\begin{aligned}g(R(X,Y)PZ,\,PW) &= g(R^*(X,Y)PZ,\,PW)\\ &\quad + C(X,PZ)g(A^*_\xi Y,PW) - C(Y,PZ)g(A^*_\xi X,PW)\end{aligned} \tag{2.19}$$

$$\begin{aligned}&\tilde{g}(R(X,Y)PZ,\,N)\\ &= (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ)\\ &\quad + C(X,PZ)\{\tau(Y)+\pi(Y)\} - C(Y,PZ)\{\tau(X)+\pi(X)\},\end{aligned} \tag{2.20}$$

$$\begin{aligned}&\tilde{g}(\widetilde{R}(X,Y)N,\,PZ)\\ &= g(-\nabla_X(A_N Y) + \nabla_Y(A_N X) + A_N[X,Y],\,PZ)\\ &\quad - \tau(Y)g(A_N X,PZ) + \tau(X)g(A_N Y,PZ),\end{aligned} \tag{2.21}$$

$$\begin{aligned}&g(R(X,Y)\xi,\,PZ)\\ &= g(-\nabla^*_X(A^*_\xi Y) + \nabla^*_Y(A^*_\xi X) + A^*_\xi[X,Y],\,PZ)\\ &\quad + \tau(Y)g(A^*_\xi X,PZ) - \tau(X)g(A^*_\xi Y,PZ).\end{aligned} \tag{2.22}$$

A complete simply connected semi-Riemannian manifold $\widetilde{M}$ of constant curvature c is called a *semi-Riemannian space form* and denote it by $\widetilde{M}(c)$. In this case, the curvature tensor $\widetilde{R}$ of $\widetilde{M}(c)$ is given by

$$\widetilde{R}(X,Y)Z = c\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\}, \tag{2.23}$$

for all X, Y, $Z \in \Gamma(T\widetilde{M})$.

3. Two characterization theorems

Lemma 3.1 [11] ∼ [14]. *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection. If the structure vector field ζ is tangent to M, then ζ satisfies*

$$B(X,\zeta) = \pi(A^*_\xi X) = 0. \tag{3.1}$$

Proof. From the two representations of (2.18), we obtain

$$B(X, A_N Y) - B(Y, A_N X) = C(Y, A^*_\xi X) - C(X, A^*_\xi Y).$$

Substituting (2.12) and (2.13) into this equation, we get

$$\pi(A^*_\xi X)\eta(Y) = \pi(A^*_\xi Y)\eta(X).$$

Replacing Y by ξ to this and using $(2.14)_2$, we have (3.1).

Definition 1. A lightlike hypersurface M of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection is called *screen quasi-conformal* [18] if B and C satisfy

$$C(X, PY) = \varphi B(X,Y) + \eta(X)\pi(PY), \tag{3.2}$$

where φ is a non-vanishing function on a neighborhood $\mathcal{U}$ in M.

From (2.12) and (2.13), we show that a necessary and sufficient condition for M to be screen quasi-conformal is

$$A_N X = \varphi A^*_\xi X - fX. \tag{3.3}$$

Theorem 3.2. *Let M be a screen quasi-conformal lightlike hypersurface of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric non-metric connection. If ζ is tangent to M but it does not belong to $S(TM)$, then $c = 1$.*

Proof. Applying ∇_Y to (3.3), we have

$$\nabla_X(A_N Y) = X[\varphi]A^*_\xi Y + \varphi\nabla_X(A^*_\xi Y) - X[f]Y - f\nabla_X Y.$$

Substituting this into (2.21) and using (2.11)∼(2.13) and (2.22), we have

$$\begin{aligned}\widetilde{g}(\widetilde{R}(X,Y)N, PZ) - \varphi\widetilde{g}(\widetilde{R}(X,Y)\xi, PZ)\\ = \{Y[\varphi] - 2\varphi\tau(Y)\}B(X,PZ)\\ - \{X[\varphi] - 2\varphi\tau(X)\}B(Y,PZ)\\ + \{X[f] - f\tau(X) - f\pi(X)\}g(Y,PZ)\\ - \{Y[f] - f\tau(Y) - f\pi(Y)\}g(X,PZ).\end{aligned}$$

Substituting (2.23) into the last equation and using (2.14), we get

$$\begin{aligned}\{X[\varphi] - 2\varphi\tau(X)\}B(Y,Z)\\ - \{Y[\varphi] - 2\varphi\tau(Y)\}B(X,Z)\\ = \{X[f] - f\pi(X) - f\tau(X) + c\eta(X)\}g(Y,Z)\\ - \{Y[f] - f\pi(Y) - f\tau(Y) + c\eta(Y)\}g(X,Z).\end{aligned} \tag{3.4}$$

Taking $X = Z = \zeta$ and $Y = \xi$ to this equation and using (3.1), we have

$$\xi[f] - f\tau(\xi) + c = 0. \tag{3.5}$$

On the other hand, substituting (2.23) into (2.16), we have

$$\begin{aligned}(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z)\\ = B(Y,Z)\{\pi(X) - \tau(X)\} - B(X,Z)\{\pi(Y) - \tau(Y)\}.\end{aligned} \tag{3.6}$$

Applying $\widetilde{\nabla}_X$ to $\eta(Y) = \widetilde{g}(Y,N)$ and using (2.1), we have

$$\begin{aligned}X(\eta(Y)) = -\pi(Y)\eta(X) - fg(X,Y) + \widetilde{g}(\nabla_X Y, N)\\ - g(A_N X, Y) + \tau(X)\eta(Y).\end{aligned}$$

Substituting this into the right term of the following equation

$$2d\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])$$

and using (2.11), (3.3) and the fact A^*_ξ is self-adjoint, we get

$$2d\eta(X,Y) = \tau(X)\eta(Y) - \tau(Y)\eta(X). \tag{3.7}$$

Substituting (2.23) into (2.17), we obtain

$$\begin{aligned}\widetilde{g}(R(X,Y)PZ,\ N) = c\{g(Y,PZ)\eta(X) - g(X,PZ)\eta(Y)\}\\ + f\{B(X,PZ)\eta(Y) - B(Y,PZ)\eta(X)\}.\end{aligned}$$

Comparing this equation and (2.20), we get

$$\begin{aligned}\{cg(Y,PZ) - fB(Y,PZ)\}\eta(X) - \{cg(X,PZ) - fB(X,PZ)\}\eta(Y)\\ = (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + C(X,PZ)\{\pi(Y) + \tau(Y)\}\\ - C(Y,PZ)\{\pi(X) + \tau(X)\}.\end{aligned} \tag{3.8}$$

Applying ∇_X to $C(Y, PZ) = \varphi B(Y, PZ) + \eta(Y)\pi(PZ)$, we have

$$(\nabla_X C)(Y, PZ) = X[\varphi]B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ) + \{X(\eta(Y)) - \eta(\nabla_X Y)\}\pi(PZ) + \eta(Y)\{X(\pi(PZ)) - \pi(\nabla^*_X PZ)\}.$$

Substituting this into (3.8) and using (3.2), (3.4), (3.6) and (3.7), we get

$$\begin{aligned} f\{\eta(Y)B(X, PZ) - \eta(X)B(Y, PZ)\} &= \{X[f] - f\pi(X) - f\tau(X)\}g(Y, PZ) \\ &\quad - \{Y[f] - f\pi(Y) - f\tau(Y)\}g(X, PZ) \\ &\quad + \eta(Y)\{X(\pi(PZ)) - \pi(\nabla^*_X PZ)\} \\ &\quad - \eta(X)\{Y(\pi(PZ)) - \pi(\nabla^*_Y PZ)\}. \end{aligned} \tag{3.9}$$

Applying ∇_X to $\pi(PZ) = g(\zeta, PZ)$ and using (2.10) and (3.1), we have

$$X(\pi(PZ)) - \pi(\nabla^*_X PZ) = -g(X, PZ) - \pi(X)\pi(PZ) + fB(X, PZ) + g(\nabla_X \zeta, PZ).$$

Substituting this equation into (3.9), we obtain

$$\begin{aligned} &\{X[f] - f\pi(X) - f\tau(X)\}g(Y, PZ) \\ &\quad - \{Y[f] - f\pi(Y) - f\tau(Y)\}g(X, PZ) \\ &\quad + \eta(X)\{g(Y, PZ) + \pi(Y)\pi(PZ) - g(\nabla_Y \zeta, PZ)\} \\ &\quad - \eta(Y)\{g(X, PZ) + \pi(X)\pi(PZ) - g(\nabla_X \zeta, PZ)\} = 0. \end{aligned} \tag{3.10}$$

Applying ∇_X to $g(\zeta, \zeta) = 1$ and using (2.10) and (3.1), we have

$$g(\nabla_X \zeta, \zeta) = \pi(X). \tag{3.11}$$

Taking $X = \xi$ and $Y = Z = \zeta$ to (3.10) and using (3.11), we get

$$\xi[f] - f\tau(\xi) + 1 = 0. \tag{3.12}$$

From this result and (3.5), we show that $c = 1$.

Corollary 1. There exist no screen quasi-conformal lightlike hypersurfaces M of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric non-metric connection such that ζ belongs to $S(TM)$.

Proof. If ζ belongs to $S(TM)$, then $f = \widetilde{g}(\zeta, N) = 0$. It follows from (3.12) that $1 = 0$. It is a contradiction. Thus there exist no screen quasi-conformal lightlike hypersurfaces M of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric non-metric connection such that ζ belongs to $S(TM)$.

Remark 1. For any lightlike submanifolds M of indefinite almost contact manifolds $\widetilde{M}$ such that the structure vector field ζ of $\widetilde{M}$ is tangent to M, if ζ belongs to $Rad(TM)$, then ζ is decompose as $\zeta = a\xi$ and $a \neq 0$. Using this, we have $1 = \widetilde{g}(\zeta, \zeta) = a^2\widetilde{g}(\xi, \xi) = 0$. It is a contradiction. Thus ζ does not belong to $Rad(TM)$. This enables one to choose a screen distribution $S(TM)$ which contains ζ. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^\sharp = TM/Rad(TM)$ [17]. Thus all screen distributions are mutually isomorphic. This implies that *if ζ is tangent to M, then it belongs to* $S(TM)$. Călin [2] proved this result. Duggal and Sahin also proved this result (see p.318 - 319 of [9]). After Călin's work, many earlier works [7, 8, 16], which have been written on lightlike submanifolds of indefinite almost contact manifolds or lightlike submanifolds of semi-Riemannian manifolds admitting semi-symmetric non-metric connections, obtained their results by using the afore cited Călin's result. However, we regret to indicate that Călin's result is not true for any lightlike hypersurfaces M of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric non-metric connection by Theorem 3.2 and its corollary.

Definition 2. A lightlike hypersurface M of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection is *screen conformal* [5, 6, 9] if the second fundamental forms B and C satisfy

$$C(X, PY) = \varphi B(X, Y), \tag{3.13}$$

where φ is a non-vanishing function on a neighborhood $\mathcal{U}$ in M.

Theorem 3.3. *Let M be a lightlike hypersurface of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric non-metric connection such that ζ is tangent to M. If M is screen conformal, then $c = 0$.*

Proof. Applying ∇_X to $C(Y, PZ) = \varphi B(Y, PZ)$, we have

$$(\nabla_X C)(Y, PZ) = X[\varphi]B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ).$$

Substituting this equation into (2.20) and using (3.6), we have

$$\begin{aligned}&\widetilde{g}(R(X, Y)PZ, N)\\ &= \{X[\varphi] - 2\varphi\tau(X)\}B(Y, PZ) - \{Y[\varphi] - 2\varphi\tau(Y)\}B(X, PZ).\end{aligned}$$

Substituting this equation and (2.23) into (2.17), we get

$$\begin{aligned}&c\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\}\\ &\quad = \{X[\varphi] - 2\varphi\tau(X) + f\eta(X)\}B(Y, PZ)\\ &\qquad - \{Y[\varphi] - 2\varphi\tau(Y) + f\eta(Y)\}B(X, PZ).\end{aligned}$$

Taking $X = \xi$ and $Y = Z = \zeta$ to this and using (3.1), we have $c = 0$.

Jin [12] proved the following result: Under the same assumption in Theorem 3.5, if M is screen conformal and $\tau = 0$, then $c = 0$.

Remark 2. From Theorem 3.2 and Theorem 3.3, we show that the two screen conformalities, which are called *screen conformal* and *screen quasi-conformal*, of M are not mutually dependent to each other but not mutually independent.

4. Einstein lightlike hypersurfaces

Let $\widetilde{Ric}$ be the Ricci curvature tensor of $\widetilde{M}$ and $R^{(0,2)}$ the induced Ricci type tensor on M given respectively by

$$\begin{aligned} \widetilde{Ric}(X,Y) &= trace\{Z \to \widetilde{R}(Z,X)Y\}, \quad \forall\, X,\, Y \in \Gamma(T\widetilde{M}), \\ R^{(0,2)}(X,Y) &= trace\{Z \to R(Z,X)Y\}\,, \quad \forall\, X,\, Y \in \Gamma(TM). \end{aligned}$$

Consider a quasi-orthonormal frame field $\{\xi; W_a\}$ on M, where $Rad(TM) = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$ and let $E = \{\xi, N, W_a\}$ be the corresponding frame field on $\widetilde{M}$. Using this frame field, we obtain

$$\begin{aligned} R^{(0,2)}(X,Y) &= \widetilde{Ric}(X,Y) + B(X,Y)trA_{_N} - g(A_{_N}X, A^*_\xi Y) \\ &\quad - \widetilde{g}(\widetilde{R}(\xi,Y)X,\, N), \quad \forall\, X,\, Y \in \Gamma(TM). \end{aligned}$$

This shows that $R^{(0,2)}$ is not symmetric. The tensor field $R^{(0,2)}$ is called its *induced Ricci tensor* [5, 6], denoted by *Ric*, of M if it is symmetric. It is known [13] that $R^{(0,2)}$ is symmetric if and only if the 1-form τ is closed, i.e., $d\tau = 0$, for any coordinate neighborhood $\mathcal{U} \subset M$.

Remark. If $R^{(0,2)}$ is symmetric, then there exists a null pair $\{\xi,\, N\}$ such that the corresponding 1-form τ satisfies $\tau = 0$ [4], which called a *canonical null pair* of M. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^\sharp = TM/Rad(TM)$ [17]. This implies that all screen distribution are mutually isomorphic. For this reason, in case $d\tau = 0$ we consider only lightlike hypersurfaces M endow with the canonical null pair.

M is called an *Einstein manifold* if the Ricci tensor of M satisfies

$$Ric = \kappa g. \tag{4.1}$$

It is well-known that if dim $M > 2$, then κ is a constant. For dim $M = 2$, any manifold M is Einstein but κ is not necessarily constant.

In case $\widetilde{M}$ is a space form $\widetilde{M}(c)$, $R^{(0,2)}$ is given by

$$R^{(0,2)}(X,Y) = mc\, g(X,Y) + B(X,Y)trA_{_N} - g(A_{_N}X, A^*_\xi Y). \tag{4.2}$$

Theorem 5.1 [13]. *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric metric connection. Then the following assertions are equivalent:*

(1) *The screen distribution $S(TM)$ is an integrable distribution.*
(2) *C is symmetric, i.e., $C(X,Y) = C(Y,X)$ for all $X, Y \in \Gamma(S(TM))$.*
(3) *The shape operator A_N is self-adjoint with respect to g, i.e.,*

$$g(A_N X, Y) = g(X, A_N Y), \quad \forall\, X,\, Y \in \Gamma(S(TM)).$$

Remark. Just as in the well-known case of locally product Riemannian or semi-Riemannian manifolds [4, 5, 6, 19], if $S(TM)$ is an integrable distribution, then and M is locally a product manifold $\mathcal{C} \times M^*$ where $\mathcal{C}$ is a null curve tangent to $Rad(TM)$ and M^* is a leaf of $S(TM)$.

Theorem 5.2. *Let M be a screen quasi-conformal Einstein lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ admitting a semi-symmetric non-metric connection. If ζ is tangent to M but it does not belong to $S(TM)$ and the mean curvature of M is constant, then M is locally a product manifold $M = \mathcal{C} \times M_1 \times M_2$, where $\mathcal{C}$ is a null curve tangent to $Rad(TM)$, M_1 is an Euclidean space and M_2 is a totally umbilical Riemannian space.*

Proof. From (3.3), (4.2) and the fact A^*_ξ is self-adjoint, we show that $R^{(0,2)}$ is symmetric and $S(TM)$ is an integrable distribution. As $g(A^*_\xi\zeta,\, X) = B(\zeta, X) = 0$ and $S(TM)$ is non-degenerate, we have

$$A^*_\xi \zeta = 0. \tag{4.3}$$

Using (2.12), (3.3), (4.1) and the fact $c = 1$, from (4.2) we have

$$g(A^*_\xi X, A^*_\xi Y) - \alpha g(A^*_\xi X, Y) + \varphi^{-1}(\kappa - m) g(X,Y) = 0, \tag{4.4}$$

for all $X,\, Y \in \Gamma(TM)$ due to $c = 1$, where $\alpha = trA^*_\xi - fm\varphi^{-1}$. Taking $X = Y = \zeta$ to (4.4) and using (4.3), we have $\kappa = m$. (4.4) becomes

$$g(A^*_\xi X, A^*_\xi Y) - \alpha g(A^*_\xi X, Y) = 0. \tag{4.5}$$

As $\widetilde{M}$ is Lorentzian manifold, $S(TM)$ is a Riemannian vector bundle. Since ξ is an eigenvector field of A^*_ξ corresponding to the eigenvalue 0 due to $(2.14)_2$ and A^*_ξ is $S(TM)$-valued real self-adjoint operator, A^*_ξ have m real orthonormal eigenvector fields in $S(TM)$ and is diagonalizable. Consider a frame field of eigenvectors $\{\xi, E_1, \ldots, E_m\}$ of A^*_ξ such that $\{E_1, \ldots, E_m\}$ is an orthonormal frame field of $S(TM)$ and $A^*_\xi E_i = \lambda_i E_i$.

Put $X = Y = E_i$ in (4.5), each eigenvalue λ_i is a solution of the equation

$$x^2 - \alpha x = 0.$$

As this equation has at most two distinct solutions 0 and α, there exists $p \in \{0, 1, \ldots, m\}$ such that $\lambda_1 = \cdots = \lambda_p = 0$ and $\lambda_{p+1} = \cdots = \lambda_m = \alpha (\neq 0)$, by renumbering if necessary. As $tr A^*_\xi = 0p + (m-p)\alpha$, we have

$$(m - p - 1)\alpha = fm\varphi^{-1}.$$

Consider four distributions D_o, D_α, D^s_o and D^s_α on $S(TM)$ given by

$$D_o = \{X \in \Gamma(TM) \mid A^*_\xi X = 0\}, \qquad D^s_o = D_o \cap S(TM),$$
$$D_\alpha = \{U \in \Gamma(TM) \mid A^*_\xi U = \alpha PU\}, \qquad D^s_\alpha = D_\alpha \cap S(TM).$$

Clearly we show that $D_o \cap D_\alpha = Rad(TM)$, $D^s_o \cap D^s_\alpha = \{0\}$ as $\alpha \neq 0$ and $D^s_o = PD_o$, $D^s_\alpha = D_\alpha$. In the sequel, we take the vector fields $X, Y \in \Gamma(D_o)$, $U, V \in \Gamma(D_\alpha)$ and $Z, W \in \Gamma(TM)$. Denote $X^* = PX$, $Y^* = PY$, $U^* = PU$ and $V^* = PV$. Then $X^*, Y^* \in \Gamma(D^s_o)$ and $U^*, V^* \in \Gamma(D^s_\alpha)$. Since X^* and U^* are eigenvector fields of the real self-adjoint operator A^*_ξ corresponding to the different eigenvalues 0 and α respectively, $X^* \perp U^*$ and $g(X, U) = g(X^*, U^*) = 0$, that is, $D_o \perp_g D_\alpha$. Also, since $B(X, U) = g(A^*_\xi X, U) = 0$, we show that $D_\alpha \perp_B D_o$. Since $\{E_i\}_{1 \le i \le p}$ and $\{E_a\}_{p+1 \le a \le m}$ are vector fields of D^s_o and D^s_α respectively and D^s_o and D^s_α are mutually orthogonal, we show that D^s_o and D^s_α are non-degenerate distributions of rank p and rank $(m-p)$ respectively. Thus $S(TM)$ is decomposed as $S(TM) = D^s_\alpha \oplus_{orth} D^s_o$.

From (4.5), we get $A^*_\xi(A^*_\alpha - \alpha P) = 0$. Let $W \in Im\, A^*_\xi$. Then there exists $Z \in \Gamma(TM)$ such that $W = A^*_\xi Z$. Then $(A^*_\xi - \alpha P)W = 0$ and $W \in \Gamma(D_\alpha)$. Thus $Im\, A^*_\xi \subset \Gamma(D_\alpha)$. By duality, $Im(A^*_\xi - \alpha P) \subset \Gamma(D_o)$.

Applying ∇_X to $B(Y, U) = 0$ and using (2.12), we obtain

$$(\nabla_X B)(Y, U) = -g(A^*_\xi \nabla_X Y, U).$$

Using this, (2.11), (3.6) and the facts $A^*_\xi X = A^*_\xi Y = 0$, we get

$$g(A^*_\xi [X, Y], U) = 0.$$

As $Im\, A^*_\xi \subset \Gamma(D_\alpha)$ and D_α is non-degenerate, $A^*_\xi [X, Y] = 0$. Thus $[X, Y] \in \Gamma(D_o)$ and D_o is integrable. This result implies $[X^*, Y^*] \in \Gamma(D_o)$. On the other hand, since $S(TM)$ is integrable, $[X^*, Y^*] \in \Gamma(S(TM))$. Thus $[X^*, Y^*] \in \Gamma(D^s_o)$. Thus D^s_o is also integrable.

Applying ∇_V to $B(U, Y) = 0$ and using $A^*_\xi Y = 0$ and $A^*_\xi U = \alpha PU$, we get

$$(\nabla_V B)(U, Y) = -\alpha g(\nabla_V Y, U).$$

Substituting this into (3.6) and using the fact $\alpha \neq 0$, we obtain

$$g(\nabla_V Y, U) = g(V, \nabla_U Y).$$

Applying ∇_V to $g(Y, U) = 0$ and using (2.10), we have

$$\pi(Y)g(U, V) - B(V, U)\eta(Y) - g(\nabla_V Y, U) = g(Y, \nabla_V U).$$

Taking the skew-symmetric part of this and using (2.11), we have

$$g([V, U], Y) = 0, \quad \forall\, Y \in \Gamma(D_o) \text{ and } U, V \in \Gamma(D_\alpha).$$

From this, we get $g([V^*, U^*], Y^*) = 0$ for all $Y^* \in \Gamma(D_o^s)$ and $U^*, V^* \in \Gamma(D_\alpha^s)$. As D_o^s and D_α^s are mutually orthogonal non-degenerate distributions, we show that $[V^*, U^*] \in \Gamma(D_\alpha^S)$. Thus D_α^s is also integrable.

Applying ∇_U to $B(X, Y) = 0$ and ∇_X to $B(U, Y) = 0$, we have

$$(\nabla_U B)(X, Y) = 0, \quad (\nabla_X B)(U, Y) = -\alpha g(\nabla_X Y, U).$$

Substituting these equations into (3.6), we have $\alpha g(\nabla_X Y, U) = 0$. As

$$g(A_\xi^* \nabla_X Y, U) = B(\nabla_X Y, U) = \alpha g(\nabla_X Y, U) = 0$$

and $Im\, A_\xi^* \subset \Gamma(D_\alpha)$ and D_α is non-degenerate, we get $A_\xi^* \nabla_X Y = 0$. This implies $\nabla_X Y \in \Gamma(D_o)$. Thus D_o is an auto-parallel distribution on $S(TM)$. This implies that $\nabla_{X^*} Y^* \in \Gamma(D_o)$ for any $X^*, Y^* \in \Gamma(D_o^s)$. As $C(X^*, Y^*) = \varphi B(X^*, Y^*) + \eta(X^*)\pi(Y^*) = 0$, we have $\nabla_{X^*} Y^* = \nabla^*_{X^*} Y^* \in \Gamma(S(TM))$. Thus $\nabla_{X^*} Y^* \in \Gamma(D_o^s)$ and D_o^s is also an auto-parallel distribution.

As $A_\xi^* \zeta = 0$, ζ belongs to D_o. Thus $\pi(U) = 0$ for any $U \in \Gamma(D_\alpha)$. Applying ∇_X to $g(U, Y) = 0$ and using (2.10) and the fact D_o is auto-parallel, we get $g(\nabla_X U, Y) = 0$. This implies $\nabla_X U \in \Gamma(D_\alpha)$.

Assume that the mean curvature vector field

$$\mu = \frac{1}{m} g(A_\xi^* E_a, E_a) = \frac{m-p}{m}\alpha$$

of M is constant. Then α is a constant. Applying ∇_X to $B(U, V) = \alpha g(U, V)$ and ∇_U to $B(X, V) = 0$, we have

$$(\nabla_X B)(U, V) = 0, \quad (\nabla_U B)(X, V) = -\alpha g(\nabla_U X, V).$$

Substituting this two equations into (3.6) and using $D_o \perp_B D_\alpha$, we have

$$g(\nabla_U X, V) = \pi(X) g(U, V).$$

Applying ∇_U to $g(X, V) = 0$ and using (2.10), we obtain

$$g(X, \nabla_U V) = 0.$$

From this, we get $g(X^*, \nabla_{U^*}V^*) = 0$ for all $X^* \in \Gamma(D_o^s)$ and $U^*, V^* \in \Gamma(D_\alpha^s)$. As D_o^s and D_α^s are mutually orthogonal non-degenerate distributions, $\nabla_{U^*}V^* \in \Gamma(D_\alpha^S)$ and D_α^s is auto-parallel distribution.

Since the leaf M^* of $S(TM)$ is a Riemannian manifold and $S(TM) = D_\alpha^s \oplus_{orth} D_o^s$, where D_α^s and D_o^s are auto-parallel distributions of M^*, by the decomposition theorem of de Rham [3] we have $M^* = M_1 \times M_2$, where M_1 is a totally geodesic leaf of D_o^s and M_2 is a totally umbilical leaf of D_α^s. Consider the frame field of eigenvectors $\{\xi, E_1, \ldots, E_m\}$ of A_ξ^* such that $\{E_i\}_i$ is an orthonormal frame field of $S(TM)$, then $B(E_i, E_j) = C(E_i, E_j) = 0$ for $1 \leq i < j \leq m$ and $B(E_i, E_i) = C(E_i, E_i) = 0$ for $1 \leq i \leq m-1$. From (2.15) and (2.19), we have $\widetilde{g}(\widetilde{R}(E_i, E_j)E_j, E_i) = g(R^*(E_i, E_j)E_j, E_i) = 0$. Thus the sectional curvature K of the leaf $M^\natural$ of D_o^s is given by

$$K(E_i, E_j) = \frac{g(R^*(E_i, E_j)E_j, E_i)}{g(E_i, E_i)g(E_j, E_j) - g^2(E_i, E_j)} = 0.$$

Thus M is locally a product manifold $M = \mathcal{C} \times M_1 \times M_2$, where $\mathcal{C}$ is a null curve tangent to $Rad(TM)$, M_1 is an Euclidean space and M_2 is a totally umbilical Riemannian space.

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