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The Unit Ball of $\mathcal{L}_s(^2d_*(1,w)^2)$

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Abstract. First we present the explicit formula for the norm of a symmetric bilinear form on the 2-dimensional real predual of the Lorentz sequence space $d_*(1,w)^2$. Using this formula, we classify the extreme points of the unit ball of $\mathcal{L}_s(^2d_*(1,w)^2)$

1. Introduction

Let $n \in \mathbb{N}$. We write B_E and S_E for the closed unit ball and sphere of a real Banach space E respectively and the dual space of E is denoted by E^*. A unit vector x in E is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. We denote by $extB_E$ the sets of all the extreme points of B_E. We denote by $\mathcal{L}_s(^nE)$ the Banach space of all continuous symmetric n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1, 1\le k\le n} |T(x_1, \cdots, x_n)|$. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists $T \in \mathcal{L}_s(^nE)$ such that $P(x) = T(x, \cdots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [7]. We will denote by $T((x_1, y_1),\ (x_2, y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1)$ and $P(x, y) = ax^2 + by^2 + cxy$ a symmetric bilinear form and a 2-homogeneous polynomial on a real Banach space of dimension 2 respectively.

Since 1998, many authors have been developing the problem of characterizing extreme points of the unit balls of $\mathcal{P}(^nE)$ for some classical real Banach spaces. Choi, Ki and the author [2, Theorem 2.4] showed that a sufficient and necessary condition on the coefficients a, b and c for $P(x, y)$ defined on the real space l_1^2 to have norm 1, is,

(i) ($|a| = 1$ or $|b| = 1$) and $|c| \le 2$

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or
(ii) $|a| < 1,\ |b| < 1,\ 2 < |c| \leq 4$ and $4|c| - c^2 = 4(|a + b| - ab)$.
It was also proved in [2, Theorem 2.6] that $P \in extB_{\mathcal{P}(^2l_1^2)}$ if and only if

$$(|a| = |b| = 1,\ |c| = 2) \text{ or } a = -b,\ 2 < |c| \leq 4,\ 4a^2 = 4|c| - c^2.$$

Choi and the author [3, Theorem 2.2] showed that $P \in extB_{\mathcal{P}(^2l_2^2)}$ if and only if

$$(|a| = |b| = 1,\ |c| = 0) \text{ or } a = -b,\ 0 < |c| \leq 2,\ 4a^2 = 4 - c^2.$$

Later, B. Grecu [9] classified the sets $extB_{\mathcal{P}(^2l_p^2)}$ for $1 < p < 2$ or $2 < p < \infty$. We denote the 2-dimensional real predual of the Lorentz sequence space with a positive weight $0 < w < 1$ by

$$d_*(1, w)^2 := \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_{d_*} := \max\{|x|,\ |y|,\ \frac{|x| + |y|}{1 + w}\ \}.$$

Very recently, the author [13] characterize the extreme points of the unit ball of $\mathcal{P}(^2d_*(1, w)^2)$. In fact, we show that the extreme points of the unit ball of $\mathcal{P}(^2d_*(1, w)^2)$ are

$$\begin{aligned}
&\pm x^2,\ \ \pm y^2,\ \ \pm \frac{1}{1 + w^2}(x^2 + y^2), \pm\frac{1}{(1 + w)^2}(x^2 + y^2 \pm 2xy),\\
&\pm \{ax^2 - ay^2\ \pm\ 2\sqrt{a(1 - a)}xy\ \}(\forall \frac{1}{1 + w^2} \leq a \leq 1),\\
&\pm [ax^2 - ay^2 \pm \{\ \frac{2}{(1 + w)^2} + 2\sqrt{\frac{1}{(1 + w)^4} - a^2}\ \}xy](\forall\ 0 \leq a \leq \frac{1 - w}{(1 + w)(1 + w^2)}).
\end{aligned}$$

Notice that $\mathcal{P}(^nE)$ and $\mathcal{L}_s(^nE)$ are not isometric in general. It is natural to ask the following question: what are extreme points of the unit ball of $\mathcal{L}_s(^nE)$?

In 2009, the author [12] started the study of characterizing extreme points of the unit balls of $\mathcal{L}_s(^nE)$ and classified the extreme points of the unit ball of $\mathcal{L}_s(^2l_\infty^2)$. We refer to ([1–6, 8–18] and references therein) for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces.

Continuing the problem of characterizing extreme points of the unit balls of $\mathcal{L}_s(^nE)$, in this paper, we focus on the space $\mathcal{L}_s(^2d_*(1, w)^2)$. First we present the explicit formula for the norm of a symmetric bilinear form in $\mathcal{L}_s(^2d_*(1, w)^2)$. Using this formula, we can classify the extreme points of the unit ball of $\mathcal{L}_s(^2d_*(1, w)^2)$ by the method of step by step.

2. Main Results

Let $T((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s({}^2d_*(1,w)^2)$ for some reals a,b,c. For simplicity we will write $T((x_1,y_1),\ (x_2,y_2)) = (a,b,c,c)$. By substituting $((x_1,y_1),(x_2,y_2))$ in T for $((x_1,y_1),(-x_2,-y_2))$ or $((x_1,-y_1),(x_2,-y_2))$ or $((y_1,x_1),(y_2,x_2))$, we may assume that $|b| \le a,\ c \ge 0$.

Theorem 2.1. *Let* $T((x_1,y_1),\ (x_2,y_2)) := (a,b,c,c) \in \mathcal{L}_s({}^2d_*(1,w)^2)$ *with* $|b| \le a,\ c \ge 0$. *Then*

$$\|T\| = \max\{bw^2 + 2cw + a,\ a - bw^2,\ (a+b)w + c(1+w^2),\ (a-b)w + c(1-w^2)\}.$$

In fact, we have the following:
Case 1: $b \ge 0$
Subcase 1: $c > a$
If $w \le \frac{c-a}{c-b}$, *then* $\|T\| = (a+b)w + c(1+w^2)$.
If $w > \frac{c-a}{c-b}$, *then* $\|T\| = bw^2 + 2cw + a$.
Subcase 2: *If* $c \le a$, $\|T\| = bw^2 + 2cw + a$.
Case 2: $b < 0$
Subcase 1: $c < |b|$
If $w \le \frac{c}{|b|}$, *then* $\|T\| = \max\{bw^2 + 2cw + a,\ (a-b)w + c(1-w^2)\}$.
If $w > \frac{c}{|b|}$, *then* $\|T\| = \max\{a - bw^2,\ (a-b)w + c(1-w^2)\}$.
Subcase 2: $c \ge |b|$
If $w \le \frac{|b|}{c}$, *then* $\|T\| = \max\{bw^2 + 2cw + a,\ (a-b)w + c(1-w^2)\}$.
If $w > \frac{|b|}{c}$, *then* $\|T\| = \max\{bw^2 + 2cw + a,\ (a+b)w + c(1+w^2)\}$.

Proof. Since $\{(\pm1,\ \pm w),\ (\pm w,\ \pm 1)\}$ is the set of all extreme points of the unit ball of $d_*(1,w)^2$ and T is bilinear,

$$\begin{aligned}\|T\| &= \max\{|T((\pm1,\ \pm w),(\pm1,\ \pm w))|, |T((\pm1,\ \pm w),(\pm w,\ \pm 1))|,\\ &\qquad |T((\pm w,\ \pm 1),(\pm w,\ \pm 1))|\}.\end{aligned}$$

It follows that, by symmetry of T,

$$\begin{aligned}\|T\| =& \max\{|T((1,\ w),\ (1,\ w))|,\ |T((1,\ w),\ (1,\ -w))|,\ |T((1,\ -w),\ (1,\ -w))|,\\ & |T((1,\ w),\ (w,\ 1))|,\ |T((1,\ w),\ (w,\ -1))|,\ |T((1,\ -w),\ (w,\ 1))|,\\ & |T((1,\ -w),\ (w,\ -1))|,\ |T((w,\ 1),\ (w,\ 1))|,\ |T((w,\ 1),\ (w,\ -1))|,\\ & |T((w,\ -1),\ (w,\ -1))|\}\\ =& \max\{bw^2 + 2cw + a,\ a - bw^2,\ (a+b)w + c(1+w^2),\ (a-b)w + c(1-w^2)\}.\end{aligned}$$

$\square$

By Theorem 2.1, if $\|T\| = 1$, then $|a| \le 1,\ |b| \le 1,\ |c| \le \frac{1}{1+w^2}$.

Lemma 2.2. *Let $T((x_1, y_1),\ (x_2, y_2)) = (a, b, c, c) \in \mathcal{L}_s({}^2 d_*(1, w)^2)$. Then the following are equivalent:*
(a) *(a, b, c, c) is extreme.*
(b) *$(-a, -b, -c, -c)$ is extreme.*
(c) *$(a, b, -c, -c)$ is extreme.*
(d) *(b, a, c, c) is extreme.*

Proof. Let $S((x_1, y_1), (x_2, y_2)) := T((u_1, v_1), (u_2, v_2))$ for some $((u_1, v_1), (u_2, v_2)) = ((x_1, y_1), (-x_2, -y_2))$ or $((x_1, -y_1), (x_2, -y_2))$ or $((y_1, x_1), (y_2, x_2))$. Then $S \in \mathcal{L}_s({}^2 d_*(1, w)^2)$ and T is extreme if and only if S is extreme. □

Using Theorem 2.1 and Lemma 2.2 we classify the extreme symmetric bilinear forms of the unit ball of $\mathcal{L}_s({}^2 d_*(1, w)^2)$.

Theorem 2.3. *Let $T((x_1, y_1),\ (x_2, y_2)) = (a, b, c, c) \in \mathcal{L}_s({}^2 d_*(1, w)^2)$. Then:*
(a) *Let $w < \sqrt{2} - 1$. Then (a, b, c, c) is extreme if and only if*

$$\begin{aligned}(a, b, c, c) \in \{&\pm(1, 0, 0, 0), \pm(0, 1, 0, 0), \pm\frac{1}{1+w^2}(1, 1, 0, 0),\\ &\pm\frac{1}{(1+w)^2}(1, 1, \pm 1, \pm 1), \pm\frac{1}{1+w^2}(1, -1, \pm w, \pm w), \pm\frac{1}{1+w^2}(w, -w, \pm 1, \pm 1),\\ &\pm\frac{1}{1+2w-w^2}(1, -1, \pm 1, \pm 1), \pm\frac{1}{(1+w)^2(1-w)}(1 - w - w^2, -w, \pm 1, \pm 1),\\ &\pm\frac{1}{(1+w)^2(1-w)}(w, -(1 - w - w^2), \pm 1, \pm 1)\}.\end{aligned}$$

(b) *Let $w = \sqrt{2} - 1$. Then (a, b, c, c) is extreme if and only if*

$$\begin{aligned}(a, b, c, c) \in \{&\pm(1, 0, 0, 0), \pm(0, 1, 0, 0), \pm\frac{2+\sqrt{2}}{4}(1, 1, 0, 0), \pm\frac{1}{2}(1, 1, \pm 1, \pm 1),\\ &\pm\frac{\sqrt{2}}{4}(1, -1, \pm(\sqrt{2}+1), \pm(\sqrt{2}+1)), \pm\frac{\sqrt{2}}{4}(\sqrt{2}+1, -(\sqrt{2}+1), \pm 1, \pm 1)\}.\end{aligned}$$

(c) *Let $w > \sqrt{2} - 1$. Then (a, b, c, c) is extreme if and only if*

$$\begin{aligned}(a, b, c, c) \in \{&\pm(1, 0, 0, 0), \pm(0, 1, 0, 0), \pm\frac{1}{1+w^2}(1, 1, 0, 0),\\ &\pm\frac{1}{(1+w)^2}(1, 1, \pm 1, \pm 1), \pm\frac{1}{1+2w-w^2}(1, -1, \pm 1, \pm 1),\\ &\pm\frac{1}{1+w^2}(1, -1, \pm\frac{1-w}{1+w}, \pm\frac{1-w}{1+w}), \pm\frac{1}{1+w^2}(\frac{1-w}{1+w}, -\frac{1-w}{1+w}, \pm 1, \pm 1),\\ &\pm\frac{1}{2+2w}(2 + w, -\frac{1}{w}, \pm 1, \pm 1), \pm\frac{1}{2+2w}(\frac{1}{w}, -(2 + w), \pm 1, \pm 1)\}.\end{aligned}$$

Proof. Suppose T is extreme. By Lemma 2.2, without loss of generality, we may assume that $|b| \leq a,\ c \geq 0$. We will find out T by considering and checking all the cases of a, b, c in the statements of Theorem 2.1.

Case 1: $b \geq 0$

We will show that if $b \geq 0$, then $0 \leq c \leq a$. Assume $c > a \geq 0$. Then $(c > a = b \geq 0)$ or $(c > a > b \geq 0, w < \frac{c-a}{c-b})$ or $(c > a > b \geq 0, w > \frac{c-a}{c-b})$ or $(c > a > b \geq 0, w = \frac{c-a}{c-b})$.

If $c > a = b$, then $1 = \|T\| = 2aw + c(1+w^2)$. If $c > a = b = 0$, let $0 < \epsilon < \min\{\frac{w}{1+w^2}, \frac{1-w}{(1+w)(1+w^2)}\}$ and $R = (\epsilon, -\epsilon, \frac{1}{1+w^2}, \frac{1}{1+w^2}), S = (-\epsilon, \epsilon, \frac{1}{1+w^2}, \frac{1}{1+w^2})$. Then $\|R\| = 1 = \|S\|, T = \frac{1}{2}(R+S)$, which is impossible because T is extreme. If $c > a = b > 0$, let $\epsilon > 0$ such that $0 < a - \epsilon < a + \epsilon < c, w < \frac{c-a-\epsilon}{c-a+\epsilon}$ and $R = T + (\epsilon, -\epsilon, 0, 0), S = T - (\epsilon, -\epsilon, 0, 0)$. Then $\|R\| = 1 = \|S\|, T = \frac{1}{2}(R+S)$, which is impossible.

If $c > a > b > 0, w < \frac{c-a}{c-b}$, let $\epsilon > 0$ such that

$$0 < b - \epsilon < b + \epsilon < a - \epsilon < a + \epsilon < c,\ w < \frac{c-a-\epsilon}{c-b+\epsilon}.$$

Let $R = T + \epsilon(1, -1, 0, 0), S = T - \epsilon(1, -1, 0, 0)$. By Theorem 2.1, $\|R\| = 1 = \|S\|,\ T = \frac{1}{2}(R+S)$, which is impossible.

If $c > a > b = 0, w < \frac{c-a}{c-b}$, let $\epsilon > 0$ such that

$$0 < a - \epsilon < a + \epsilon < c - \frac{w}{1+w^2}\epsilon,\ w < \frac{c - a - \frac{1+w+w^2}{1+w^2}\epsilon}{c + \frac{w}{1+w^2}\epsilon}.$$

Let $R = T + \epsilon(1, 0, -\frac{w}{1+w^2}, -\frac{w}{1+w^2}), S = T - \epsilon(1, 0, -\frac{w}{1+w^2}, -\frac{w}{1+w^2})$. Then $\|R\| = 1 = \|S\|, T = \frac{1}{2}(R+S)$, which is impossible.

If $c > a > b > 0, w > \frac{c-a}{c-b}$, let $\epsilon > 0$ such that

$$0 < b - \frac{1}{w^2}\epsilon < b + \frac{1}{w^2}\epsilon < a - \epsilon < a + \epsilon < c - \frac{1}{w}\epsilon,\ w > \frac{c - a + (\frac{1}{w}+1)\epsilon}{c - b - (\frac{1}{w} + \frac{1}{w^2})\epsilon}.$$

Let $R = T + \epsilon(1, \frac{1}{w^2}, -\frac{1}{w}, -\frac{1}{w}), S = T - \epsilon(1, \frac{1}{w^2}, -\frac{1}{w}, -\frac{1}{w})$. Then $\|R\| = 1 = \|S\|, T = \frac{1}{2}(R+S)$, which is impossible. If $c > a > b = 0, w > \frac{c-a}{c-b}$, let $\epsilon > 0$ such that

$$0 < a - \epsilon < a + \epsilon < c - \frac{1}{2w}\epsilon,\ w > \frac{c - a + (\frac{1}{2w}+1)\epsilon}{c - \frac{1}{2w}\epsilon}.$$

Let $R = T + \epsilon(1, 0, -\frac{1}{2w}, -\frac{1}{2w}), S = T - \epsilon(1, 0, -\frac{1}{2w}, -\frac{1}{2w})$. Then $\|R\| = 1 = \|S\|, T = \frac{1}{2}(R+S)$, which is impossible.

If $c > a > b, w = \frac{c-a}{c-b}$, then $T = (a, \frac{1}{w^2}a + \frac{w-1}{w^2(1+w)}, -\frac{1}{w}a + \frac{1}{w(1+w)}, -\frac{1}{w}a + \frac{1}{w(1+w)})$ with $\frac{1-w}{1+w} \leq a < \frac{1}{(1+w)^2}$. If $a = \frac{1-w}{1+w}$, then $T = \frac{1}{1+w}(1-w, 0, 1, 1)$. There exists $\epsilon > 0$ such that $\|R\| = 1 = \|S\|$, where $R = T + \epsilon(1, \frac{1}{w^2}, -\frac{1}{w}, -\frac{1}{w}), S = T - \epsilon(1, \frac{1}{w^2}, -\frac{1}{w}, -\frac{1}{w})$, which is impossible. If $\frac{1-w}{1+w} < a < \frac{1}{(1+w)^2}$, let $a_1, a_2 \in \mathbb{R}$ such that $\frac{1-w}{1+w} < a_1 < a < a_2 < \frac{1}{(1+w)^2}$, $a = \frac{1}{2}(a_1 + a_2)$. Define $R = (a_1, \frac{1}{w^2}a_1 + \frac{w-1}{w^2(1+w)}, -\frac{1}{w}a_1 + \frac{1}{w(1+w)}, -\frac{1}{w}a_1 + \frac{1}{w(1+w)})$ and $S = (a_2, \frac{1}{w^2}a_2 + \frac{w-1}{w^2(1+w)}, -\frac{1}{w}a_2 + \frac{1}{w(1+w)}, -\frac{1}{w}a_2 + \frac{1}{w(1+w)})$. Then $\|R\| = 1 = \|S\|, T = \frac{1}{2}(R + S)$, which is impossible. Therefore, if $b \geq 0$, then $0 \leq c \leq a$. In this case we will show that $T = \frac{1}{1+w^2}(1, 1, 0, 0)$ or $(1, 0, 0, 0)$ or $\frac{1}{(1+w)^2}(1, 1, 1, 1)$. Suppose that $b \geq 0$ and $0 \leq c \leq a$. Then $(b \geq 0, 0 \leq c < a)$ or $(b \geq 0, 0 \leq c = a)$, which are divided into nine cases; $(c = 0,\ 0 < b < a)$ or $(c = 0, 0 < b = a)$ or $(c = 0, 0 = b < a)$ or $(0 < c < a, 0 < b < a)$ or $(0 < c < a, 0 < b = a)$ or $(0 < c < a, 0 = b < a)$ or $(c = 0, 0 < b < a)$ or $(c = 0, 0 < b = a)$ or $(c = 0, 0 = b < a)$.

If $c = 0$ and $0 < b < a$, let $\epsilon > 0$ such that

$$0 < b - \frac{1}{w^2}\epsilon < b + \frac{1}{w^2}\epsilon < a - \epsilon < a + \epsilon < 1.$$

Let $R = T + \epsilon(1, -\frac{1}{w^2}, 0, 0), S = T - \epsilon(1, -\frac{1}{w^2}, 0, 0)$. Then $\|R\| = 1 = \|S\|$, $T = \frac{1}{2}(R + S)$, which is impossible. If $c = 0, 0 < b = a$, then $T = \frac{1}{1+w^2}(1, 1, 0, 0)$. We will show that T is extreme. Let $R = T + (\epsilon, \gamma, \delta, \delta), S = T - (\epsilon, \gamma, \delta, \delta)$ for some $\epsilon \geq 0, \gamma, \delta \in \mathbb{R}$ with $\|R\| = 1 = \|S\|$. Since $1 \geq |R((1, w), (1, w))|$ and $1 \geq |S((1, w), (1, w))|$, $\epsilon + w^2\gamma + 2w\delta = 0$. Since $1 \geq |R((w, 1), (w, 1))|$ and $1 \geq |S((w, 1), (w, 1))|$, $w^2\epsilon + \gamma + 2w\delta = 0$. Hence, $\gamma = \epsilon$. Since $1 \geq |R(1, -w), (1, -w))|$ and $1 \geq |S((1, -w), (1, -w))|$, $\epsilon + w^2\gamma - 2w\delta = 0$, which shows that $\epsilon = 0 = \gamma = \delta$. If $c = 0$ and $0 = b < a$, then $T = (1, 0, 0, 0)$ is extreme. Indeed, let $R = T + (\epsilon, \gamma, \delta, \delta), S = T - (\epsilon, \gamma, \delta, \delta)$ for some $\epsilon \geq 0, \gamma, \delta \in \mathbb{R}$ with $\|R\| = 1 = \|S\|$. Since

$$1 \geq |R((1, 0), (1, 0))| = 1 + \epsilon,$$

$\epsilon = 0$. Since $1 \geq |R((1, \pm w), (1, \pm w))|$ and $1 \geq |S((1, \pm w), (1, \pm w))|$, $\gamma = 0 = \delta$.

If $0 < c < a$ and $0 < b < a$, let $\epsilon > 0$ such that

$$0 < b - \frac{1}{w^2}\epsilon < b + \frac{1}{w^2}\epsilon < a - \epsilon,\ 0 < c - \frac{1}{w}\epsilon < c + \frac{1}{w}\epsilon < a - \epsilon < a + \epsilon < 1.$$

Let $R = T + \epsilon(1, \frac{1}{w^2}, -\frac{1}{w}, -\frac{1}{w}), S = T - \epsilon(1, \frac{1}{w^2}, -\frac{1}{w}, -\frac{1}{w})$. Then $\|R\| = 1 = \|S\|, T = \frac{1}{2}(R + S)$, which is impossible.

If $0 < c < a = b$ and $0 < b < a$, then

$$T = (a, a, \frac{1 - a(1 + w^2)}{2w}, \frac{1 - a(1 + w^2)}{2w}) \text{ for } \frac{1}{(1 + w)^2} < a < \frac{1}{1 + w^2}.$$

Let $a_1, a_2 \in \mathbb{R}$ such that $\frac{1}{(1+w)^2} < a_1 < a < a_2 < \frac{1}{1+w^2}, a = \frac{a_1+a_2}{2}$. Let $R_i = (a_i, a_i, \frac{1-a_i(1+w^2)}{2w}, \frac{1-a_i(1+w^2)}{2w})$ for $i = 1, 2$. Then $\|R_i\| = 1$ and $T = \frac{1}{2}(R_1 + R_2)$, which is impossible.

If $0 < c < a = b$ and $0 = b < a$, let $\epsilon > 0$ such that

$$0 < c - \frac{1}{2w}\epsilon < c + \frac{1}{2w}\epsilon < a - \epsilon < a + \epsilon < 1.$$

Let $R = T + \epsilon(1, 0, -\frac{1}{2w}, -\frac{1}{2w}), S = T - \epsilon(1, 0, -\frac{1}{2w}, -\frac{1}{2w})$. Then $\|R\| = 1 = \|S\|,\ T = \frac{1}{2}(R + S)$, which is impossible.

If $c = a$ and $0 < b < a$, let $\epsilon > 0$ such that

$$\frac{\epsilon}{a - b - \frac{1}{w^2}\epsilon} < w,\ 0 < b - \frac{1}{w^2}\epsilon < b + \frac{1}{w^2}\epsilon < a - \epsilon < a + \epsilon < 1.$$

Let $R = T + \epsilon(1, -\frac{1}{w^2}, 0, 0), S = T - \epsilon(1, -\frac{1}{w^2}, 0, 0)$. Then $\|R\| = 1 = \|S\|,\ T = \frac{1}{2}(R + S)$, which is impossible.

If $c = a$ and $0 = b < a$, then $T = \frac{1}{1+2w}(1, 0, 1, 1)$. We show that T is not extreme. Indeed, let $\epsilon > 0$ such that

$$\frac{1}{1 + 2w} > \max\{\epsilon, \frac{1}{2w}\epsilon\}, w > \frac{(1 + 2w)^2\epsilon}{2w + (1 + 2w)\epsilon}.$$

Let $R = T + \epsilon(1, 0, -\frac{1}{2w}, -\frac{1}{2w}), S = T - \epsilon(1, 0, -\frac{1}{2w}, -\frac{1}{2w})$. Then $\|R\| = 1 = \|S\|,\ T = \frac{1}{2}(R + S)$, which shows that T is not extreme.

If $c = a$ and $0 < b = a$, then $T = \frac{1}{(1+w)^2}(1, 1, 1, 1)$. We will show that T is extreme. Let $R = T + (\epsilon, \gamma, \delta, \delta), S = T - (\epsilon, \gamma, \delta, \delta)$ for some $\epsilon \geq 0, \gamma, \delta \in \mathbb{R}$ with $\|R\| = 1 = \|S\|$. Since $1 \geq |R((1, w), (1, w))|$ and $1 \geq |S((1, w), (1, w))|$, $\epsilon + w^2\gamma + 2w\delta = 0$. Since $1 \geq |R((w, 1), (w, 1))|$ and $1 \geq |S((w, 1), (w, 1))|$, $w^2\epsilon + \gamma + 2w\delta = 0$. Hence, $\gamma = \epsilon$ and $\delta = -\frac{1+w^2}{2w}\epsilon$. Note that $1 \geq |S((1, w), (w, 1))| = 1 + \frac{(1-w^2)^2}{2w}\epsilon$, which shows that $\epsilon = 0 = \gamma = \delta$.

Case 2: $b < 0$

Notice that $0 < a < 1$ and $c < |b|$ or $c \geq |b|$. Suppose $c < |b|$. Then we have 5 cases; $(c < |b| < a, w < \frac{c}{|b|})$ or $(c < |b| < a, w > \frac{c}{|b|})$ or $(c < |b| < a, w = \frac{c}{|b|})$ or $(c < |b| = a, w \leq \frac{c}{|b|})$ or $(c < |b| = a, w > \frac{c}{|b|})$.

If $c < |b| < a, w < \frac{c}{|b|}$, we can find $\epsilon > 0$ such that

$$c + \frac{1 + w^2}{3w - w^3}\epsilon < |b| - \frac{|1 - 3w^2|}{3w^2 - w^4}\epsilon < |b| + \frac{|1 - 3w^2|}{3w^2 - w^4}\epsilon < a - \epsilon,$$

$$b + \frac{|1 - 3w^2|}{3w^2 - w^4}\epsilon < 0, \quad w < \frac{c - \frac{1+w^2}{3w-w^3}\epsilon}{|b| + \frac{|1-3w^2|}{3w^2-w^4}\epsilon}.$$

Let $R = T + \epsilon(1, \frac{1-3w^2}{w^4-3w^2}, \frac{1+w^2}{w^3-3w}, \frac{1+w^2}{w^3-3w})$ and $S = T - \epsilon(1, \frac{1-3w^2}{w^4-3w^2}, \frac{1+w^2}{w^3-3w}, \frac{1+w^2}{w^3-3w})$. Then $\|R\| = 1 = \|S\|,\ T = \frac{1}{2}(R + S)$, which is impossible.

If $c < |b| < a, w > \frac{c}{|b|}$, we can find $\epsilon > 0$ such that

$$c + \frac{1}{w}\epsilon < |b| - \frac{1}{w^2}\epsilon < |b| + \frac{1}{w^2}\epsilon < a - \epsilon,\ b + \frac{1}{w^2}\epsilon < 0,\ w > \frac{c + \frac{1}{w}\epsilon}{|b| - \frac{1}{w^2}\epsilon}.$$

Let $R = T + \epsilon(1, \frac{1}{w^2}, \frac{1}{w}, \frac{1}{w})$ and $S = T - \epsilon(1, \frac{1}{w^2}, \frac{1}{w}, \frac{1}{w})$. Then $\|R\| = 1 = \|S\|,\ T = \frac{1}{2}(R + S)$, which is impossible.

If $c < |b| < a, w = \frac{c}{|b|}$, then $T = \frac{1}{2+2w}(2 + w, -\frac{1}{w}, 1, 1)$ for $w > \sqrt{2} - 1$. Indeed, note that

$$1 = \|T\| = \max\{bw^2 + 2wc + a, (a - b)w + c(1 - w^2)\}.$$

Hence, $bw^2 + 2wc + a = 1 = (a - b)w + c(1 - w^2)$. Then $T = \frac{1}{2+2w}(2 + w, -\frac{1}{w}, 1)$ for $w > \sqrt{2} - 1$. We will show that $T = \frac{1}{2+2w}(2 + w, -\frac{1}{w}, 1, 1)$ with $w > \sqrt{2} - 1$ is extreme. Define $R = T + (\epsilon, \gamma, \delta, \delta)$ and $S = T - (\epsilon, \gamma, \delta, \delta)$ for some $\epsilon \ge 0, \gamma, \delta \in \mathbb{R}$ with $\|R\| = 1 = \|S\|$. Then $\gamma = \frac{3w^2-1}{w^2(3-w^2)}\epsilon, \delta = \frac{1+w^2}{w^3-3w}\epsilon$. Since, by Theorem 2.1,

$$1 = \|R\| \ge 1 + \frac{4 - 4w^2}{3 - w^2}\epsilon,$$

so $\epsilon = 0 = \gamma = \delta$.

Suppose that $c < |b| = a, w \le \frac{c}{|b|}$. If $a(1 - w^2) + 2wc = 1 = c(1 - w^2) + 2wa$, then $T = \frac{1}{1+2w-w^2}(1, -1, 1, 1)$, which is impossible since $c < a$. Therefore, $1 = a(1 - w^2) + 2wc > c(1 - w^2) + 2wa$ or $a(1 - w^2) + 2wc < c(1 - w^2) + 2wa = 1$. If $1 = a(1 - w^2) + 2wc > c(1 - w^2) + 2wa$, then

$$T = (a, -a, \frac{1 - a(1 - w^2)}{2w}, \frac{1 - a(1 - w^2)}{2w})$$

for $\frac{1}{1+w^2} \le a < \frac{w^2}{(1+2w-w^2)(-1+2w+w^2)}$. If $a = \frac{1}{1+w^2}$, then $T = \frac{1}{1+w^2}(1, -1, w, w)$ with $w < \sqrt{2} - 1$ is extreme since

$$\begin{aligned} 1 &= |T((1, w), (1, w))| = |T((1, w), (1, -w))| = |T((w, 1), (w, -1))| \\ &= |T((w, -1), (w, -1))|. \end{aligned}$$

If $a(1 - w^2) + 2wc < c(1 - w^2) + 2wa = 1$, then

$$T = (a, -a, \frac{1 - 2wa}{1 - w^2}, \frac{1 - 2wa}{1 - w^2}) \text{ for } 0 < a < \frac{w^2}{1 + 2w - w^2},$$

which is impossible. If $c < |b| = a, w > \frac{c}{|b|}$, then $a(1 + w^2) = 1 = c(1 - w^2) + 2wa$ and $T = \frac{1}{1+w^2}(1, -1, \frac{1-w}{1+w}, \frac{1-w}{1+w})$ with $w \ge \sqrt{2} - 1$. We will show that T is extreme since

$$1 = |T((1, w), (1, -w))| = |T((1, -w), (w, 1))| = |T((w, 1), (w, -1))|.$$

Suppose $c \geq |b|$. Then $c > |b|$ or $c = |b|$. Suppose $c > |b|$. Then we have 9 cases; $(c > a > |b|, w < \frac{|b|}{c})$ or $(c > a > |b|, w > \frac{|b|}{c})$ or $(c > a > |b|, w = \frac{|b|}{c})$ or $(c > a = |b|, w < \frac{|b|}{c})$ or $(c > a = |b|, w > \frac{|b|}{c})$ or $(c > a = |b|, w = \frac{|b|}{c})$ or $(c = a > |b|)$ or $(a > c > |b|, w > \frac{|b|}{c})$ or $(a > c > |b|, w \leq \frac{|b|}{c})$.

If $c > a > |b|, w < \frac{|b|}{c}$, we can find $\epsilon > 0$ such that

$$c - \frac{1+w^2}{3w-w^3}\epsilon > a+\epsilon > a-\epsilon > |b| + \frac{|1-3w^2|}{3w^2-w^4}\epsilon,\ w < \frac{|b| - \frac{|1-3w^2|}{3w^2-w^4}\epsilon}{c + \frac{1+w^2}{3w-w^3}\epsilon}.$$

Let $R = T + \epsilon(1, \frac{1-3w^2}{w^4-3w^2}, \frac{1+w^2}{w^3-3w}, \frac{1+w^2}{w^3-3w})$ and $S = T - \epsilon(1, \frac{1-3w^2}{w^4-3w^2}, \frac{1+w^2}{w^3-3w}, \frac{1+w^2}{w^3-3w})$. Then $\|R\| = 1 = \|S\|,\ T = \frac{1}{2}(R+S)$, which is impossible.

If $c > a > |b|, w > \frac{|b|}{c}$, we can find an $\epsilon > 0$ such that

$$c - \frac{1}{w}\epsilon > a+\epsilon > a-\epsilon > |b| + \frac{1}{w^2}\epsilon,\ w > \frac{|b| + \frac{1}{w^2}\epsilon}{c - \frac{1}{w}\epsilon}.$$

Let $R = T + \epsilon(1, \frac{1}{w^2}, -\frac{1}{w}, -\frac{1}{w})$ and $S = T - \epsilon(1, \frac{1}{w^2}, -\frac{1}{w}, -\frac{1}{w})$. Then $\|R\| = 1 = \|S\|,\ T = \frac{1}{2}(R+S)$, which is impossible. If $c > a > |b|, w = \frac{|b|}{c}$, then

$$1 = -|b|w^2 + 2cw + a = -cw^2 + 2cw + a, 1 = (a+|b|)w + c(1-w^2) = c + aw.$$

Hence, $T = \frac{1}{(1+w)^2(1-w)}(1-w-w^2, -w, 1, 1)$ for $w < \sqrt{2}-1$. We will show that T is extreme. Let $R = T + (\epsilon, \gamma, \delta, \delta), S = T - (\epsilon, \gamma, \delta, \delta)$ for some $\epsilon \geq 0, \gamma, \delta \in \mathbb{R}$ with $\|R\| = 1 = \|S\|$. Since $1 \geq |R((1,w),(1,w))|$ and $1 \geq |S((1,w),(1,w))|$, $\epsilon + w^2\gamma + 2w\delta = 0$. Since $1 \geq |R((1,w),(w,1))|$ and $1 \geq |S((1,w),(w,1))|$, $w\epsilon + w\gamma + (1+w^2)\delta = 0$. Since $1 \geq |R((1,-w),(w,1))|$ and $1 \geq |S((1,-w),(w,1))|$, $w\epsilon - w\gamma + (1-w^2)\delta = 0$, which shows that $\epsilon = 0 = \gamma = \delta$.

Suppose that $c > |b| = a, w < \frac{|b|}{c}$. We will show that $T = \frac{1}{1+w^2}(w, -w, 1, 1)$ for $w < \sqrt{2}-1$ or $T = \frac{1}{1+w^2}(\frac{1-w}{1+w}, \frac{-1+w}{1+w}, 1, 1)$ for $w \geq \sqrt{2}-1$. If $1 = a(1-w^2) + 2cw = c(1-w^2) + 2aw$, then $w = \sqrt{2}-1$ and

$$T = (a, -a, \frac{\sqrt{2}+1}{2} - a, \frac{\sqrt{2}+1}{2} - a) \ \text{ for } \frac{\sqrt{2}}{4} < a < \frac{\sqrt{2}+1}{4},$$

which is impossible. Therefore, $1 = a(1-w^2) + 2cw > c(1-w^2) + 2aw$ or $a(1-w^2) + 2cw < c(1-w^2) + 2aw = 1$. If $1 = a(1-w^2) + 2cw > c(1-w^2) + 2aw$, then $w > \sqrt{2}-1$ and

$$T = (a, -a, \frac{1-a(1-w^2)}{2w}, \frac{1-a(1-w^2)}{2w}) \ \text{ for } \frac{1}{3-w^2} < a < \frac{1}{1+2w-w^2},$$

which is impossible. If $a(1-w^2) + 2cw < c(1-w^2) + 2aw = 1$, then $w < \sqrt{2}-1$ and

$$T = (a, -a, \frac{1-2aw}{1-w^2}, \frac{1-2aw}{1-w^2}) \ \text{ for } \frac{w}{1+w^2} < a < \frac{1}{1+2w-w^2},$$

which is impossible. Suppose $c > |b| = a, w \geq \frac{a}{c}$. If $w > \frac{a}{c}$ and $1 = a(1-w^2)+2cw > c(1+w^2)$, then $w > \sqrt{2}-1$ and

$$T = (a, -a, \frac{1-a(1-w^2)}{2w}, \frac{1-a(1-w^2)}{2w}) \text{ for } \frac{1-w}{(1+w^2)(1+w)} < a < \frac{1}{3-w^2},$$

which is impossible. If $w = \frac{a}{c}$ and $1 = a(1-w^2)+2cw > c(1+w^2)$, then $w > \sqrt{2}-1$ and $T = (\frac{1}{3-w^2}, -\frac{1}{3-w^2}, \frac{1}{w(3-w^2)})$, which is impossible since, by Theorem 2.1, we can choose $\epsilon > 0$ such that $\|R\| = 1 = \|S\|$, where

$$R = (\frac{1}{3-w^2} + \epsilon, -\frac{1}{3-w^2} - \epsilon, \frac{1}{w(3-w^2)} - \frac{1-w^2}{2w}\epsilon, \frac{1}{w(3-w^2)} - \frac{1-w^2}{2w}\epsilon)$$

and

$$S = (\frac{1}{3-w^2} - \epsilon, -\frac{1}{3-w^2} + \epsilon, \frac{1}{w(3-w^2)} + \frac{1-w^2}{2w}\epsilon, \frac{1}{w(3-w^2)} + \frac{1-w^2}{2w}\epsilon).$$

Suppose that $w > \frac{a}{c}$ and $a(1-w^2) + 2cw < c(1+w^2) = 1$. Then $w < \sqrt{2}-1$ and

$$T = (a, -a, \frac{1}{1+w^2}, \frac{1}{1+w^2}) \text{ for } 0 < a < \frac{w}{1+w^2},$$

which is impossible. If $w = \frac{a}{c}$ and $a(1-w^2) + 2cw < c(1+w^2) = 1$, then $w < \sqrt{2}-1$ and $T = (\frac{w}{1+w^2}, -\frac{w}{1+w^2}, \frac{1}{1+w^2}, \frac{1}{1+w^2})$. We will show that $T = (\frac{w}{1+w^2}, -\frac{w}{1+w^2}, \frac{1}{1+w^2}, \frac{1}{1+w^2})$ for $w < \sqrt{2}-1$ is extreme. Indeed, let $R = T+(\epsilon, \gamma, \delta, \delta)$ and $S = T - (\epsilon, \gamma, \delta, \delta)$ with $\|R\| = 1 = \|S\|$ for $\epsilon \geq 0, \gamma, \delta \in \mathbb{R}$. Since

$$1 = |T((1, w), (w, 1))| = |T((1, -w), (w, 1))| = |T((1, -w), (w, -1))|,$$

$$\begin{aligned} 0 &= \epsilon w + \gamma w + \delta(1+w^2) \\ 0 &= \epsilon w - \gamma w + \delta(1-w^2) \\ 0 &= \epsilon w + \gamma w - \delta(1+w^2), \end{aligned}$$

which imply that $0 = \epsilon = \gamma = \delta$. Suppose that $w \geq \frac{a}{c}$ and $1 = a(1-w^2) + 2cw = c(1+w^2)$. Then $T = \frac{1}{1+w^2}(\frac{1-w}{1+w}, \frac{-1+w}{1+w}, 1, 1)$ for $w \geq \sqrt{2}-1$. We will show that $T = \frac{1}{1+w^2}(\frac{1-w}{1+w}, \frac{-1+w}{1+w}, 1, 1)$ is extreme if and only if $w \geq \sqrt{2}-1$

By Theorem 2.1, if $w < \sqrt{2}-1$, then $\|T\| > 1$, so T can not be extreme. Suppose that $w \geq \sqrt{2}-1$. Let $R = T+(\epsilon, \gamma, \delta, \delta), S = T-(\epsilon, \gamma, \delta, \delta)$ for some $\epsilon \geq 0, \gamma, \delta \in \mathbb{R}$ with $\|R\| = 1 = \|S\|$. Then $\gamma = -\epsilon, \delta = \frac{w^2-1}{2w}$. Since

$$1 \geq \|S\| \geq 1 + \frac{1-w^4}{2w}\epsilon,$$

so $\epsilon = 0 = \gamma = \delta$.

If $c = a > |b|$, then

$$1 = \|T\| = -|b|w^2 + 2cw + a > \max\{(a+|b|)w + c(1-w^2), (a-|b|)w + c(1+w^2)\}.$$

From it, there exists a sufficiently small $\epsilon > 0$ such that $\|R\| = 1 = \|S\|$, if $R = T + \epsilon(1, 0, -\frac{1}{2w}, -\frac{1}{2w})$ and $S = T - \epsilon(1, 0, -\frac{1}{2w}, -\frac{1}{2w})$, which is impossible.

If $a > c > |b|, w > \frac{|b|}{c}$, then

$$\begin{aligned} 1 &= \|T\| = -|b|w^2 + 2cw + a \\ &> \max\{(a+|b|)w + c(1-w^2), (a-|b|)w + c(1+w^2), a - bw^2\}, \end{aligned}$$

which is impossible. If $a > c > |b|, w \leq \frac{|b|}{c}$, then we claim that $1 = -|b|w^2 + 2cw + a > (a+|b|)w + c(1-w^2)$ or $1 = (a+|b|)w + c(1-w^2) > -|b|w^2 + 2cw + a$. Otherwise. Then $T = (a, \frac{1-2w-w^2}{w^2(3-w^2)} + \frac{3w^2-1}{w^2(3-w^2)}a, \frac{1+w}{w(3-w^2)} - \frac{1+w^2}{w(3-w^2)}a, \frac{1+w}{w(3-w^2)} - \frac{1+w^2}{w(3-w^2)}a)$ and $\frac{1}{1+2w-w^2} < a < \frac{1}{1+2w-w^2}$, which is impossible. Since $1 = -|b|w^2 + 2cw + a > \max\{(a+|b|)w + c(1-w^2), a - bw^2\}$ or $1 = (a+|b|)w + c(1-w^2) > \max\{-|b|w^2 + 2cw + a, (a-|b|)w + c(1+w^2)\}$, T is not extreme, which is a contradiction. Suppose $c = |b|$. Then $c = |b| < a$ or $c = |b| = a$. If $|b| = c < a$, then

$$\begin{aligned} 1 &= \|T\| = bw^2 + 2cw + a = -cw^2 + 2cw + a \\ &> \max\{a - bw^2, (a+b)w + c(1+w^2), (a-b)w + c(1-w^2)\}. \end{aligned}$$

We can find $\epsilon > 0$ such that $\|R\| = 1 = \|S\|$, where $R = T + \epsilon(1, -\frac{1}{w^2}, 0, 0), S = T - \epsilon(1, -\frac{1}{w^2}, 0, 0)$, which is impossible. If $c = a = |b|$, then $T = \frac{1}{1+2w-w^2}(1, -1, 1, 1)$ for $w \neq \sqrt{2} - 1$. We will show that $T = \frac{1}{1+2w-w^2}(1, -1, 1, 1) \in extB_{\mathcal{L}_s({}^2d_*(1,w))}$ for $w \neq \sqrt{2} - 1$. Indeed, let $R = T + (\epsilon, \gamma, \delta, \delta), S = T - (\epsilon, \gamma, \delta, \delta)$ for some $\epsilon \geq 0, \gamma, \delta \in \mathbb{R}$ with $\|R\| = 1 = \|S\|$. Since $1 \geq |R((1,w),(1,w))|$ and $1 \geq |S((1,w),(1,w))|$, $\epsilon + w^2\gamma + 2w\delta = 0$. Since $1 \geq |R((w,-1),(w,-1))|$ and $1 \geq |S((w,-1),(w,-1))|$, $w^2\epsilon + \gamma - 2w\delta = 0$. Hence, $\gamma = -\epsilon$ and $\delta = \frac{w^2-1}{2w}\epsilon$. Since $1 \geq |R((-1,w),(w,1))|$ and $1 \geq |S((-1,w),(w,1))|$, $\epsilon(\frac{(1-w^2)^2}{2w} - 2w) = 0$, which shows that $\epsilon = 0 = \gamma = \delta$. Therefore, it completes the proof. $\square$

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