KYUNGPOOK Math. J. 53(2013), 257-271
http://dx.doi.org/10.5666/KMJ.2013.53.2.257

Multiple Unbounded Positive Solutions for the Boundary Value Problems of the Singular Fractional Differential Equations

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Abstract. In this article, we establish the existence of at least three unbounded positive solutions to a boundary-value problem of the nonlinear singular fractional differential equation. Our analysis relies on the well known fixed point theorems in the cones.

1. Introduction

Fractional differential equations have many applications in modeling of physical and chemical processes and in engineering and have been of great interest recently. In its turn, mathematical aspects of studies on fractional differential equations were discussed by many authors, see the text books [1,2], the survey papers [3,4] and papers [5-12] and the references therein.

The use of cone theoretic techniques in the study of the existence of solutions to

* Corresponding Author.
Received April 22, 2011; accepted August 1, 2011.
2010 Mathematics Subject Classification: 92D25, 34A37, 34K15.
Key words and phrases: Positive solution, fractional differential equation, fixed-point theorem.
The first and second authors are supported by the Natural Science Foundation of Guangdong province (No: S2011010001900) and the Guangdong Higher Education Foundation for High-level talents and the third author supported by the Natural Science Foundation of Hunan province (No: 12JJ6006) and the Science Foundation of Department of Science and Technology of Hunan province (No. 2012FJ3107).

boundary value problems has a rich and diverse history. Motivated by this reason, in this paper, we discuss the existence of three positive solutions to the boundary value problem of the nonlinear fractional differential equation of the form

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t))=0,\ t\in(0,1), 1<\alpha<2,\\ \left[I_{0+}^{2-\alpha}u(t)\right]'\Big|_{t=0}=0\,,\\ u(1)=0, \end{cases} \tag{1.1}$$

where D_{0+}^{α} (D^{α} for short) is the Riemann-Liouville fractional derivative of order α, and $f:(0,1)\times[0,\infty)\to[0,\infty)$ is continuous, f may be singular at $t=0$ or $t=1$.

We obtain the existence results for two and three unbounded positive solutions about this boundary-value problem by using the fixed point theorems in the cones.

2. Preliminary results

For the convenience of the reader, we present here the necessary definitions from fixed point theory and fractional calculus theory. These definitions and results can be found in the literatures [13] and [1,2].

Definition 2.1. Let X be a real Banach space. The nonempty convex closed subset P of X is called a cone in X if $ax\in P$ for all $x\in P$ and $a\geq 0$, $x\in X$ and $-x\in X$ imply $x=0$.

Definition 2.2. A map $\psi:P\to[0,+\infty)$ is a nonnegative continuous concave or convex functional map provided ψ is nonnegative, continuous and satisfies

$$\psi(tx+(1-t)y)\geq t\psi(x)+(1-t)\psi(y),$$

or

$$\psi(tx+(1-t)y)\leq t\psi(x)+(1-t)\psi(y),$$

for all $x,y\in P$ and $t\in[0,1]$.

Definition 2.3. An operator $T:X\to X$ is completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Let ψ be a nonnegative functional on a cone P of a real Banach space X. Define the sets by

$$\begin{aligned} &P_r=\{y\in P:\ \ ||y||<r\},\\ &P(\psi;a,b)=\{y\in P:\ \ a\leq\psi(y),\ ||y||<b\},\\ &P(\psi,d):=\{x\in P:\psi(x)<d\}. \end{aligned}$$

Lemma 2.1. *Let $T:\ \overline{P}_c\to\overline{P}_c$ be a completely continuous operator and let ψ be a nonnegative continuous concave functional on P such that $\psi(y)\leq||y||$ for all $y\in\overline{P}_c$. Suppose that there exist $0<a<b<d\leq c$ such that*

(E1) *$\{y \in P(\psi; b, d) | \psi(y) > b\} \neq \emptyset$ and $\psi(Ty) > b$ for $y \in P(\psi; b, d)$;*

(E2) *$||Ty|| < a$ for $||y|| \leq a$;*

(E3) *$\psi(Ty) > b$ for $y \in P(\psi; b, c)$ with $||Ty|| > d$.*

Then T has at least three fixed points y_1, y_2 and y_3 such that $||y_1|| < a$, $b < \psi(y_2)$ and $||y_3|| > a$ with $\psi(y_3) < b$.

Lemma 2.2. *Suppose P is a cone in a real Banach space X, $\alpha, \gamma : P \to I_0$ be two continuous increasing functionals, $\theta : P \to I_0$ be a continuous functional and there exist positive numbers $M, c > 0$ such that*

(i) *$T : \overline{P}(\gamma, c) \to P$ is a completely continuous operator;*

(ii) *$\theta(0) = 0$ and $\gamma(x) \leq \theta(x) \leq \alpha(x)$, $||x|| \leq M\gamma(x)$ for all $x \in \overline{P}(\gamma, c)$;*

(iii) *there exist constants $0 < a < b < c$ such that $\theta(\lambda x) \leq \lambda\theta(x)$ for all $\lambda \in [0, 1]$ and $x \in \partial P(\theta, b)$;*

(iv) *$\gamma(Tx) > c$ for all $x \in \partial P(\gamma, c)$; $\theta(Tx) < b$ for all $x \in \partial P(\theta, b)$; $P(\alpha, a) \neq \emptyset$ and $\alpha(Tx) > a$ for all $x \in \partial P(\alpha, a)$.*

Then T has two fixed points x_1, x_2 in $P(\gamma, c)$ such that

$$\alpha(x_1) > a, \ \theta(x_1) < b < \theta(x_2), \ \ \gamma(x_2 < c.$$

Lemma 2.3. *Suppose P is a cone in a real Banach space X, $\alpha, \gamma : P \to I_0$ be two continuous increasing functionals, $\theta : P \to I_0$ be a continuous functional and there exist positive numbers $M, c > 0$ such that (i), (ii) and (iii) in Lemma 2.4 hold and*

(iv) *$\gamma(Tx) < c$ for all $x \in \partial P(\gamma, c)$; $\theta(Tx) > b$ for all $x \in \partial P(\theta, b)$; $P(\alpha, a) \neq \emptyset$ and $\alpha(Tx) < a$ for all $x \in \partial P(\alpha, a)$.*

Then T has two fixed points x_1, x_2 in $P(\gamma, c)$ such that

$$\alpha(x_1) > a, \ \theta(x_1) < b < \theta(x_2), \ \ \gamma(x_2 < c.$$

Definition 2.4. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to R$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right-hand side exists.

Definition 2.5. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to R$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n - 1 < \alpha \leq n$, provided that the right-hand side is point-wise defined on $(0, \infty)$.

Lemma 2.4. *Let $n-1<\alpha\le n$, $u\in C^0(0,1)\bigcap L^1(0,1)$. Then*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_nt^{\alpha-n},$$

where $C_i\in R$, $i=1,2,\ldots n$.

Lemma 2.5. *For $\alpha\ge 0$ and $\mu>-1$, the relations*

$$I_{0+}^{\alpha}t^{\mu}=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}t^{\mu+\alpha},\ \ D_{0+}^{\alpha}t^{\mu}=\frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}$$

are valid.

Lemma 2.6. *Suppose that $h\in L^1(0,1)$. Then the unique solution of*

(2.1)
$$\begin{cases} D^{\alpha}u(t)+h(t)=0, 0<t<1,\\ \left[I_0^{2-\alpha}u(t)\right]'\Big|_{t=0}=0\,,\\ u(1)=0,\end{cases}$$

is

(2.2)
$$u(t)=\frac{1}{\Gamma(\alpha)}\left[\int_0^t\left[t^{\alpha-2}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}\right]h(s)ds+t^{\alpha-2}\int_t^1(1-s)^{\alpha-1}h(s)ds\right].$$

Proof. We may apply Lemma 2.4 to reduce BVP(2.1) to an equivalent integral equation

$$u(t)=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+c_1t^{\alpha-1}+c_2t^{\alpha-2}$$

for some $c_i\in R, i=1,2$. We get by using Lemma 2.5 that

$$[I_0^{2-\alpha}u(t)]'=-\int_0^t h(s)ds+c_1\Gamma(\alpha).$$

Then $\left[I_0^{2-\alpha}u(t)\right]'\Big|_{t=0}=0$ implies $c_1=0$. Since $u(1)=0$,, we get

$$c_2=\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds.$$

Therefore, the unique solution of BVP(2.1) is

$$u(t)=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+t^{\alpha-2}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds.$$

Then (2.2) holds. Reciprocally, let u satisfy (2.2). Then

$$u(1)=0,\ \left[I_0^{2-\alpha}u(t)\right]'\Big|_{t=0}=0,$$

furthermore, we have $D^{\alpha}u(t) = -h(t)$. The proof is complete. □

Lemma 2.7. *Suppose that $\beta \in (0,1)$ and $h \in L^1(0,1)$ is nonnegative. If u is the solution of BVP(2.1), then*

$$\inf_{t\in[0,\beta]} t^{2-\alpha}u(t) \geq [1-\beta^{2-\alpha}] \sup_{t\in[0,1]} t^{2-\alpha}u(t). \tag{2.3}$$

Proof. One sees from Lemma 2.6 that u satisfies (2.2). Let

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-2}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 < s \leq t \leq 1, \\ t^{\alpha-2}(1-s)^{\alpha-1}, & 0 < t \leq s \leq 1. \end{cases}$$

It follows that $G(t,s) \geq 0$ for all $t, s \in (0,1]$ and $u(t) = \int_0^1 G(t,s)h(s)ds$.

One sees that

$$t^{2-\alpha}G(t,s) \leq \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1} \text{ for } s,t \in (0,1]. \tag{2.4}$$

For $s \geq t$, we get

$$t^{2-\alpha}G(t,s) = \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1} \geq \frac{1}{\Gamma(\alpha)}[1-\beta^{2-\alpha}](1-s)^{\alpha-1}.$$

For $s \leq t \leq \beta$, we have

$$t^{2-\alpha}G(t,s) = \frac{1}{\Gamma(\alpha)}\left[(1-s)^{\alpha-1} - t^{2-\alpha}(t-s)^{\alpha-1}\right] \geq \frac{1}{\Gamma(\alpha)}[1-\beta^{2-\alpha}](1-s)^{\alpha-1}.$$

Hence

$$t^{2-\alpha}G(t,s) \geq \frac{1}{\Gamma(\alpha)}[1-\beta^{2-\alpha}](1-s)^{\alpha-1}, t \in (0,\beta], s \in (0,1]. \tag{2.5}$$

It follows from (2.2) that $u(t) = \int_0^1 G(t,s)h(s)ds \geq 0$ for all $t \in (0,1]$. Hence (2.4) and (2.5) imply

$$\begin{aligned} \inf_{t\in[0,\beta]} t^{2-\alpha}u(t) &= \min_{t\in[0,\beta]} \int_0^1 t^{2-\alpha}G(t,s)h(s)ds \\ &\geq \frac{1}{\Gamma(\alpha)} \int_0^1 [1-\beta^{2-\alpha}](1-s)^{\alpha-1}h(s)ds \\ &\geq [1-\beta^{2-\alpha}] \int_0^1 G(t,s)h(s)ds. \end{aligned}$$

Hence

$$\inf_{t\in[0,\beta]} t^{2-\alpha}u(t) \geq [1-\beta^{2-\alpha}] \sup_{t\in[0,1]} t^{2-\alpha}u(t).$$

Then (2.3) holds. The proof is completed. □

We use the Banach space $C[0,1]$ with the norm

$$||u||_\infty = \max_{t\in[0,1]} |u(t)|.$$

Let

$$X = C^{\alpha-1}[0,1] := \left\{u \ : \ u(t) = I_{0^+}^{\alpha-1}x(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2}, \ t \in (0,1], \ x \in C[0,1]\right\}.$$

It is easy to show that $t^{2-\alpha}u \in C[0,1]$ is bounded. For $u \in X$, define the norm

$$||u|| = \sup_{t\in[0,1]} |t^{2-\alpha}u(t)|.$$

By means of the linear functional analysis theory, we can prove that X is a Banach space.

We seek solutions of BVP(1.1) that lie in the cone

$$P = \left\{u \in X : u(t) \geq 0, t \in [0,1], \ \inf_{t\in[0,\beta]} t^{2-\alpha}u(t) \geq [1-\beta^{2-\alpha}] \sup_{t\in[0,1]} t^{2-\alpha}u(t)\right\}.$$

Define the operator T on P by

$$(Tu)(t) = \int_0^1 G(t,s)f(s,u(s))ds.$$

Lemma 2.8. *Suppose that $f : (0,1) \times [0,\infty) \to [0,\infty)$ is continuous and satisfies that for each $r > 0$ there exists $\phi_r \in L^1(0,1)$ such that $|f(t,t^{\alpha-2}x)| \leq \phi_r(t)$ for all $t \in (0,1)$ and $|x| \leq r$. Then $T : P \to P$ is completely continuous.*

Proof. We divide the proof into four steps.

Step 1. We prove that $T : P \to P$ is well defined.

For $u \in P$, we find $u(t) \geq 0$ for all $t \in [0,1]$ and $t^{2-\alpha}u(t)$ is continuous on $[0,1]$. Hence there exits $r > 0$ such that

$$||u|| = \sup_{t\in[0,1]} t^{2-\alpha}|u(t)| < r.$$

Then there exists $\phi_r \in L^1(0,1)$ such that $0 \leq f(t,t^{\alpha-2}x(t)) \leq \phi_r(t)$ for all $t \in (0,1)$. Then $(Tu)(t) \geq 0$ for all $t \in [0,1]$ and

$$\begin{aligned} t^{2-\alpha}|(Tu)(t)| &= t^{2-\alpha}\left|\int_0^1 G(t,s)f(s,u(s))ds\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_r(s)ds. \end{aligned}$$

By the method used in Lemma 2.7, we get $Tu \in P$. So $T : P \to P$ is well defined.

Step 2. T is continuous.
Let $\{y_n\}$ be a sequence such that $y_n \to y$ in X. Let

$$r = \max\left\{\sup_{t\in[0,1]} t^{2-\alpha}|y_n(t)|, \sup_{t\in[0,1]} t^{2-\alpha}|y(t)|\right\}.$$

Then for $t \in [0,1]$, we have $\phi_r \in L^1(0,1)$ such that $0 \le f(t, t^{\alpha-2}x) \le \phi_r(t)$ for all $t \in (0,1)$ and $|x| \le r$. So

$$\begin{aligned}
& t^{2-\alpha}|(Ty_n)(t) - (Ty)(t)| \\
= & \left|\int_0^1 t^{2-\alpha}G(t,s)f(s,y_n(s))ds - \int_0^1 t^{2-\alpha}G(t,s)f(s,y(s))ds\right| \\
\le & \int_0^1 t^{2-\alpha}G(t,s)|f(s,y_n(s)) - f(s,y(s))|ds \\
\le & \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}|f(s,s^{\alpha-2}s^{2-\alpha}y_n(s)) - f(s,s^{\alpha-2}s^{2-\alpha}y(s))|ds \\
\le & 2\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_r(s)ds.
\end{aligned}$$

Since $f(t, s^{\alpha-2}x)$ is continuous in x, we have $||Ty_n - Ty|| \to 0$ as $n \to \infty$.
Step 3. T maps bounded sets into bounded sets in X.

It suffices to show that for each $r > 0$, there exists a positive number $L > 0$ such that each $x \in M = \{y \in X : ||y|| \le r\}$, we have $||Ty|| \le L$. By the definition of T, we get

$$\begin{aligned}
t^{2-\alpha}|(Ty)(t)| &= \int_0^1 t^{2-\alpha}G(t,s)f(s,y(s))ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^1 f(s,s^{\alpha-2}s^{2-\alpha}y(s))ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_l(s)ds.
\end{aligned}$$

It follows that

$$||Ty|| \le \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_l(s)ds \text{ for each } y \in \{y \in X : ||y|| \le l\}.$$

So T maps bounded sets into bounded sets in X.
Step 4. T maps bounded sets into equicontinuous sets in X.

We prove that T is equicontinuous on compact sub interval of $(0,1]$. Let $t_1, t_2 \in$

$[0,1]$ with $t_1 < t_2$ and $y \in M = \{y \in X : ||y|| \le l\}$ defined in Step 2. We have

$$\begin{aligned}
& |t_1^{2-\alpha}(Ty)(t_1) - t_2^{2-\alpha}(Ty)(t_2)| \\
= & \left|\int_0^1 t_1^{2-\alpha}G(t_1,s)f(s,y(s))ds - \int_0^1 t_2^{2-\alpha}G(t_2,s)f(s,y(s))ds\right| \\
\le & \int_0^1 |t_1^{2-\alpha}G(t_1,s) - t_2^{2-\alpha}G(t_2,s)|f(s,s^{\alpha-2}s^{2-\alpha}y(s))ds \\
\le & \int_0^{t_1} \frac{|t_1^{2-\alpha}(t_1-s)^{\alpha-1} - t_2^{2-\alpha}(t_2-s)^{\alpha-1}|}{\Gamma(\alpha)} f(s,s^{\alpha-2}s^{2-\alpha}y(s))ds \\
& + \int_{t_1}^{t_2} \frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,s^{\alpha-2}s^{2-\alpha}y(s))ds \\
& + \int_{t_2}^1 \frac{t_1^{2-\alpha}t_1^{\alpha-2}(1-s)^{\alpha-1} - t_2^{2-\alpha}t_2^{\alpha-2}(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,s^{\alpha-2}s^{2-\alpha}y(s))ds \\
\le & \int_0^1 \left|\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1} - t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\right| \phi_l(s)ds \\
& + \frac{|t_1-t_2|}{\Gamma(\alpha)} \int_0^1 \phi_l(s)ds.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. Therefore, T is equicontinuous on $[0,1]$. Then T is completely continuous. The proof is complete. □

3. Main results

In this section, we prove the main results. Let

$$M = \frac{1}{\Gamma(\alpha+1)}, \quad W = \frac{(1-\beta^{2-\alpha})[1-(1-\beta)^\alpha]}{\alpha\Gamma(\alpha)}.$$

(A). $f : (0,1) \times [0,\infty) \to [0,\infty)$ is continuous with $\int_0^\beta (1-s)^{\alpha-1} f(s,1)\, ds > 0$ and satisfies that for each $r > 0$ there exists $\phi_r \in L^1(0,1)$ such that $|f(t, t^{\alpha-2}x)| \le \phi_r(t)$ for all $t \in (0,1)$ and $|x| \le r$.

(B). there exits real numbers $\lambda < 0 < \mu$ and $\sigma_2 > \sigma_1 > 0$ such that

$$f(t,cx) \ge c^\lambda f(t,x), \quad c \ge \sigma_2, t \in (0,\beta], x \in [0,\sigma_1]$$

and

$$f(t,c) \ge c^\mu f(t,1), \quad 0 < c < \sigma_1, t \in (0,\beta].$$

Theorem 3.1. *Suppose that (A) and (B) hold. Furthermore, there exist constants e_1, e_2 and c such that*

$$0 < e_1 < e_2 < \frac{e_2}{1-\beta^{2-\alpha}} < c, \quad Wc > Me_2,$$

and

(D1) $f(t, t^{\alpha-2}u) \leq \frac{c}{M}$ *for* $t \in (0,1)$, $u \in [0,c]$;

(D2) $f(t, t^{\alpha-2}u) \leq \frac{e_1}{M}$ *for* $t \in (0,1)$ *and* $u \in [0,e_1]$;

(D3) $f(t, t^{\alpha-2}u) \geq \frac{e_2}{W}$ *for* $t \in (0,\beta]$ *and* $u \in \left[e_2, \frac{e_2}{1-\beta^{2-\alpha}}\right]$.

Then BVP(1.1) has at least three positive solutions x_1, x_2 *and* x_3 *satisfying*

$$\sup_{t\in[0,1]} t^{2-\alpha}x_1(t) < e_1, \quad \inf_{t\in[0,\beta]} t^{2-\alpha}x_2(t) > e_2 \tag{3.1}$$

and

$$\sup_{t\in[0,1]} t^{2-\alpha}x_3(t) > e_1, \quad \inf_{t\in[0,\beta]} t^{2-\alpha}x_3(t) < e_2. \tag{3.2}$$

Proof. Define the functional ψ by

$$\psi(x) = \inf_{t\in[0,\beta]} t^{2-\alpha}x(t) \text{ for } x \in P.$$

It is easy to see that ψ is a nonnegative convex continuous functional on the cone P. $\psi(y) \leq ||y||$ for all $y \in P$. For $x \in P$, it follows from (A) and Lemma 2.8 that $TP \subseteq P$ and $T : P \to P$ is completely continuous.

Corresponding to Lemma 2.1, choose

$$d = \frac{e_2}{1-\beta^{2-\alpha}}, \quad b = e_2, \quad a = e_1.$$

Then $0 < a < b < d < c$. We divide the remainder of the proof into four steps.

Step 1. Prove that $T(\overline{P_c}) \subset \overline{P_c}$.

For $x \in \overline{P_c}$, one has $||x|| \leq c$. Then

$$0 \leq t^{2-\alpha}x(t) \leq c, \ t \in [0,1].$$

It follows from (D1) that

$$f(t,x(t)) = f(t, t^{\alpha-2}t^{2-\alpha}x(t)) \leq \frac{c}{M}, t \in (0,1).$$

Then $Tx \in P$ implies that

$$\begin{aligned}
||Tx|| &= \sup_{t\in[0,1]} t^{2-\alpha}(Tx)(t) \\
&= \sup_{t\in[0,1]} \int_0^1 t^{2-\alpha}G(t,s)f(s,x(s))ds \\
&\leq \sup_{t\in(0,1]} \int_0^1 t^{2-\alpha}G(t,s)\frac{c}{M}ds \\
&\leq \frac{c}{M}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&\leq \frac{1}{\Gamma(\alpha+1)}\frac{c}{M} \\
&= c.
\end{aligned}$$

Then $Tx \in \overline{P_c}$, Hence $T(\overline{P_c})$. This completes the proof of Step 1.

Step 2. Prove that

$$\{y \in P(\psi; b, d) | \psi(y) > b\} = \{y \in P\left(\psi; e_2, \frac{e_2}{1-\beta^{2-\alpha}}\right) | \psi(y) > e_2\} \neq \emptyset$$

and $\psi(Ty) > b = e_2$ for $y \in P\left(\psi; e_2, \frac{e_2}{1-\beta^{2-\alpha}}\right)$.

It is easy to see that $\left\{x \in P\left(\psi; e_2, \frac{e_2}{1-\beta^{2-\alpha}}\right), \psi(x) > e_2\right\} \neq \emptyset$. For $x \in P(\psi, e_2, e_2/\beta^{\alpha})$, then $\psi(x) \geq e_2$ and $||x|| \leq \frac{e_2}{1-\beta^{2-\alpha}}$. Then

$$\inf_{t\in[0,\beta]} t^{2-\alpha}x(t) \geq e_2, \quad \sup_{t\in[0,1]} x(t) \leq \frac{e_2}{1-\beta^{2-\alpha}}.$$

Hence

$$e_2 \leq t^{2-\alpha}x(t) \leq \frac{e_2}{1-\beta^{2-\alpha}}, \ t \in [0, \beta].$$

Hence (D3) implies that

$$f(t, x(t)) = f(t, t^{\alpha-2}t^{2-\alpha}x(t) \geq \frac{e_2}{W}, t \in (0, \beta].$$

We get

$$\begin{aligned}
\psi(Tx) &= \inf_{t\to[0,\beta]} \int_0^1 t^{2-\alpha}G(t,s)f(s,x(s))ds \\
&> \inf_{t\to[0,\beta]} \int_0^\beta t^{2-\alpha}G(t,s)f(s,x(s))ds \\
&\geq \int_0^\beta \frac{(1-s)^{\alpha-1}(1-\beta^{2-\alpha})}{\Gamma(\alpha)} f(s,x(s))ds \\
&\geq \int_0^\beta \frac{(1-s)^{\alpha-1}(1-\beta^{2-\alpha})}{\Gamma(\alpha)} \frac{e_2}{W} ds \\
&\geq e_2.
\end{aligned}$$

This completes the proof of Step 2.

Step 3. Prove that $||Ty|| < a = e_1$ for $y \in P$ with $||y|| \leq a$.

For $x \in \overline{P_{e_1}}$, we have

$$\sup_{t\in[0,1]} t^{2-\alpha}x(t) \leq e_1 = a.$$

It follows from (D2) and $Tx \in P$ that

$$f(t, x(t)) = f(t, t^{\alpha-2}t^{2-\alpha}x(t)) \leq \frac{e_1}{M}, \ t \in (0, 1).$$

The proof is similar to that of Step 1. Then $||Ty|| < e_1$ for $||y|| \leq e_1$. This completes that proof of Step 3.

Step 4. Prove that $\psi(Ty) > b$ for $y \in P(\psi; b, c)$ with $||Ty|| > d$.

For $x \in P(\psi; b, c) = P(\psi, e_2, c)$ and $||Tx|| > d = \frac{e_2}{1-\beta^{2-\alpha}}$, then

$$\inf_{t\in[0,\beta]} t^{2-\alpha}x(t) \geq e_2, \quad \sup_{t\in[0,1]} t^{2-\alpha}(Tx)(t) \geq \frac{e_2}{1-\beta^{2-\alpha}} \text{ and } ||x|| = \sup_{t\in(0,1]} t^{2-\alpha}x(t) \leq c.$$

Hence we have from $Tx \in P$ that

$$\begin{aligned}
\psi(Tx) &= \inf_{t\in[0,\beta]} t^{2-\alpha}(Tx)(t) \\
&= (1-\beta^{2-\alpha}) \sup_{t\in[0,1]} t^{2-\alpha}(Tx)(t) \\
&\geq (1-\beta^{2-\alpha})\frac{e_2}{1-\beta^{2-\alpha}} \\
&= b.
\end{aligned}$$

This completes the proof of Step 4.

From above steps, (E1), (E2) and (E3) of Lemma 2.1 are satisfied. Then, by Lemma 2.1, T has three fixed points x_1, x_2 and $x_3 \in \overline{P_c}$ such that

$$||x_1|| < a, \ \psi(x_2) > b, \ ||x_3|| \geq a, \ \psi(x_3) \leq b,$$

i.e., x_1, x_2 and x_3 satisfy (3.1) and (3.2). Hence BVP(1.1) has at least three positive solutions x_1, x_2 and x_3.

Finally, we prove that x_1, x_2 and x_3 are unbounded.

In fact, if x_i is bounded on $(0,1]$, then we have a positive number $r > 0$ such that $0 \leq x_i^{-\lambda}(t) \leq r$ for all $t \in (0,1]$. Choose $c > 0$ sufficiently large such that $c||x_i|| > \sigma_2$ and $\frac{1}{c} < \sigma_1$.

Since $t^{2-\alpha}x_i(t)$ is continuous on $[0,1]$, we see that there is $t_0 \in [0,1]$ such that $t_0^{2-\alpha}x_i(t_0) = ||x_i||$. So $x(t_0) = \frac{||x_i||}{t_0^{2-\alpha}} \geq ||x_i||$. Then for $t \in [0,\beta]$, we have

$$cx_i(t) = c\frac{t^{2-\alpha}x_i(t)}{t^{2-\alpha}} \geq ct^{2-\alpha}x_i(t) \geq c(1-\beta^{2-\alpha})||x_i|| \geq (1-\beta^{2-\alpha})\sigma_2.$$

Using (B), then

$$\begin{aligned}
t^{2-\alpha}x_i(t) &= t^{2-\alpha}\int_0^1 G(t,s)f(s,x_i(s))ds \\
&\geq \int_0^\beta t^{2-\alpha}G(t,s)f(s,x_i(s))ds \\
&\geq \frac{1-\beta^{2-\alpha}}{\Gamma(\alpha)}\int_0^\beta (1-s)^{\alpha-1}f\left(s, cx_i(s)\frac{1}{c}\right)ds \\
&\geq \frac{1-\beta^{2-\alpha}}{\Gamma(\alpha)}\int_0^\beta c^\lambda x_i^\lambda(s)(1-s)^{\alpha-1}f\left(s,\frac{1}{c}\right)ds \\
&\geq \frac{1-\beta^{2-\alpha}}{\Gamma(\alpha)}\int_0^\beta c^\lambda x_i^\lambda(s)(1-s)^{\alpha-1}\frac{1}{c^\mu}f(s,1)\,ds \\
&\geq \frac{1-\beta^{2-\alpha}}{\Gamma(\alpha)}\int_0^\beta c^{\lambda-\mu}\frac{(1-\beta^{2-\alpha})^\lambda\sigma_2^\lambda}{c^\lambda}(1-s)^{\alpha-1}f(s,1)\,ds \\
&= \frac{c^{-\mu}(1-\beta^{2-\alpha})^{\lambda+1}\sigma_2^\lambda}{\Gamma(\alpha)}\int_0^\beta (1-s)^{\alpha-1}f(s,1)\,ds.
\end{aligned}$$

Then

$$r^{\frac{1}{-\lambda}} \geq x_i(t) \geq \frac{c^{-\mu}(1-\beta^{2-\alpha})^{\lambda+1}\sigma_2^\lambda}{\Gamma(\alpha)}\int_0^\beta (1-s)^{\alpha-1}f(s,1)\,ds\frac{1}{t^{2-\alpha}}.$$

Let $t \to 0$, we get from (A) that $r^{\frac{1}{-\lambda}} \geq \infty$, a contraction. So x_1, x_2 and x_3 are unbounded. The proof is complete. □

Theorem 3.2. *Suppose that (A) and (B) hold. Furthermore, there exist positive numbers $a < b < c$ such that $Wb > Ma$, and*

(E1) $f(t, t^{\alpha-2}u) \geq \frac{c}{W}$ *for* $t \in (0,\beta]$, $u \in [c, c/(1-\beta^{2-\alpha}]$;

(E2) $f(t, t^{\alpha-2}u) \leq \frac{b}{M}$ *for* $t \in (0,1)$ *and* $u \in [0,b]$;

(E3) $f(t, t^{\alpha-2}u) \geq \frac{a}{W}$ *for* $t \in (0\beta]$ *and* $u \in \big[(1-\beta^{2-\alpha})a, a\big]$.

Then BVP(1.1) has at least two positive solutions x_1 and x_2 satisfying

$$\begin{aligned}
&\sup_{t\in[0,1]} t^{\alpha-2}x_1(t) > a, \ \sup_{t\in[0,1]} t^{\alpha-2}x_1(t) < b, \\
&\sup_{t\in[0,1]} t^{\alpha-2}x_2(t) > b, \ \inf_{t\in[0,\beta]} t^{\alpha-2}x_2(t) < c.
\end{aligned} \tag{3.3}$$

Proof. Define the nonnegative, increasing and continuous functionals $\gamma, \theta, \alpha : P \to I$ by

$$\begin{aligned}
\gamma(x) &= \inf_{t\in[0,\beta]} t^{\alpha-2}x(t), \ \ x \in P, \\
\theta(x) &= \sup_{t\in[0,1]} t^{\alpha-2}x(t), \ \ x \in P, \\
\alpha(x) &= \sup_{t\in[0,1]} t^{\alpha-2}x(t), \ \ x \in P.
\end{aligned}$$

It is easy to see that $\theta(0) = 0$ and

$$\gamma(x) \le \theta(x) \le \alpha(x), \quad x \in P$$

and for $x \in P$ we have $\gamma(x) \ge (1-\beta^{2-\alpha})||x||$, $\theta(\nu x) \le \nu\theta(x)$ for all $\nu \in [0,1]$ and $x \in P$. From (A) and Lemma 2.8, we have $TP \subset P$ and T is completely continuous. Hence (i)-(iii) in Lemma 2.2 hold. To obtain two positive solutions of BVP(1.1), it suffices to show that the condition (iv) in Lemma 2.2 holds.

First, we verify that

$$\gamma(Tx) > c \text{ for alll } x \in \partial P(\gamma, c). \tag{3.4}$$

Since $x \in \partial P(\gamma, c)$, we get $\min_{t\in[0,\beta]} t^{2-\alpha}x(t) = c$. Then $||x|| \le \frac{1}{1-\beta^{2-\alpha}}\gamma(x) \le \frac{c}{1-\beta^{2-\alpha}}$. Then $c \le t^{2-\alpha}x(t) \le \frac{c}{1-\beta^{2-\alpha}}$ for all $t \in [0,\beta]$. Hence (E1) implies

$$f(t,x(t)) = f(t, t^{\alpha}t^{2-\alpha}x(t)) \ge \frac{c}{W}, \quad t \in [0,\beta]\,.$$

So we get from $Tx \in P$ that

$$\begin{aligned}
\gamma(Tx)(t) &= \inf_{t\in[0,\beta]} \int_0^1 t^{2-\alpha}G(t,s)f(s,x(s))ds \\
&> \inf_{t\in[0,\beta]} \int_0^{\beta} t^{2-\alpha}G(t,s)f(s,x(s))ds \\
&\ge \int_0^{\beta} \frac{(1-\beta^{2-\alpha})(1-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{c}{W}ds \\
&\ge c.
\end{aligned}$$

Secondly, we prove that

$$\theta(Tx) < b \text{ for all } x \in \partial P(\theta, b). \tag{3.5}$$

Since $\theta(x) = b$, we get $\sup_{t\in[0,1]} t^{2-\alpha}x(t) = b$. Then

$$t^{2-\alpha}x(t) \le b \text{ for all } t \in [0,1].$$

Hence (E2) implies

$$f(t,x(t)) = f(t, t^{\alpha-2}t^{2-\alpha}x(t)) \le \frac{b}{M}, t \in (0,1).$$

So the definition of T imply

$$\begin{aligned}\theta(Tx) &= \sup_{t\in[0,1]} t^{2-\alpha}(Tx)(t)\\ &\le \sup_{t\in[0,1]}\int_0^1 t^{2-\alpha}G(t,s)f(s,x(s))ds\\ &\le \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{b}{M}ds\\ &\le \frac{1}{\Gamma(\alpha+1)}\frac{b}{M}\\ &= b.\end{aligned}$$

Finally, we prove that

$$P(\alpha,a)\neq\emptyset,\ \ \alpha(Tx)>a \text{ for all } x\in\partial P(\alpha,a). \tag{3.6}$$

It is easy to see that $P(\alpha,a)\neq\emptyset$. For $x\in\partial P(\alpha,a)$, we have $\sup_{t\in[0,1]} t^{2-\alpha}x(t)=a$. Then

$$(1-\beta^{2-\alpha}a \le t^{2-\alpha}x(t)\le a \text{ for all } t\in[0,\beta]\,.$$

Then (E3) implies

$$f(t,x(t)) = f(t,2^{\alpha-2}t^{2-\alpha}x(t)) \ge \frac{a}{W}, t\in[0,\beta]\,.$$

Similarly to the first step, we can prove that $\alpha(Tx)>a$. It follows from above discussion that all conditions in Lemma 2.2 are satisfied. Then T has two fixed points x_1,x_2 in P satisfying (3.3). So BVP(1.1) has two positive solutions x_1 and x_2 satisfying (3.3). The proof is complete. □

Theorem 3.3. *Suppose that (A) and (B) hold. Furthermore, there exist positive numbers $a<\beta^\alpha b<b<c$ such that $Wc>Mb$, and*

(E4) $f(t,t^{\alpha-2}u)\le\frac{c}{M}$ *for* $t\in(0,1)$, $u\in[0,c/(1-\beta^{2-\alpha}]$;

(E5) $f(t,t^{\alpha-2}u)\ge\frac{b}{W}$ *for* $t\in(0,\beta]$ *and* $u\in[(1-\beta^{2-\alpha}b,b]$;

(E6) $f(t,t^{\alpha-2}u)\le\frac{a}{M}$ *for* $t\in(0,1)$ *and* $u\in[0,a]$.

Then BVP(1.1) has at least two positive solutions x_1 and x_2 satisfying

$$\begin{aligned}&\sup_{t\in[0,1]} t^{\alpha-2}x_1(t)>a,\ \sup_{t\in[0,1]} t^{\alpha-2}x_1(t)<b,\\ &\sup_{t\in[0,1]} t^{\alpha-2}x_2(t)>b,\ \inf_{t\in[0,\beta]} t^{\alpha-2}x_2(t)<c.\end{aligned} \tag{3.7}$$

Proof. Let the nonnegative, increasing and continuous functionals $\gamma,\theta,\alpha:P\to I$ be defined in the proof of Theorem 3.2. By using Lemma 2.3, the remainder of the proof is similar to that of the proof of Theorem 3.2 and is omitted. □

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