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Ulam Stability Generalizations of 4^{th}- Order Ternary Derivations Associated to a Jmrassias Quartic Functional Equation on Fréchet Algebras

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Abstract. Let $\mathcal{A}$ be a Banach ternary algebra over a scalar field R or C and $\mathcal{X}$ be a ternary Banach $\mathcal{A}$-module. A quartic mapping $D : (\mathcal{A}, [\ ]_{\mathcal{A}}) \to (\mathcal{X}, [\ ]_{\mathcal{X}})$ is called a 4^{th}-order ternary derivation if $D([x, y, z]) = [D(x), y^4, z^4] + [x^4, D(y), z^4] + [x^4, y^4, D(z)]$ for all $x, y, z \in \mathcal{A}$. In this paper, we prove Ulam stability generalizations of $4^{th}-$ order ternary derivations associated to the following JMRassias quartic functional equation on fréchet algebras:

$$f(kx + y) + f(kx - y) = k^2[f(x + y) + f(x - y)] + 2k^2(k^2 - 1)f(x) - 2(k^2 - 1)f(y).$$

1. Introduction

Ternary algebraic operations were considered in the 19 th century by several mathematicians such as A. Cayley [5] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 ([30]). The comments on physical applications of ternary structures can be found in [1, 31, 32, 33, 34, 48, 51].

A nonempty set G with a ternary operation $[.,.,.] : G^3 \to G$ is called a ternary groupoid and is denoted by $(G, [.,.,.])$. The ternary groupoid $(G, [.,.,.])$ is called commutative if $[x_1, x_2, x_3] = [x_{\delta(1)}, x_{\delta(2)}, x_{\delta(3)}]$ for all $x_1, x_2, x_3 \in G$ and all permutations δ of $\{1, 2, 3\}$. If a binary operation $\circ$ is defined on G such that $[x, y, z] = (x \circ y) \circ z$ for all $x, y, z \in G$, then we say that $[.,.,.]$ is derived from $\circ$. We say that $(G, [.,.,.])$ is a ternary semigroup if the operation $[.,.,.]$ is associative, i.e., if $[[x, y, z], u, v] = [x, [y, z, u], v] = [x, y, [z, u, v]]$ holds for all $x, y, z, u, v \in G$ (see Ref. [4],[5] and [51]).

Let $\mathcal{A}$ be a Banach ternary algebra and $\mathcal{X}$ be a Banach space. Then $\mathcal{X}$ is called a ternary Banach $\mathcal{A}$-module, if module operations $\mathcal{A} \times \mathcal{A} \times \mathcal{X} \to \mathcal{X}$, $\mathcal{A} \times \mathcal{X} \times \mathcal{A} \to \mathcal{X}$,

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and $\mathcal{X} \times \mathcal{A} \times \mathcal{A} \to \mathcal{X}$ which are C-linear in every variable. Moreover satisfy

$$[[xab]_{\mathcal{X}}\ cd]_{\mathcal{X}} = [x[abc]_{\mathcal{A}}\ d]_{\mathcal{X}} = [xa[bcd]_{\mathcal{A}}]_{\mathcal{X}},$$

$$[[axb]_{\mathcal{X}}\ cd]_{\mathcal{X}} = [a[xbc]_{\mathcal{X}}\ d]_{\mathcal{X}} = [ax[bcd]_{\mathcal{A}}]_{\mathcal{X}},$$

$$[[abx]_{\mathcal{X}}\ cd]_{\mathcal{X}} = [a[bxc]_{\mathcal{X}}\ d]_{\mathcal{X}} = [ab[xcd]_{\mathcal{X}}]_{\mathcal{X}},$$

$$[abc]_{\mathcal{A}}\ xd]_{\mathcal{X}} = [a[bcx]_{\mathcal{X}}\ d]_{\mathcal{X}} = [ab[cxd]_{\mathcal{X}}]_{\mathcal{X}},$$

$$[[abc]_{\mathcal{A}}\ dx]_{\mathcal{X}} = [a[bcd]_{\mathcal{A}}\ x]_{\mathcal{X}} = [ab[cdx]_{\mathcal{X}}]_{\mathcal{X}}$$

for all $x \in \mathcal{X}$ and all $a, b, c, d \in \mathcal{A}$,

$$\max\{\|xab\|, \|axb\|, \|abx\|\} \leq \|a\|\|b\|\|x\|$$

for all $x \in \mathcal{X}$ and all $a, b \in \mathcal{A}$.

In functional analysis and related areas of mathematics, Fréchet spaces, named after Maurice Fréchet, are special topological vector spaces. They are generalizations of Banach spaces (normed vector spaces which are complete with respect to the metric induced by the norm). Fréchet spaces can be defined in two equivalent ways: the first employs a translation invariant metric, the second a countable family of semi-norms.

A topological vector space X is a Fréchet space if and only if it satisfies the following three properties:

1) It is complete as a uniform space;

2) It is locally convex;

3) It is topology can be induced by a translation invariant metric, i.e. a metric $d : X \times X \to \mathbb{R}$ such that $d(x, y) = d(x + a, y + a)$ for all $a, x, y \in X$.

For more detailed definitions of such terminologies, we can refer to [16].

The study of stability problems originated from a famous talk given by S. M. Ulam [50] in 1940 : "under what condition does there exist a homomorphism near an approximate homomorphism?" In the next year 1941, D. H. Hyers [28] answered affirmatively the question of Ulam. This stability phenomenon is called the Ulam stability of the additive functional equation $g(x + y) = g(x) + g(y)$. A generalized version of the Theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [43], this new concept is known as generalized Ulam stability of functional equations. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. (see [2], [6], [9], [10],[21]–[24],[27],[29],[37]–[46] and [49]).

For a mapping $f : A \to B$, consider the quartic functional equation:

$$f(kx + y) + f(kx - y) + 2(k^2 - 1)f(y) = k^2[f(x + y) + f(x - y)] + 2k^2(k^2 - 1)f(x). \tag{1}$$

In 1999, J. M. Rassias [37] proved the stability of the Eq. (1). Recently, Chang and Sahoo [8] solved Eq. (1). On the other hand, it is easy to see that the solution f of (1) is even, thus the above equation can be written in the following way;

$$f(kx+y)+f(kx-y)=k^2[f(x+y)+f(x-y)]+2k^2(k^2-1)f(x) \\ -2(k^2-1)f(y).$$

In [36], Prak and Bae, considered functional equation (1). In fact, they proved that a function f between two real vector spaces X and Y is a solution of (1) if and only if there exists a unique symmetric multi–additive function $M : X\times X\times X\times X \to Y$ such that $f(x) = M(x,x,x,x)$ for all x. We refer the interested readers for more information on such problems to the papers [1],[8] and [15].

Recently, M. Bavand Savadkouhi, M. Eshaghi Gordji, J. M. Rassias and N. Ghobadipour in [3], proved Approximate ternary Jordan derivations on Banach ternary algebras. For more detailed definitions of such terminologies, we can refer to [7], [10]–[12], [17]–[20], [25], [26], [35] and [47].

In this paper, we prove the generalized Ulam stability of 4^{th}-order ternary derivations on fréchet algebras, associated with the following quartic functional equation

$$f(kx+y)+f(kx-y)=k^2[f(x+y)+f(x-y)]+2k^2(k^2-1)f(x) \\ -2(k^2-1)f(y).$$

2. Main results

Definition 1. Let $\mathcal{A}$ be a Banach ternary algebra over a scalar field R or C and $\mathcal{X}$ be a ternary Banach $\mathcal{A}$-module. A quartic mapping $D : (\mathcal{A}, [\ ]_{\mathcal{A}}) \to (\mathcal{X}, [\ ]_{\mathcal{X}})$ is called a 4^{th}-order ternary derivation if $D([x,y,z]) = [D(x), y^4, z^4] + [x^4, D(y), z^4] + [x^4, y^4, D(z)]$ for all $x, y, z \in \mathcal{A}$.

Now, we investigate the generalized Ulam stability of 4^{th}-order ternary derivations on Fréchet algebras.

Theorem 1. *Let A and B will be two Fréchet algebras by metrics d_1 and d_2, respectively. Suppose $f : A \to B$ be a mapping for which there exist a function $\psi : A\times A\times A \to [0,\infty)$ such that*

$$\sum_{j=0}^{\infty} \frac{1}{k^{4j}} \psi(k^j x, k^j y, k^j z) < \infty, \tag{2}$$

$$d_2(f(kx+y)+f(kx-y)+2(k^2-1)f(y), k^2 f(x+y) \\ + k^2 f(x-y) + 2k^2(k^2-1)f(x)) \le \psi(x,y,0), \tag{3}$$

$$d_2(f([x,y,z]), [f(x), y^4, z^4] + [x^4, f(y), z^4] + [x^4, y^4, f(z)]) \\ \le \psi(x,y,z), \tag{4}$$

for all $x, y, z \in A$. Then there exists a unique 4^{th}-order ternary derivation $D : A \to B$ such that

$$d_2(f(x), D(x)) \leq \frac{1}{2k^4} \sum_{j=0}^{\infty} \frac{\psi(k^j x, 0, 0)}{k^{4j}}, \tag{5}$$

for all $x \in A$.

Proof. Putting $x = y = 0$ in (3) we get $f(0) = 0$. If we putting $y = 0$ in (3), we get

$$d_2(2f(kx), 2k^4 f(x)) \leq \psi(x, 0, 0), \tag{6}$$

for all $x \in A$. Now multiply both sides of (6) by $\frac{1}{2k^4}$, we get

$$d_2(\frac{f(kx)}{k^4}, f(x)) \leq \frac{\psi(x, 0, 0)}{2k^4}, \tag{7}$$

for all $x \in A$. One can use induction on n to show that

$$d_2(\frac{f(k^n x)}{k^{4n}}, f(x)) \leq \frac{1}{2k^4} \sum_{j=0}^{n-1} \frac{\psi(k^j x, 0, 0)}{k^{4j}}, \tag{8}$$

for all $x \in A$ and all non-negative integers n. Hence,

$$d_2(\frac{f(k^{n+m} x)}{k^{4(n+m)}}, \frac{f(k^m x)}{k^{4m}}) \leq \frac{1}{2k^4} \sum_{j=m}^{n+m-1} \frac{\psi(k^j x, 0, 0)}{k^{4j}},$$

for all non-negative integers n and m with $n \geq m$ and all $x \in A$. It follows from the convergence (2) that the sequence $\{\frac{f(k^n x)}{k^{4n}}\}$ is Cauchy. Due to the completeness of B, this sequence is convergent. So one can define the mapping $D : A \to B$ by Set

$$D(x) := \lim_{n\to\infty} \frac{f(k^n x)}{k^{4n}} \tag{9}$$

for all $x \in A$. Replacing x, y by $k^n x, k^n y$, respectively, in (3) and multiply both sides of (3) by $\frac{1}{k^{4n}}$, we get

$$\begin{aligned}
&d_2(D(kx+y) + D(kx-y) + 2(k^2-1)D(y), k^2 D(x+y) + k^2 D(x-y) \\
&+ 2k^2(k^2-1)D(x)) = \lim_{n\to\infty} \frac{1}{k^{4n}} d_2(f(k^n(kx+y)) + f(k^n(kx-y)) \\
&+ 2(k^2-1)f(k^n y), k^2 f(k^n(x+y)) + k^2 f(k^n(x-y)) \\
&+ 2k^2(k^2-1)f(k^n x)) \leq \lim_{n\to\infty} \frac{\psi(k^n x, k^n y, 0)}{k^{4n}}
\end{aligned}$$

for all $x, y \in A$ and all non-negative integers n. Taking the limit as $n \to \infty$ we obtain

$$\begin{aligned}
D(kx+y) + D(kx-y) + 2(k^2-1)D(y) &= k^2 D(x+y) + k^2 D(x-y) \\
&+ 2k^2(k^2-1)D(x),
\end{aligned}$$

for all $x, y \in A$. Moreover, it follows from (8) and (9) that

$$d_2(f(x), D(x)) \leq \frac{1}{2k^4}\sum_{j=0}^{\infty}\frac{\psi(k^j x, 0, 0)}{k^{4j}}$$

for all $x \in A$. It follows from (4) we get

$$\begin{aligned}
&d_2(D([x,y,z]), [D(x), y^4, z^4] + [x^4, D(y), z^4] + [x^4, y^4, D(z)]) \\
&= \lim_{n\to\infty}\frac{1}{k^{12n}} d_2(f(k^{3n}[x,y,z]), [f(k^n x), (k^n y)^4, (k^n z)^4] \\
&+ [(k^n x)^4, f(k^n y), (k^n z)^4] + [(k^n x)^4, (k^n y)^4, f(k^n z)]) \\
&\leq \lim_{n\to\infty}\frac{\psi(k^n x, k^n y, k^n z)}{k^{12n}}
\end{aligned}$$

for all $x, y, z \in A$ and all non-negative integers n. Taking the limit as $n \to \infty$ we obtain

$$D([x,y,z]) = [D(x), y^4, z^4] + [x^4, D(y), z^4] + [x^4, y^4, D(z)],$$

for all $x, y, z \in A$.
Now, let $D' : A \to B$ be another 4^{th}-order ternary derivation satisfying (5). Then we have

$$\begin{aligned}
d_2(D(x), D'(x)) &= \frac{1}{k^{4n}} d_2(D(k^n x), D'(k^n x)) \\
&\leq \frac{1}{k^{4n}}(d_2(D(k^n x), f(k^n x)) + d_2(f(k^n x), D'(k^n x))) \\
&\leq \frac{1}{k^4}\sum_{j=n}^{\infty}\frac{\psi(k^j x, 0, 0)}{k^{4j}}
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $D(x) = D'(x)$ for all $x \in A$. This proves the uniqueness of D. Thus, the mapping $D : A \to B$ is a unique 4^{th}-order ternary derivation satisfying (5). □

Theorem 2. *Let A and B will be two Fréchet algebras by metrics d_1 and d_2, respectively. Suppose $f : A \to B$ be a mapping for which there exist a function $\psi : A \times A \times A \to [0, \infty)$ satisfying* (3), (4) *and*

$$\sum_{j=1}^{\infty} k^{4j}\psi\left(\frac{x}{k^j}, \frac{y}{k^j}, \frac{z}{k^j}\right) < \infty \tag{10}$$

for all $x, y, z \in A$. Then there exists a unique 4^{th}-order ternary derivation $D : A \to B$ such that

$$d_2(f(x), D(x)) \leq \frac{1}{2k^4}\sum_{j=1}^{\infty} k^{4j}\psi\left(\frac{x}{k^j}, 0, 0\right), \tag{11}$$

for all $x \in A$.

Proof. Now by (6), replacing x by $\frac{x}{k}$, and multiply both side of (6) by $\frac{1}{2}$, we get

$$d_2(k^4 f(\frac{x}{k}), f(x)) \leq \frac{1}{2}\psi(\frac{x}{k}, 0, 0), \tag{12}$$

for all $x \in A$. One can use induction on n to show that

$$\begin{aligned} d_2(k^{4n} f(\frac{x}{k^n}), f(x)) &\leq \frac{1}{2}\sum_{j=0}^{n-1} k^{4j}\psi(\frac{x}{k^{j+1}}, 0, 0) \\ &= \frac{1}{2k^4}\sum_{j=1}^{n} k^{4j}\psi(\frac{x}{k^j}, 0, 0), \end{aligned} \tag{13}$$

for all $x \in A$ and all non-negative integers n. Hence,

$$d_2(k^{4(n+m)} f(\frac{x}{k^{n+m}}), k^{4m} f(\frac{x}{k^m})) \leq \frac{1}{2k^4}\sum_{j=1+m}^{n+m} k^{4j}\psi(\frac{x}{k^j}, 0, 0),$$

for all non-negative integers n and m with $n \geq m$ and all $x \in A$. It follows from the convergence (10) that the sequence $\{k^{4n} f(\frac{x}{k^n})\}$ is Cauchy. Due to the completeness of B, this sequence is convergent. So one can define the mapping $D : A \to B$ by Set

$$D(x) := \lim_{n\to\infty} k^{4n} f(\frac{x}{k^n}) \tag{14}$$

for all $x \in A$. Replacing x, y by $\frac{x}{k^n}, \frac{y}{k^n}$, respectively, in (3) and multiply both sides of (3) by k^{4n}, we get

$$\begin{aligned} &d_2(D(kx+y) + D(kx-y) + 2(k^2-1)D(y), k^2 D(x+y) + k^2 D(x-y) \\ &+ 2k^2(k^2-1)D(x)) = \lim_{n\to\infty} k^{4n} d_2(f(\frac{kx+y}{k^n}) + f(\frac{kx-y}{k^n}) \\ &+ 2(k^2-1)f(\frac{y}{k^n}), k^2 f(\frac{x+y}{k^n}) + k^2 f(\frac{x-y}{k^n}) + 2k^2(k^2-1)f(\frac{x}{k^n})) \\ &\leq \lim_{n\to\infty} k^{4n}\psi(\frac{x}{k^n}, \frac{y}{k^n}, 0) \end{aligned}$$

for all $x, y \in A$ and all non-negative integers n. Taking the limit as $n \to \infty$ we obtain

$$\begin{aligned} D(kx+y) + D(kx-y) + 2(k^2-1)D(y) = k^2 D(x+y) + k^2 D(x-y) \\ + 2k^2(k^2-1)D(x), \end{aligned}$$

for all $x, y \in A$. Moreover, it follows from (13) and (14) that

$$d_2(f(x), D(x)) \leq \frac{1}{2k^4}\sum_{j=1}^{\infty} k^{4j}\psi(\frac{x}{k^j}, 0, 0)$$

for all $x \in A$. It follows from (4) we get

$$
\begin{aligned}
& d_2(D([x,y,z]),[D(x),y^4,z^4]+[x^4,D(y),z^4]+[x^4,y^4,D(z)]) \\
& = \lim_{n\to\infty} k^{12n} d_2(f(\frac{[x,y,z]}{k^{3n}}),[f(\frac{x}{k^n}),(\frac{y}{k^n})^4,(\frac{z}{k^n})^4] \\
& + [(\frac{x}{k^n})^4, f(\frac{y}{k^n}),(\frac{z}{k^n})^4] + [(\frac{x}{k^n})^4,(\frac{y}{k^n})^4, f(\frac{z}{k^n})]) \\
& \leq \lim_{n\to\infty} k^{12n}\psi(\frac{x}{k^n},\frac{y}{k^n},\frac{z}{k^n})
\end{aligned}
$$

for all $x, y, z \in A$ and all non-negative integers n. Taking the limit as $n \to \infty$ we obtain

$$D([x,y,z]) = [D(x),y^4,z^4]+[x^4,D(y),z^4]+[x^4,y^4,D(z)],$$

for all $x, y, z \in A$.
Now, let $D' : A \to B$ be another 4^{th}-order ternary derivation satisfying (11). Then we have

$$
\begin{aligned}
& d_2(D(x), D'(x)) = k^{4n} d_2(D(\frac{x}{k^n}), D'(\frac{x}{k^n})) \\
& \leq k^{4n}(d_2(D(\frac{x}{k^n}), f(\frac{x}{k^n})) + d_2(f(\frac{x}{k^n}), D'(\frac{x}{k^n}))) \\
& \leq \frac{1}{k^4}\sum_{j=1+n}^{\infty} k^{4j}\psi(\frac{x}{k^j},0,0)
\end{aligned}
$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $D(x) = D'(x)$ for all $x \in A$. This proves the uniqueness of D. Thus, the mapping $D : A \to B$ is a unique 4^{th}-order ternary derivation satisfying (11). □

By Theorems 1 and 2 we solve the following Ulam stability of 4^{th}-order ternary derivations on Fréchet algebras.

Corollary 1. *Let A and B will be two Fréchet algebras by metrics d_1 and d_2, respectively. Let $p \geq 0$ be given with $p \neq 4$. Let ϵ be non-negative real number, and let $f : A \to B$ be a mapping such that*

$$
\begin{aligned}
& d_2(f(kx+y)+f(kx-y)+2(k^2-1)f(y), k^2 f(x+y) \\
& + k^2 f(x-y) + 2k^2(k^2-1)f(x)) \leq \epsilon(d_1(x,0)^p + d_1(y,0)^p),
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
& d_2(f([x,y,z]),[f(x),y^4,z^4]+[x^4,f(y),z^4]+[x^4,y^4,f(z)]) \\
& \qquad \leq \epsilon(d_1(x,0)^p + d_1(y,0)^p + d_1(z,0)^p),
\end{aligned} \tag{16}
$$

for all $x, y, z \in A$. Then there exists a unique 4^{th}-order ternary derivation $D : A \to B$ such that

$$d_2(D(x), f(x)) \leq \frac{\epsilon d_1(x,0)^p}{2(k^4-k^p)}, \tag{17}$$

holds for all $x \in A$, *where* $p < 4$, *or the inequality*

$$d_2(D(x), f(x)) \leq \frac{\epsilon d_1(x,0)^p}{2(k^p - k^4)}, \tag{18}$$

holds for all $x \in A$, *where* $p > 4$.

Proof. It follows from Theorems 1 and 2. By putting $\psi(x,y,z) := \epsilon(d_1(x,0)^p + d_1(y,0)^p + d_1(z,0)^p)$ for all $x, y, z \in A$. □

By Theorem 1, we prove the following generalized Ulam stability of 4^{th}-order ternary derivations in ternary Banach algebras.

Corollary 2. *Let* A *and* B *will be two ternary Banach algebras. Suppose* $f : A \to B$ *be a mapping for which there exists a function* $\psi : A \times A \times A \to [0, \infty)$ *satisfying* (2),

$$\begin{aligned} &\|(f(kx+y) + f(kx-y) - k^2 f(x+y) - k^2 f(x-y) \\ &- 2k^2(k^2-1)f(x) + 2(k^2-1)f(y)\| \leq \psi(x,y,0), \end{aligned} \tag{19}$$

$$\begin{aligned} &\|f([x,y,z]) - [f(x), y^4, z^4] - [x^4, f(y), z^4] - [x^4, y^4, f(z)]\| \\ &\leq \psi(x,y,z), \end{aligned} \tag{20}$$

for all $x, y, z \in A$. *Then there exists a unique* 4^{th}*-order ternary derivation* $D : A \to B$ *such that*

$$\|f(x) - D(x)\| \leq \frac{1}{2k^4} \sum_{j=0}^{\infty} \frac{1}{k^{4j}} \psi(k^j x, 0, 0),$$

for all $x \in A$.

Proof. It follows from Theorem 1. By putting $d_1(x,y) := \|x - y\|$ for all $x, y \in A$.□

Corollary 3. *Let* A *and* B *will be two ternary Banach algebras. Suppose* $f : A \to B$ *be a mapping for which there exists a function* $\psi : A \times A \times A \to [0, \infty)$ *satisfying* (10), (19) *and* (20) *for all* $x, y, z \in A$. *Then there exists a unique* 4^{th}*-order ternary derivation* $D : A \to B$ *such that*

$$\|f(x) - D(x)\| \leq \frac{1}{2k^4} \sum_{j=1}^{\infty} k^{4j} \psi(\frac{x}{k^j}, 0, 0),$$

for all $x \in A$.

Proof. It follows from Theorem 2. By putting $d_1(x,y) := \|x - y\|$ for all $x, y \in A$.□

Corollary 4. *Let* A *and* B *will be two ternary Banach algebras. Let* $p \geq 0$ *be given* *with $p \neq 4$. Let ϵ be non-negative real number, and let $f : A \to B$ be a mapping such that*

$$\|(f(kx+y) + f(kx-y) - k^2 f(x+y) - k^2 f(x-y) - 2k^2(k^2-1)f(x) \\ + 2(k^2-1)f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p),$$

$$\|f([x,y,z]) - [f(x), y^4, z^4] - [x^4, f(y), z^4] - [x^4, y^4, f(z)]\| \\ \leq \epsilon(\|x\|^p + \|y\|^p + \|z\|^p),$$

for all $x, y, z \in A$. Then there exists a unique 4^{th}-order ternary derivation $D : A \to B$ such that

$$\|D(x) - f(x)\| \leq \frac{\epsilon}{2(k^4 - k^p)}\|x\|^p,$$

holds for all $x \in A$, where $p < 4$, or the inequality

$$\|D(x) - f(x)\| \leq \frac{\epsilon}{2(k^p - k^4)}\|x\|^p,$$

holds for all $x \in A$, where $p > 4$.

Proof. In Theorems 1 and 2, by putting $d_1(x,y) = \|x-y\|$ and $d_2(x,y) = \|x-y\|$ for all $x, y \in A$, we obtain the conclusion of the Theorem. □

The following corollary is Ulam stability of 4^{th}-order ternary derivations in ternary Banach algebras.

Corollary 5. *Let A and B will be two ternary Banach algebras. Let ϵ be non-negative real number, and let $f : A \to B$ be a mapping such that*

$$\|(f(kx+y) + f(kx-y) - k^2 f(x+y) - k^2 f(x-y) - 2k^2(k^2-1)f(x) \\ + 2(k^2-1)f(y)\| \leq \epsilon,$$

$$\|f([x,y,z]) - [f(x), y^4, z^4] - [x^4, f(y), z^4] - [x^4, y^4, f(z)]\| \leq \epsilon,$$

for all $x, y, z \in A$. Then there exists a unique 4^{th}-order ternary derivation $D : A \to B$ such that

$$\|D(x) - f(x)\| \leq \frac{\epsilon}{2(k^4 - 1)},$$

holds for all $x \in A$.

Proof. In Corollary 4, by putting $p := 0$, we obtain the conclusion of the corollary. □

The following corollary is JMRassias stability for 4^{th}-order ternary derivations associated to the mixed type product-sum function $(x,y) \to \epsilon(\|x\|^r\|y\|^s + \|x\|^p + \|y\|^p)$.

Corollary 6. *Let r, s, p be real numbers such that $p \neq 4$, $r + s \neq 4$, $\epsilon > 0$. Let A*

and B will be two ternary Banach algebras and let $f : A \to B$ be a mapping such that

$$\|(f(kx+y)+f(kx-y)-k^2f(x+y)-k^2f(x-y)-2k^2(k^2-1)f(x) \\ +2(k^2-1)f(y)\| \leq \epsilon(\|x\|^r\|y\|^s+\|x\|^p+\|y\|^p),$$

$$\|f([x,y,z])-[f(x),y^4,z^4]-[x^4,f(y),z^4]-[x^4,y^4,f(z)]\| \\ \leq \epsilon(\|x\|^r\|y\|^s+\|x\|^p+\|y\|^p),$$

for all $x, y, z \in A$. Then there exists a unique 4^{th}-order ternary derivation $D : A \to B$ such that

$$\|D(x)-f(x)\| \leq \frac{\epsilon}{2|k^4-k^p|}\|x\|^p$$

holds for all $x \in A$.

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