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On BN-algebras

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Abstract. In this paper, we introduce a BN-algebra, and we prove that a BN-algebra is 0-commutative, and an algebra X is a BN-algebra if and only if it is a 0-commutative BF-algebra. And we introduce a quotient BN-algebra, and we investigate some relations between BN-algebras and several algebras.

1. Introduction

The notion of B-algebra was introduced by J. Neggers and H. S. Kim ([7]). They defined a *B-algebra* as an algebra $(X, *, 0)$ of type (2,0) (i.e., a non-empty set with a binary operation $*$ and a constant 0) satisfying the following axioms:

(B1) $x * x = 0$,

(B2) $x * 0 = x$,

(B) $(x * y) * z = x * [z * (0 * y)]$

for any $x, y, z \in X$.

Recently, C. B. Kim and H. S. Kim ([3]) defined a BG-algebra, which is a generalization of B-algebra. An algebra $(X, *, 0)$ of type (2,0) is called a *BG-algebra* if it satisfies $(B1), (B2)$, and

(BG) $x = (x * y) * (0 * y)$

for any $x, y \in X$. Also they introduced a BM-algebra. An algebra $(X, *, 0)$ of type (2,0) is called a *BM-algebra* ([4]) if it satisfies $(B2)$ and

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(BM) $(z * x) * (z * y) = y * x$

for any $x, y, z \in X$.

Y. B. Jun, E. H. Roh and H. S. Kim ([2]) introduced the notion of a BH-algebra which is a generalization of $BCK/BCI/BCH$-algebras. An algebra $(X, *, 0)$ of type (2,0) is called a *BH-algebra* if it satisfies $(B1), (B2)$, and

(BH) $x * y = y * x = 0$ implies $x = y$

for any $x, y \in X$.

In [12], A. Walendziak introduced $BF/BF_1/BF_2$-algebra. An algebra $(X, *, 0)$ of type (2,0) is a *BF-algebra* if it satisfies $(B1), (B2)$ and

(BF) $0 * (x * y) = y * x$.

for any $x, y \in X$. A BF-algebra is called a *BF_1-algebra* (resp., a *BF_2-algebra*) if it satisfies (BG) (resp., (BH)). In this paper, we define a BN-algebra and investigate some relations between BN-algebras and several algebras, i.e., B-algebras, BM-algebras, BF-algebras, Coxeter-algebras, etc..

2. BN-algebras

In this section, we define a BN-algebra and investigate some relations between BN-algebras and other algebras.

Definition 2.1. A *BN-algebra* is an algebra $(X, *, 0)$ of type (2,0) satisfying $(B1), (B2)$, and the following axiom:

(BN) $(x * y) * z = (0 * z) * (y * x)$

for any $x, y, z \in X$.

Example 2.2. Let $X := \{0, 1, 2\}$ be a set. If we define a binary operation "$*$" on X as follows:

$*$	0	1	2
0	0	1	2
1	1	0	1
2	2	1	0

then $(X, *, 0)$ is a BN-algebra.

Theorem 2.3. *If $(X, *, 0)$ is a BN-algebra, then $(X, *, 0)$ is a BF-algebra.*

Proof. If we let $z := 0$ in (BN), we obtain $x * y = 0 * (y * x)$. Hence $(X, *, 0)$ is a BF-algebra. □

Remark. The converse of Theorem 2.3 does not hold in general.

Example 2.4. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	2	1	3
1	1	0	1	2
2	2	2	0	2
3	3	1	1	0

Then $(X, *, 0)$ is a BF-algebra, but not a BN-algebra, since $(3 * 1) * 3 = 2 \neq (0 * 3) * (1 * 3) = 1$.

Theorem 2.5. *If* $(X, *, 0)$ *is a* BN*-algebra, then*

(i) $0 * (0 * x) = x$,

(ii) $y * x = (0 * x) * (0 * y)$,

(iii) $(0 * x) * y = (0 * y) * x$,

(iv) $x * y = 0 \Longrightarrow y * x = 0$,

(v) $0 * x = 0 * y \Longrightarrow x = y$,

(vi) $(x * z) * (y * z) = (z * y) * (z * x)$,

for any $x, y, z \in X$.

Proof. (i). If we let $y := 0, z := 0$ in (BN), then $(x * 0) * 0 = (0 * 0) * (0 * x)$. By applying $(B1)$ and $(B2)$, we obtain $0 * (0 * x) = x$.
(ii). By $(B2)$ and (BN), $y * x = (y * 0) * x = (0 * x) * (0 * y)$.
(iii). By applying $(B2)$ and (BN), we obtain $(0 * x) * y = (0 * y) * x$.
(iv). By Theorem 2.3, $0 = 0 * 0 = 0 * (x * y) = y * x$.
(v). If $0 * x = 0 * y$, then by (i) we have $x = 0 * (0 * x) = 0 * (0 * y) = y$.
(vi). $(x * z) * (y * z) = (0 * (y * z)) * (z * x) = (z * y) * (z * x)$ by (BN) and Theorem 2.3. □

Definition 2.6. An algebra $(X, *, 0)$ is said to be 0-*commutative* if $x * (0 * y) = y * (0 * x)$ for all $x, y \in X$.

Theorem 2.7. *If* $(X, *, 0)$ *is a* BN*-algebra, then it is* 0*-commutative.*

Proof. Let x and y be any elements of X. Then

$$\begin{aligned} x * (0 * y) &= [(0 * (0 * x)) * (0 * y)] && \text{[Theorem 2.5(i)]} \\ &= [0 * (0 * y)] * [(0 * x) * 0] && \text{[(BN)]} \\ &= y * (0 * x). && \text{[Theorem 2.5(i) and (B2)]} \end{aligned}$$ □

Theorem 2.8. *If* $(X, *, 0)$ *is a* 0*-commutative* BF*-algebra, then it is a* BN*-algebra.*

Proof. Let x, y, z be any elements of X. Then

$$\begin{aligned} (0 * z) * (y * x) &= (0 * z) * (0 * (x * y)) && \text{[(BF)]} \\ &= (x * y) * (0 * (0 * z)) && \text{[0-commutative]} \\ &= (x * y) * z && \text{[(BF) and (B2)].} \end{aligned}$$

Hence $(X, *, 0)$ is a BN-algebra. □

Using Theorem 2.3, Theorem 2.7 and Theorem 2.8, we obtain the following result.

Theorem 2.9. *$(X, *, 0)$ is a 0-commutative BF-algebra if and only if it is a BN-algebra.*

Proposition 2.10([4])**.** *If $(X, *, 0)$ is a 0-commutative B-algebra, then*

$$(0 * x) * (0 * y) = y * x$$

for any $x, y \in X$.

Proposition 2.11([11])**.** *If $(X, *, 0)$ is a B-algebra, then*

$$0 * (x * y) = y * x$$

for any $x, y \in X$.

Corollary 2.12. *If $(X, *, 0)$ is a 0-commutative B-algebra, then it is a BN-algebra.*

Proof. It follows immediately from Theorem 2.9 and Proposition 2.11. □

Remark. The converse of Corollary 2.12 does not hold in general.

Example 2.13. Let $X := \{0, 1, 2\}$ be a set with the table given in Example 2.2. Then it is easy to show that $(X, *, 0)$ is a BN-algebra, but not a B-algebra, since $(1 * 1) * 2 = 2 \neq 1 * [2 * (0 * 1)] = 0$.

The condition, 0-commutativity, is very necessary for B-algebras to be BN-algebras. Consider the following example.

Example 2.14. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a set with the following table:

*	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	5	3	4
2	2	1	0	4	5	3
3	3	5	4	0	1	2
4	4	3	5	2	0	1
5	5	4	3	1	2	0

Then it is easy to show that $(X, *, 0)$ is a B-algebra, but it is not 0-commutative, since $1 * (0 * 3) = 1 * 3 = 5 \neq 3 * (0 * 1) = 3 * 2 = 4$. Moreover, we know that $(X, *, 0)$ is not a BN-algebra, since $(2 * 3) * 5 = 1 \neq (0 * 5) * (3 * 2) = 2$.

Every abelian group determines a BN-algebra. Consider the following theorem.

Theorem 2.15. *Let $(X;\circ,0)$ be an abelian group. If we define $x*y := x\circ y^{-1}, \forall x,y\in X$, then $(X;*,0)$ is a BN-algebra.*

Proof. We see that $x*x = x\circ x^{-1} = 0$ and $x*0 = x\circ 0^{-1} = x\circ 0 = x$. For any $x,y,z\in X$,

$$\begin{aligned}(x*y)*z &= (x\circ y^{-1})\circ z^{-1}\\ &= z^{-1}\circ(x\circ y^{-1})\\ &= z^{-1}\circ(y^{-1}\circ x)\\ &= z^{-1}\circ(x^{-1}\circ y)^{-1}\\ &= z^{-1}*(x^{-1}\circ y)\\ &= (0*z)*(y\circ x^{-1})\\ &= (0*z)*(y*x).\end{aligned}$$

Hence $(X,*,0)$ is a BN-algebra. □

Theorem 2.16. *Let $(X,*,0)$ be an algebra with $0*(0*x)=x$ for any $x\in X$. Then $(X,*,0)$ is 0-commutative if and only if $(0*x)*(0*y)=y*x$ for any $x,y\in X$.*

Proof. If $(X,*,0)$ is 0-commutative, then

$$\begin{aligned}(0*x)*(0*y) &= y*(0*(0*x)) \quad [X\text{: 0-commutative}]\\ &= y*x\end{aligned}$$

for any $x,y\in X$. Conversely, if $(0*x)*(0*y)=y*x$ for any $x,y\in X$, then

$$\begin{aligned}x*(0*y) &= (0*(0*x))*(0*y)\\ &= y*(0*x),\end{aligned}$$

proving the theorem. □

Definition 2.17. An algebra $(X,*,0)$ is said to have a *condition* (D) if $(x*y)*z = x*(z*y)$ for any $x,y,z\in X$.

Theorem 2.18. *If $(X,*,0)$ is a BN-algebra with the condition (D), then*

(i) $0*x=x$,

(ii) $x*y=y*x$

for any $x,y\in X$.

Proof. (i). If we let $x:=0, z:=0$ in (D), then $0*y = 0*(0*y) = y$ by Theorem 2.5(i). (ii). $x*y = x*(0*y) = y*(0*x) = y*x$ by Theorem 2.7 and (i). □

Example 2.19. Let $X := \{0,1,2,3\}$ be a set with the following table:

$*$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

Then $(X, *, 0)$ is a BN-algebra with the condition (D).

Theorem 2.20. *If $(X, *, 0)$ is a BN-algebra with the condition (D), then it is a B-algebra.*

Proof. Let x, y, z be any elements of X. Then

$$\begin{aligned} x * (z * (0 * y)) &= x * (z * y) \quad \text{[Theorem 2.18(i)]} \\ &= (x * y) * z. \quad [(D)] \end{aligned}$$

Hence $(X, *, 0)$ is a B-algebra. □

Theorem 2.21. *Let $(X, *, 0)$ be a BN-algebra with the condition (D), then $(X, *, 0)$ is an abelian group.*

Proof. Since $(X, *, 0)$ is a BN-algebra, $x * x = 0$ for any $x \in X$. We may regard x as its own inverse, i.e., $x^{-1} = x$. By applying $(B2)$ and Theorem 2.18(i), we obtain $x * 0 = 0 * x = x$, i.e., 0 is the identity element for A. Since $(x * y) * z = x * (z * y) = x * (y * z)$ by Theorem 2.18(ii), the associative law holds. Theorem 2.18(ii) shows that $(X, *, 0)$ is an abelian group. □

Theorem 2.22. *If $(X, *, 0)$ is a BN-algebra with the condition (D), then it is a BH-algebra.*

Proof. Let $x * y = 0$ and $y * x = 0$. Then, by Theorem 2.18, we have $x = x * 0 = x * (y * x) = (x * x) * y = 0 * y = y$. Hence $(X, *, 0)$ is a BH-algebra. □

Definition 2.23([5])**.** A *Coxeter algebra* is a non-empty set X with a constant 0 and a binary operation " $*$" satisfying the axioms $(B1), (B2)$, and (As) $(x * y) * z = x * (y * z)$ for any $x, y, z \in X$.

It is known that a Coxeter algebra is a special type of abelian groups (see [5]).

Proposition 2.24([5])**.** *If $(X, *, 0)$ is a Coxeter algebra, then*

(i) $0 * x = x$,

(ii) $x * y = y * x$

for any $x, y \in X$.

Proposition 2.25. *Every Coxeter algebra is a BN-algebra.*

Proof. Let $(X, *, 0)$ be a Coxeter algebra. Given $x, y, z \in X$, $(0 * z) * (y * x) = (y*x)*(0*z) = (x*y)*z$, by Proposition 2.24. Hence $(X, *, 0)$ is a BN-algebra. □

Remark. The converse of Theorem 2.25 does not hold in general.

Example 2.26. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	1	0	1
3	3	1	1	0

Then $(X, *, 0)$ is a BN-algebra, but not a Coxeter algebra, since $(1 * 1) * 2 = 2 \neq 1 * (1 * 2) = 0$.

Proposition 2.27. *If $(X, *, 0)$ is a BN-algebra with the condition (D) if and only if it is a Coxeter algebra.*

Proof. For any $x, y, z \in x$,

$$\begin{aligned}(x * y) * z &= x * (z * y) \quad [(D)] \\ &= x * (y * z) \quad [\text{Theorem 2.18(ii)}]\end{aligned}$$

Hence $(X, *, 0)$ is a Coxeter algebra.

Conversely, assume that $(X, *, 0)$ is a Coxeter algebra. By Theorem 2.15, it is enough to show the condition (D). For any $x, y, z \in X$,

$$\begin{aligned}(x * y) * z &= x * (y * z) \quad [X\text{: Coxeter algebra}] \\ &= x * (z * y) \quad [\text{Proposition 2.24(ii)}]\end{aligned}$$

This completes the proof. □

C. B. Kim and H. S. Kim ([4]) introduced and investigated BM-algebras. A *BM-algebra* is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(B2) $x * 0 = x$

(BM) $(z * x) * (z * y) = y * x$

for any $x, y, z \in X$.

It is known that the concept of a BM-algebra is equivalent to the concept of a 0-commutative B-algebra (see [4]).

By Theorem 3.3 and Theorem 3.5 of [4], $(X, *, 0)$ is a Coxeter algebra if and only if $(X, *, 0)$ is a BM-algebra with $0 * x = x$ for any $x \in X$. So we have following the result.

Corollary 2.28. $(X, *, 0)$ *is a* BN*-algebra with the condition* (D) *if and only if* $(X, *, 0)$ *is a* BM*-algebra with* $0 * x = x$ *for any* $x \in X$.

By applying Theorem 2.3, Theorem 2.12 and the results of [4] mentioned above, we obtain the following relation:

The class of Coxeter algebras $\subset$ *the class of* BM*-algebras* $\subset$ *the class of* BN*-algebras* $\subset$ *the class of* BF*-algebras.*

3. Quotient BN-algebras

In this section, we construct the quotient BN-algebra and investigate their properties.

Definition 3.1. Let $(X, *, 0)$ be a BN-algebra and let $\emptyset \neq S \subseteq X$. S is said to be a *subalgebra* of X if $x * y \in S$ for any $x, y \in S$. S is said to be *normal* of X if $(x * a) * (y * b) \in S$, whenever $x * y, a * b \in S$.

Example 3.2. In Example 2.26, $S := \{0, 3\}$ is a normal subset of X.

Theorem 3.3. *Every normal subset* S *of a* BN*-algebra* $(X, *, 0)$ *is a subalgebra of* X.

Proof. If $x, y \in S$, then $x * 0, y * 0 \in S$. Since S is normal, $x * y = (x * y) * (0 * 0) \in S$. Thus S is a subalgebra of A. □

Lemma 3.4. *Let* S *be a normal subalgebra of a* BN*-algebra* $(X, *, 0)$. *If* $x * y \in S$, *then* $y * x \in S$.

Proof. Let $x * y \in S$. Since $y * y = 0 \in S$ and S is normal, $y * x = (y * x) * (y * y) \in S$. □

We construct a quotient BN-algebra using the notion of normal subalgebra as follows. Let $(X, *, 0)$ be a BN-algebra and let S be a normal subalgebra of X. Define a relation $\sim_S$ on X by $x \sim_S y$ if and only if $x * y \in S$, where $x, y \in X$. Then it is easy to show that $\sim_S$ is an equivalence relation on X. Denote the equivalence class containing x by $[x]_S$, i.e., $[x]_S := \{y \in X \mid x \sim_S y\}$ and let $X/S := \{[x]_S \mid x \in X\}$.

Theorem 3.5. *Let* S *be a normal subalgebra of a* BN*-algebra* $(X, *, 0)$. *Then* X/S *is a* BN*-algebra.*

Proof. If we define $[x]_S * [y]_S = [x * y]_S$, then the operation " $*$ " is well-defined, since if $x \sim_S p$ and $y \sim_S q$, then $x * p \in S$ and $y * q \in S$ implies $(x * y) * (p * q) \in S$ by normality of S. So $x * y \sim_S p * q$ and so $[x * y]_S = [p * q]_S$. Note that $[0]_S = \{x \in X \mid x \sim_S 0\} = \{x \in X \mid x * 0 \in S\} = \{x \in X \mid x \in S\} = S$. Checking remaining axioms is trivial and we omit the proof. □

The BN-algebra X/S discussed in Theorem 3.5 is called the *quotient* BN-

algebra of X by S.

Let $(X, *, 0)$ and $(Y, *, 0)$ be BN-algebras. A mapping $\phi : X \longrightarrow Y$ is called a *homomorphism* from X into Y if $\phi(x*y) = \phi(x)*\phi(y)$ for any $x, y \in X$. Observe that $\phi(0_X) = 0_Y$. Indeed, $\phi(0_X) = \phi(x*x) = \phi(x)*\phi(x) = 0_Y$. We denote by $Ker\phi$ the subset $\{x \in X \mid \phi(x) = 0_Y\}$ of X. (It is called the *kernel* of the homomorphism). The proof of Theorem 3.6 follows from the Homomorphism Theorem for Algebras([6, pp. 28-29]), and we omit the proof.

Theorem 3.6. *Let S be a normal subalgebra of a BN-algebra X. Then the mapping $\gamma : X \longrightarrow X/S$ given by $\gamma(x) := [x]_S$ is an epimorphism of BN-algebras and $Ker\gamma = S$.*

Definition 3.7. A BN-algebra $(X, *, 0)$ is said to be a BN_1-algebra if it satisfies the condition:

(BN_1) $x = (x * y) * y$

for any $x, y \in X$.

Example 3.8. Let $X := \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set as in Example 2.19. Then $(X, *, 0)$ is a BN_1-algebra.

Proposition 3.9. *If $(X, *, 0)$ is a BN_1-algebra, then it is a BG-algebra.*

Proof. Let $(X, *, 0)$ be a BN_1-algebra. If we let $y := x$ in (BN_1), then $x = (x * x) * x = 0 * x$ for any $x \in A$. Thus we obtain

$$(x * y) * (0 * y) = (x * y) * y = x.$$

Hence $(X, *, 0)$ is a BG-algebra. □

Corollary 3.10. *Let $(X, *, 0)$ be a BN_1-algebra. If $x * y = 0$, then $x = y$.*

Proof. Since $x = 0 * x$ for any $x \in X$ by the proof of Proposition 3.9, we obtain $x = (x * y) * y = 0 * y = y$. □

Theorem 3.11. *Let $\phi : X \longrightarrow Y$ be a homomorphism from a BN-algebra $(X, *, 0)$ into a BN_1-algebra $(Y, *, 0)$. Then the kernel $Ker\phi$ of ϕ is a normal subalgebra of X.*

Proof. Since $0_X \in Ker\phi$, $Ker\phi \neq \emptyset$. If $x, y \in Ker\phi$, then

$$\phi(x * y) = \phi(x) * \phi(y) = 0_B * 0_B = 0_B,$$

i.e., $x * y \in Ker\phi$. Hence $Ker\phi$ is a subalgebra of X. Let $x * y, a * b \in Ker\phi$. Then $\phi(x * y) = \phi(x) * \phi(y) = 0_B$ and $\phi(a * b) = \phi(a) * \phi(b) = 0_Y$. Since Y is a BN_1-algebra, by Corollary 3.10, $\phi(x) = \phi(y)$ and $\phi(a) = \phi(b)$. Hence

$$\begin{aligned} \phi((x * a) * (y * b)) &= \phi(x * a) * \phi(y * b) \\ &= (\phi(x) * \phi(a)) * (\phi(y) * \phi(b)) \\ &= (\phi(x) * \phi(a)) * (\phi(x) * \phi(a)) \\ &= 0_Y. \end{aligned}$$

Thus $(x * a) * (y * b) \in Ker\phi$. Hence $Ker\phi$ is a normal subalgebra of X. □

Corollary 3.12. *Let $\phi : X \longrightarrow Y$ be a homomorphism from a BN-algebra $(X, *, 0)$ into a BN_1-algebra $(Y, *, 0)$. Then $X/Ker\phi \simeq Im\phi$. In particular, if ϕ is surjective, then $X/Ker\phi \simeq Y$.*

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