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Mathematical Structures and SuanXue QiMeng

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Dedicated to Professor Park Chang Kyun on his 60th birthday

It is well known that SuanXue QiMeng has given the greatest contribution to the development of Chosun mathematics and that the topics and their presentation including TianYuanShu in the book have been one of the most important backbones in the developement. The purpose of this paper is to reveal that Zhu ShiJie emphasized decidedly mathematical structures in his SuanXue QiMeng, which in turn had a great influence to Chosun mathematicians' structural approaches to mathematics. Investigating structural approaches in Chinese mathematics books before SuanXue QiMeng, we conclude that Zhu's attitude to mathematical structures is much more developed than his precedent ones and that his mathematical structures are very close to the present ones.

Keywords: mathematical structures, Zhu ShiJie(朱世傑), SuanXue QiMeng(算學啓蒙), Liu Hui(劉徽), Yang Hui(楊輝).

MSC: 01A07, 01A25, 01A35, 11–03, 11A05

0 Introduction

Since the set theory has been established, mathematics is nowadays known to be a study of mathematical structures on sets although it contains a vicious cycle. In the eastern mathematics, the underlying set for algebraic and order structures was assumed to be the field of rational numbers since JiuZhang SuanShu(九章算術). Further, areas and volumes of various figures and theories of right triangles were the main subject for the geometrical structures. The former clearly involves a primitive approximation structure or topological structure and the latter the similarity of right triangles explained by areas.

As is well known, every mathematical subject was explained by the word problems dealing with daily topics. Thus it might be difficult at first sight to find the

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mathematical structural approaches in the eastern mathematics. In the preface of JiuZhang SuanShu by Liu Hui(劉徽), he stressed mathematical principles or mathematical structures which he meant basic ingredients appearing in various problems. JiuZhang SuanShu became the basic book for mathematical studies throughout the whole history of eastern mathematics and hence his dictum has been followed by eastern mathematicians.

SuanXue QiMeng(算學啓蒙, 1299) played the most important role for the development of Chosun mathematics. We point out that JiuZhang SuanShu was introduced to Chosun in the middle of the 19th century[17] and that SuanXue QiMeng was lost in China until 1830's. In this paper, we show that the book's contribution to the development is two fold: the subject matters including TianYuanShu(天元術) and mathematical structural approach to mathematics. We have already treated the contribution through the subject matters in [9, 11, 13] (also see [8, 15, 18]) and therefore, we deal with mathematical structures in SuanXue QiMeng in the present paper.

The paper is divided into two parts. In the first, we review the structural approaches in Chinese mathematics up to Yang Hui(楊輝) from Liu Hui and then deal with Zhu ShiJie's attitude to mathematical structures in the remaining part. Investigating SuanXue QiMeng, his concept of mathematical structures is different from his predecessors' ones and is much the same as the modern ones.

For the Chinese sources, we refer to ZhongGuo KeXue JiShu DianJi TongHui ShuXueJuan(中國科學技術典籍通彙 數學卷)[3] and ZhongGuo LiDai SuanXue JiCheng (中國歷代算學集成)[4]. Those books appeared in them will not be numbered as an indivisual reference.

1 Mathematical structures before Zhu ShiJie

Detecting basic common ingredients in various mathematical objects and then abstracting them, we introduce mathematical structures. As is already mentioned, mathematical objects or underlying sets are kept down to the field of rational numbers in the history of eastern mathematics and hence one can not consider the process of abstraction in the eastern mathematics. Thus mathematical structures are defined by the process of detecting common ingredients in various mathematical problems, which are denoted by "Lei(類)". We note that Lei has been also used for various objects.

The earliest Lei appeared in the dialogue between RongFang(榮方) and ChenZi (陳子) in the second chapter of ZhouBi SuanJing(周髀算經), where the following famous sentence in the ninth chapter of the appendix XiCi(繫辭) to YiJing(易經) is also quoted.

引而伸之 觸類而長之 天下之能事畢矣

The first part of the above quote was quoted in most of mathematics books and it leads to the mathematical structure in our sense. We note that the "Lei" here means various objects.

Lei in our sense was first mentioned by Liu Hui in his preface to JiuZhang SuanShu as follows:

事類相推 各有攸歸 故枝條雖分而同本幹知 發其一端而已

Here Shi(事) means various objects, Lei(類) their intrinsic common mathematical structure. Further, Shi and Lei generate in cycle the others respectively as "剛柔相推而生變化" in the second chapter of XiCi. This statement means precisely the mathematical structural approach to mathematics (see also [2]). For Liu Hui's general attitude to mathematics and his philosophical motivations, also see [5, 6]. Although his attitude was preserved throughout the whole history of the eastern mathematics, Liu had some restriction for a full structural approach because he presented his view as commentaries to JiuZhang SuanShu.

Since the publication of XiangJie JiuZhang SuanFa(詳解九章算法, 1261), Yang Hui explicitly revealed Liu Hui's view to Lei. As is well known, XiangJie JiuZhang SuanFa has missing chapters. In the book, he introduced problems called BiLei(比類). Using this, he showed intrinsic mathematical strucutres involved in various objects. Since the first chapter FangTian(方田) is missing in XiangJie, where the basic operations on the set of positive rational numbers are introduced, we should refer to TianMu BiLei ChengChu JieFa(田畝比類乘除捷法, 1275) for Yang's view to operations. In its first Book(上卷), Yang begins with the area of a rectangle for the multiplication and then states the following:

直田長闊相乘與萬象同 中山劉先生益 議古根源序曰 入則諸門出則直田
蓋直田能致諸用 而有是說諸家算經 皆以直田爲第一問亦默會也

He then added various BiLei problems dealing with multiplications of various cases. As in JiuZhang, Yang Hui applied the well known area of a rectangle to the other cases as "...與萬象同". After this, he obtained formulas of various polygons and then extended them to the theory of arithmetic series. TianMu BiLei ChengChu JieFa is the second book after ChengChu TongBian SuanBao(乘除通變算寶, 1274) in YangHui SuanFa(楊輝算法, 1274–1275) which is devoted to methods of operation, multiplication and division. As is mentioned above, one can easily discern the difference regarding structural approaches between two books.

We now return to BiLei problems in XiangJie JiuZhang SuanFa. The first BiLei problem in the chapter YingBuZu(盈不足) is given by the same data as the original problem but is dealing with completely different topic. This kind of BiLei problems also appears in the second Book(下卷) of TianMu BiLei ChengChu JieFa. In this case, his main problem was taken by a BiLei problem in the chapter GouGu (句股) of XiangJie how to construct and solve a quadratic equation dealing with a rectangle. The other BiLei problems have the same mathematical ingredients but deal with the different situations. We point out that the series problems on triangular number series $\sum_{k=1}^{n}\big(\sum_{i=1}^{k} i\big)$(三角垛) and the series $\sum_{k=1}^{n}\big(a+k-1\big)\big(b+k-1\big)$ $(a, b \in \mathbb{N})$ given by Shen Kuo(沈括, 1031–1095)(菓子一垛) in the chapter ShangGong(商功) are clearly somewhat variations to the original problems dealing with volumes[10]. These show that Yang's BiLei problems are also attached to the original ones in JiuZhang SuanShu and hence his view of Lei is a stage of detecting common ingredients in various problems(觸類) as Liu Hui's one.

Yang Hui made a real breakthrough on Lei when he wrote XiangJie JiuZhang SuanFa ZuanLei(詳解九章算法纂類), abbreviated to ZuanLei in the sequel, as an appendix to XiangJie. He disregarded the titles of chapters of JiuZhang and then rearranged its problems according to mathematical structures(類) as follows:

> ChengChu(乘除), HuHuan(互換), HeLu(合率), FenLu(分率), CuiFen (衰分), LeiJi(壘積), YingBuZu(盈不足), FangCheng(方程), GouGu(句股)

One can easily notice that the titles of four chapters, i.e., CuiFen, YingBuZu, FangCheng and GouGu are retained in his new classifications. ChengChu basically replaces FangTian as we mentioned above on TianMu BiLei ChengChu JieFa, HuHuan the first part of SuMi(粟米), FenLu its last part, LeiJi ShangGong(商功), respectively. Yang put the first part of ShaoGuang(少廣) in HeLu and the remaining part dealing with extractions of square and cube roots in GouGu. Finally JunShu was divided into HuHuan, HeLu, FangCheng and the obvious CuiFen. This shows Yang Hui's attitude for structural approach as Lei by modes involved in the solving process(因法推類) (see also [7]). But Yang detected ChengChu in the areas of various figures which is one of the most important subject of the classical mathematics. This shows that Yang Hui had an improved insight on structural approach.

2 Mathematical structural approaches in SuanXue QiMeng

Zhu ShiJie published SiYuan YuJian(四元玉鑑, 1303) which contains Zhu's great insight of mathematics but it is too much condensed so that one had to wait for understanding fully its contents until good commentaries were published in the 19th century. Furthermore, his structural approaches in SuanXue QiMeng would give us enough informations and hence our discussion will be limited to SuanXue QiMeng.

We begin with the following quotation from the section ShuangJu HuHuanMen (雙據互換門) in the second Book(中卷) of QiMeng.

> 雙據互換之法 學者少識所乘所除之理 前問織綿三術 返復還源備矣
> 此問與煎鹽議同 而今有之及雇車行道相類也 故引草證使習 筭者無疑矣

ShuangJu HuHuanMen is concerned with problems of compound ratio which is the first section dealing with double conditions in QiMeng. Thus Zhu took the section for introducing his view on Lei, or mathematical structures. It deals with three kinds of problems, namely weaving cottons, manufacturing salt and hiring carts and then a variation. Three problems of the first kind are given by rotating a condition in the same situation. Zhu suggested to grasp the basic ingredients(還源) and to find the next two problems are mathematically same(議同). Thus the readers must detect that they have the same mathematical structure(相類). These processes lead to a full understanding on the given problems. Contrary to the fact that Yang Hui determined mathematical structures by solving process(因法推類), Zhu decided mathematical structures involved in various situations by comparing problems or given situations(觸類而長). Furthermore, as the last problem of the section ShuangJu HuHuanMen indicates, Zhu put a few variation problems in the end of each section which have the same structure but lead to a future applications. This indicates that Zhu is an excellent educator as indicated in the preface to SiYuan YuJian by MoRuo(莫若).

Zhu ShiJie used the above process in the first part of QiMeng, where he dealt with the operations, multiplication and division. For the readers to understand mathematical structures, he made problems for the division which have the exactly same numbers with those for the multiplication but deal with different situations. He put the following statement just before the division problems:

> 通前之問 還源於除法 內訓導初學 務要演熟 乘除加減 引而伸之

We note that in the section DuiJi HuanYuanMen(堆積還源門), Zhu repeats the above process. Indeed, he introduced problems on sums of various series and then using the data from these problems, constructed equations. Finally he put a problem of an equation given by two series, the triangular number series(三角垛) and the series $\sum_{k=1}^{n} k^2$ of sequence of squares(四角垛), which leads to the great contribution to the theory of series in Zhu's SiYuan YuJian. Except this problem, he had equations directly from the formulas of series. It would be much better that Zhu introduced the TianYuanShu in this last problem. As is well known, he might defer the TianYuanShu to the last problem in the section FangCheng ZhengFuMen (方程正負門) just before the section KaiFang ShiSuoMen(開方釋鎖門) which deals with theory of equations. This indicates Zhu's systematic presentation in QiMeng. We recall that Liu Hui used freely the results from ShaoGuang and GouGu in his discussion of π in the first chapter FangTian. This would give novices a great difficulty to understand his commentary.

In this direction, we now take the theory of fractions in QiMeng. We note that Chinese mathematicians used a dual numerical systems, one in spoken language and the other by counting rods. The former is given by denominate numbers including monetary units[14]. Although Zhu introduced numerical system in spoken language, his presentation in the first Book(上卷) implicates that his system is mainly one by counting rods, which leads to a slightly awkward notation for fractions. When he used the denominate numbers, he had eventually infinite decimals. We note that 1 Liang(兩) = 24 Zhu(銖) and 1 Mu(畝) = 240 Bu(步) are not in the decimal system and contain 3 as a divisor. Zhu ShiJie managed to avoid infinite decimals for nearly all problems in the first two books by arranging conditions. He first used the division algorithm, i.e., for given natural numbers a, b, there exist unique natural numbers q, r with $a = bq + r$, $0 \leq r < b$ and then converting 2106 Zhu = 9 Jin(斤) 7 Liang 18 Zhu in the section JiuGui ChuFaMen(九歸除法門) in the first Book, where he gave many more examples about the division algorithm.

In problem 11 of the later section KuWu JieShuiMen(庫務解稅門) in the first Book, he had a fraction $3301\frac{737}{1363}$ by the division algorithm for $4,500,000 = 1,363\times 3,301 + 737$. He then freely had answers containing fractions in the second Book for infinite decimals as the above example.

Beginning with the division algorithm of the set $\mathbb{N}$ of natural numbers, Zhu ShiJie introduced fraction particularly to avoid infinite decimals and then in the first section ZhiFen QiTongMen(之分齊同門) of the third Book where he introduced operations on the set of positive fractions, he stated exactly his structural approach to the set of non-negative rational numbers as follows:

> 但有除分者餘不盡之數 不可棄之棄之則不合其源 可以爲之分言之
> 之分者乃乘除往來之數 還源則不失其本也 故九章設諸分於篇首者
> 爲何謂之分者乃開筭之戶牖也 緣其義閎遠 其術奥妙 是以學者造之鮮矣

After introducing the set $\mathbb{Q}_0$ of nonnegative rational numbers in the previous two Books, Zhu ShiJie dealt with the basic operations and order on $\mathbb{Q}_0$. This is precisely the modern concept of algebraic and order structures. Furthermore, Zhu claimed that the mathematical object $\mathbb{Q}_0$ endowed with the above structures is the starting point for mathematics(開筭之戶牖). We also note that the above procedure and view to mathematical structures are much different from Liu and Yang's ones although JiuZhang was referred here as well.

3 Conclusion

Kim SiJin(金始振, 1618–1667) republished SuanXue QiMeng in 1660 and then Chosun mathematics could have successful revival after a series of foreign invasions. Park Yul(朴繘, 1621–1668) published a book SanHak WonBon(算學原本, 1700)[12, 16] posthumously which deals mainly with the theory of equations including TianYuanShu and then Hong Jung Ha(洪正夏, 1684–?) the greatest Chosun mathematics book, GuIlJib(九一集, 1724)[1]. Both mathematicians are greatly indebted to SuanXue QiMeng for their mathematical works. This motivates our inquiry why SuanXue QiMeng has so much influences for the development of mathematics in Chosun.

We find that Zhu ShiJie wrote SuanXue QiMeng with a far more deeper insight on mathematical structures than his predecessors including Liu Hui and Yang Hui. Liu and Yang also used a concept of mathematical structures, which are strongly attached to JiuZhang SuanShu but Zhu is rather free from JiuZhang SuanShu and hence he could have built his own view on mathematical structures which are very close to modern mathematical structures and then succeeded in completing the book with structural approaches to mathematics. Hong Jung Ha fully understood Zhu's view and therefore he could extend the approaches and accomplish his own mathematics.

Although Chosun could not have a complete collection of Chinese mathematics books, Chosun was able to achieve her own mathematics based on the excellent structural approaches initiated by Zhu ShiJie in his SuanXue QiMeng.

References

1. 金容雲 編,《韓國科學技術史資料大系》, 數學編, 驪江出版社, 1985.
2. 郭書春 譯注,《九章筭術 譯注》, 上海古籍出版社, 2009.
3. 郭書春 主編,《中國科學技術典籍通彙》數學卷 全五卷, 河南教育出版社, 1993.
4. 靖玉樹 編勘,《中國歷代算學集成》上, 中, 下, 山東人民出版社, 1994.
5. 吳文俊 主編,《中國數學史大系》, 北京師范大學出版社, 1999.
6. 周瀚光, 孔國平,《劉徽評傳, 附 秦九韶, 李冶, 楊輝, 朱世傑 評傳》, 南京大學出版社, 1994.
7. 孔國平, "楊輝《詳解九章算法》初探", 數學史研究文集 第四輯(1993), 36–41, 科學出版社.

8. Her Min, "A comparison between Suanxue qimeng(Introduction to Mathematical Studies) and Muksa-jipsanbub(산학계몽과 묵사집산법의 비교)", *The Korean Journal for History of Mathematics*(한국수학사학회지), 21(1) (2008), 1–16.
9. Hong Sung Sa, "Theory of Equations in the history of Chosun Mahtematics", Proceeding Book 2, The HPM Satellite Meeting of ICME-12, (2012), 719–731.
10. Hong Sung Sa, "Finite series in Chosun Dynasty Mathematics(朝鮮 算學의 堆垛術)", *The Korean Journal for History of Mathematics*(한국수학사학회지), 19(2) (2006), 1–24.
11. Hong Sung Sa, Hong Young Hee, "Chosun Mathematics Book Suan Xue Qi Meng Ju Hae(朝鮮 算書 算學啓蒙註解)", *The Korean Journal for History of Mathematics*(한국수학사학회지), 22(2) (2009), 1–22.
12. Hong Sung Sa, Hong Young Hee, Kim Chang Il, "Approximate Solutions of Equations in Chosun Mathematics(方程式의 近似解)", *The Korean Journal for History of Mathematics*(한국수학사학회지), 25(3) (2012), 1–14.
13. Hong Sung Sa, Hong Young Hee, Kim Chang Il, "GouGuShu in the 18th century Chosun(18世紀 朝鮮의 句股術)", *The Korean Journal for History of Mathematics*(한국수학사학회지), 22(2) (2009), 1–12.
14. Hong Sung Sa, Hong Young Hee, Lee Seung On, "TianYuanShu and Numerical Systems in Eastern Asia(天元術과 記數法)", *The Korean Journal for History of Mathematics*(한국수학사학회지), 25(4) (2012), 1–10.
15. Jin Yuzi, Kim Young Wook, "Chosun mathematics in the 17th century and Muk Sa Jib San Beob(17세기 朝鮮 算學과《默思集筭法》)", *The Korean Journal for History of Mathematics*(한국수학사학회지), 22(4) (2009), 15–28.
16. Kim Young Wook, Hong Sung Sa, Hong Young Hee, "Park Yul's SanHak WonBon(朴繘의 算學原本)", *The Korean Journal for History of Mathematics*(한국수학사학회지), 18(4) (2005), 1–16.
17. Lee Chang Koo, Hong Sung Sa, "Chinese Mathematics in Chosun", *Journal for History of Mathematics*, 26(1) (2013), 1–9.
18. Yun Hye Soon, "Mathematics educational constructions and structures in SuanXue QiMeng and MukSaJibSanBub(算學啓蒙과 默思集算法의 수학 교육적 구성과 구조)", *The Korean Journal for History of Mathematics*(한국수학사학회지), 25(4) (2012), 11–19.

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ABSTRACTS

Hong Sung Sa, Hong Young Hee, Lee Seung On 홍성사, 홍영희, 이승온 Mathematical Structures and SuanXue QiMeng『數學的 構造와 算學啓蒙』

朱世傑의 算學啓蒙은 조선 산학의 발전에 가장 중요한 역할을 한 산서이다. 천원술을 비롯한 算學啓蒙의 내용은 조선 산학의 중요한 연구 대상이 되었다. 이 논문의 목적은 朱世傑이 수학적 구조를 강조하면서 算學啓蒙을 저술한 것을 보여서 조선 산학자들에게 수학적 구조에 대한 이해를 크게 확장한 것을 드러내는 것이다. 이와 함께 朱世傑이전의 산서에 나타나는 구조적 접근과 算學啓蒙의 접근을 비교하여 朱世傑의 접근이 뛰어나고 또 현대에 사용되는 구조적 접근과 일치하는 것을 보인다.

Song Min Ho 송민호 A Study on Learning Environments for Euler's formula with activities『'오일러 공식과 오일러 표수' 탐구 활동을 위한 학습 환경 연구』

오일러 공식과 오일러 표수는 다면체를 탐구하는 지표의 역할을 하기 때문에 위상적 불변량이라는 관점에서 중요한 수학적 개념이다. 우리나라는 3차부터 7차 교육과정까지 오일러 공식에 관한 내용이 교과서에 언급되었으나 이후 교육과정에서 제외되었다. 본 연구에서는 영재교육이나 방과후교실과 같은 비형식적(informal) 교육과정의 소재로 오일러 공식과 오일러 표수에 주목하였다. 본 연구에서는 먼저 오일러 공식과 오일러 표수가 가지는 의미를 수학사와 그 응용 분야, 교육과정에서 찾아본다. 이를 위해 오일러 공식과 오일러 표수의 역사, 다양한 수학 분야에 기여한 내용, 그리고 교육과정에 도입된 오일러 공식에 관한 내용을 살펴본다. 나아가 공식 암기가 아닌 탐구 활동의 대상으로 오일러 공식을 새롭게 조명할 수 있는 학습 환경을 제안하고 이를 이용한 활동을 예를 들어 살펴본다.

Choi Eun Mi 최은미 Historical analysis of System of Equations—Focused on Resultant『연립방정식 풀이의 역사발생적 고찰—종결식을 중심으로』

본 논문에서 연립일차방정식의 풀이법 연구로부터 시작하여 연립고차방정식의 해법 연구로 발전되어가는 과정을 역사발생적 관점에서 고찰한다. 연립일차방정식을 푸는데 중요한 역할을 하는 가우스 소거법과 비교하여 상대적으로 덜 알려져 있지만, 연립고차방정식에는 오일러의 소거이론과 베조의 종결식이 있다. 이러한 발전의 역사적 과정을 알아보고 특별히 종결식을 처음으로 정의한 베조의 연구 방법을 조명해 본다.

Lee Jung Oh 이정오 A Brief Study on Stanojevic's Works on the $\mathscr{L}^1$-Convergence『Stanojevic의 푸리에 급수의 $\mathscr{L}^1$-수렴성 연구의 소 계보 고찰』

본 논문은 저자의 선행 연구 결과에 따른 부가적인 연구로 '푸리에 급수의 $\mathscr{L}^1$-수렴성'에 관한 많은 업적을 남긴 세계적인 수학자인 스타노제빅(Caslav V. Stanojevic)[1]을 중심으로 20세기 후반부터 21세기 초까지(1973–2002) 30년간 그의 연구결과를 순차적으로 고찰하여 푸리에 급수의 $\mathscr{L}^1$-수렴성 연구자들의 2012년까지 소 계보를 조사한다.

1) 스타노제빅(1928–2008) ; 세르비아 (구 유고슬라비아) 출신의 세계적인 수학자.