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Length-biased Rayleigh distribution: reliability analysis, estimation of the parameter, and applications

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Abstract. In this article, a new model based on the Rayleigh distribution is introduced. This model is useful and practical in physics, reliability, and life testing. The statistical and reliability properties of this model are presented, including moments, the hazard rate, the reversed hazard rate, and mean residual life functions, among others. In addition, it is shown that the distributions of the new model are ordered regarding the strongest likelihood ratio ordering. Four estimating methods, namely, method of moment, maximum likelihood method, Bayes estimation, and uniformly minimum variance unbiased, are used to estimate the parameters of this model. Simulation is used to calculate the estimates and to study their properties. Finally, the appropriateness of this model for real data sets is shown by using the chi-square goodness of fit test and the Kolmogorov-Smirnov statistic.

Key Words*: Reliability function, hazard rate function, reversed hazard rate function, mean residual life function, moment method, maximum likelihood estimates, Bayesian estimates, uniformly minimum variance unbiased estimator, stochastic ordering*

1. INTRODUCTION

Statistical analysis is a useful tool for obtaining the reliability information of a device or process based on a limited number of samples. Sample sets are analyzed referenced to a known distribution function. Once a distribution function is assumed, the sample mean and variance are used to determine reliability information. A widely used model event that occurs in different fields such as medicine and social and natural sciences is the Rayleigh distribution (RD). In physics, for instance, an RD is used to study various types of radiation, such as sound and light measurements. The RD was originally derived in connection with a problem in acoustics, and has been used in modeling certain features of electronic waves and as the distance distribution between individuals in a spatial Poisson

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process. Most frequently, however, an RD appears as a suitable model in life testing and reliability theory. For more details on the RD, the reader is referred to Polovko (1968) and Johnson et al. (1994).

In contrast, length-biased (LB) distributions are particular cases of weighted distributions. Specifically, if Y is a nonnegative random variable with probability distribution function (pdf) $f_Y(y)$, then the weighted version of Y with weight function $w(y)$, denoted by Y_w and whose distribution is called the weighted distribution, has a pdf given by

$$f_{Y_w}(y) = \frac{w(y)\ f_Y(y)}{E[w(Y)]}, \quad y > 0, \tag{1.1}$$

assuming $0 < E[w(Y)] < \infty$. A particular case of the weighted distributions is obtained when we replace $w(y) = y$ in Eq. (1.1). In this case, Y_w is called the size-biased or length-biased version of Y, denoted by the r.v. T, which has the pdf expressed as

$$f_T(t) = \frac{t\ f_Y(t)}{\mu}; \quad t > 0,$$

where $0 < \mu = E[T] < \infty$. The distribution of T is the length biased version of the distribution of Y. The parameter that corresponds to the mean from the original distribution is incorporated in the length-biased density to obtain valid density. Actually, the mean is a function of the parameters from the original model. Thus, length-biased distribution does not incorporate new parameters and has the same number of parameters as the original distribution (cf. Jain et al., 1989).

LB distributions have been applied in various fields, such as biometry, ecology, environmental sciences, reliability, and survival analysis. A review of these distributions and their applications is included in Gupta and Kirmani (1995). LB distribution occurs naturally in many situations because sometimes it is not possible to work with a truly random sample from the population of interest. In particular, in the environmental field, Patil (2002) mentioned that observations may fall in the non-experimental, non-replicated, and nonrandom categories making random selection from the target population impossible. Thus, in this case, model specification and data interpretation problems acquire great importance. A way of confronting this problem is by considering observations selected with probability proportional to their length. The resulting distribution is called length-biased distribution, which considers the method of ascertainment by adjusting the probabilities of the actual occurrence of events to arrive at a specification of the probabilities of those events as observed and recorded. Failure to make such adjustments can lead to incorrect conclusions. LB versions for several distributions, such as Weibull, inverse Gaussian (IG), Sinh-normal (SN), and Birnbaum-Saunder (BS) distributions, have been developed in the literature (cf. Sansgiry and Akman (2001), Boudrissa and Shaban (2007), and Leiva et al. (2009)), among others.

The objectives of this paper are as follows: (i) obtain a new probability model that can be used in physics, reliability, and life testing; (ii) provide a comprehensive description of the statistical and probabilistic properties of this new model; and (iii) illustrate its applicability in different areas. The paper is organized as follows: The density and the moment of the new model are given in Section 2. In this section, we provide the reliability function and hazard (reversed hazard) functions and show that the new model is ordered regarding the strongest likelihood ratio ordering. Moment estimation, maximum likelihood estimation, and Bayesian estimation problems are considered in Section 3. To indicate the adequacy of the new model, applications using numerical examples and examples with real data are discussed in Section 4. Finally, in Section 5, we provide a brief conclusion and remarks regarding current and future research.

2. THE NEW MODEL AND ITS RELIABILITY PROPERTIES

Let X be a random variable with scale parameter $\theta > 0$. The pdf of X is

$$g(x,\theta) = \theta\, x\, e^{-\frac{\theta}{2}x^2}, \quad x \geq 0,\ \theta > 0.$$

The distribution of X is called a Rayleigh distribution (RD). Based on the RD, we define the following new model.

Definition 2.1.

A nonnegative random variable T is said to have a length-biased Rayleigh (LBR) distribution if the variable's density function is given by

$$f(t,\theta) = \sqrt{\frac{2}{\pi}}\,\theta^{\frac{3}{2}}\, t^2\, e^{-\frac{\theta}{2}t^2}; \ \ t > 0,\ \ \theta > 0. \tag{2.1}$$

For this new model, the notation $T \sim LBR(\theta)$ will be used in the sequel.

Interpretation 2.1.

Suppose that the lifetimes of a given sample of items follow the Rayleigh distribution and that the items do not have the same chance of being selected but each one is selected according to its life length. Then the resulting distribution does not follow the RD. The distribution follows the LBR distribution.

Interpretation 2.2.

Suppose that $T \sim LBR(\theta)$ and let $Z = (\theta/2)T^2$. Then Z follows the $\text{Gamma}(3/2,1)$ distribution.

Clearly, $f(t,\theta)$ vanishes as $t \to 0$ and $t \to \infty$. The function $f(t,\theta)$ is easily shown to be log-concave, and thus, the distribution is always unimodal with mode at $\sqrt{\frac{2}{\theta}}$. The shapes of the pdf for special values of the scale parameter θ are illustrated in Figure 1.

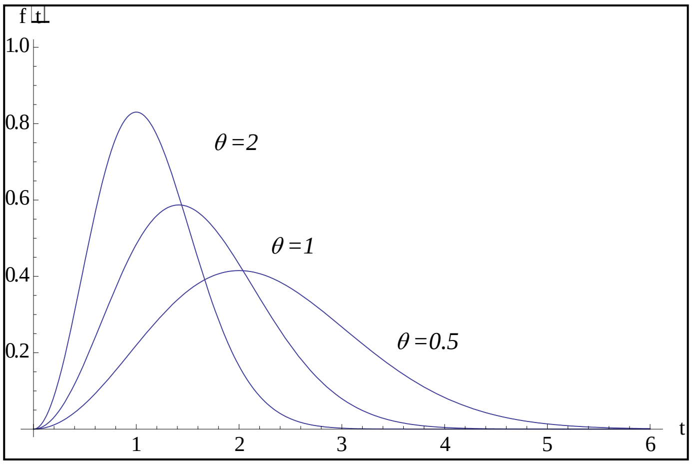

Figure 1: Plot of the pdf for different values of θ.

The positive integer moments are useful for inference and model fitting (cf. Johnson et al. (1994, p. 23) and Leiva et al. (2009)). A result that allows us to compute the moments of the LBR distribution is given in the following lemma.

Lemma 2.1.

If $T \sim LBR(t;\theta)$, then the $r^{\underline{th}}$ moment is given by

$$E\left(T^r\right)=\frac{2^{\frac{r}{2}+1}\,\Gamma\left(\frac{r+3}{2}\right)}{\sqrt{\pi}\,\theta^{\frac{r}{2}}}.$$

From the result, the mean and variance are given, respectively, by

$$E(T)=\frac{2^{\frac{3}{2}}}{\sqrt{\pi\theta}} \text{ and } Var(T)=\frac{3\pi-8}{\theta\pi}.$$

The coefficient of skewness γ_1, is defined by

$$\gamma_1=\frac{\mu_3}{\sigma^3},$$

where μ_3 is the third moment about the mean and σ is the standard deviation of the distribution. Thus, the skewness of the LBR distribution is given by

$$\gamma_1=\frac{2\sqrt{2}\left(16-5\pi\right)}{(3\pi-8)^{\frac{3}{2}}},$$

which is independent of θ. Since γ_1 is positive, then the distribution is skewed to the right. This is clear from the plot of the pdf in Figure 1. The coefficient of kurtosis γ_2, is defined by

$$\gamma_2 = \frac{\mu_4}{\sigma^4} - 3,$$

where μ_4 is the fourth moment about the mean. In the case of the LBR distribution, we have

$$\gamma_2 = 4\left[\frac{40\pi - 3\pi^2 - 96}{(3\pi - 8)^2}\right],$$

and thus, the distribution is leptokurtic since the kurtosis is positive. Let $T \sim LBR(\theta)$, The reliability function (RF) of T is given by

$$\overline{F}(t,\theta) = P_\theta(T > t) = 1 - \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{\theta}{2}t^2\right), \quad \theta > 0. \tag{2.2}$$

where $\gamma\left(\frac{3}{2}, \frac{\theta}{2}t^2\right)$ is the lower incomplete gamma function, which is defined by

$$\gamma(a,t) = \int_0^t x^{a-1}e^{-x}dx.$$

Let X and Y be two nonnegative random variables having distribution functions F_X and F_Y, respectively, denote by $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y = 1 - F_Y$ their respective survival functions, and f_X and f_Y are corresponding density functions. The hazard rate (HR) function of the nonnegative random variable X is

$$h_X(x) = -(d/dx)\left[\log \overline{F}(x)\right], \quad \text{for all } x > 0,$$

and thus, the HR function of LBR distribution is given by

$$h_T(t,\theta) = \frac{\sqrt{2}\,\theta^{\frac{3}{2}}\,t^2\,e^{-\frac{\theta}{2}t^2}}{\sqrt{\pi} - 2\gamma(\frac{3}{2}, \frac{\theta}{2}t^2)}.$$

The behavior of the HR and the HR average is the same. Since $f(t)$ is log-concave, then the HR function $h(t)$ is increasing (see Barlow and Proschan (1975)). However, in replacement and repair strategies, although the shape of the HR function plays an important role, the mean residual life (MRL) function of the random variable X, which is defined by

$$m(t) = \frac{1}{R(t)}\int_t^\infty y\, f(y)dy - t,$$

is more relevant than the HR function because the former summarizes the entire residual life function, whereas the latter considers only the risk of instantaneous failure at some time t. The MRL function of the LBR distribution is given by

$$m(t,\theta) = \frac{2^{\frac{3}{2}}\Gamma\left(2, \frac{\theta}{2}t^2\right)}{\sqrt{\theta\pi} - 2\sqrt{\theta}\,\gamma\left(\frac{3}{2}, \frac{\theta}{2}t^2\right)} - t,$$

where $\Gamma\left(2, \frac{\theta}{2}t^2\right)$ is the upper incomplete gamma function, which is defined by

$$\Gamma(a,t) = \int_t^\infty x^{a-1}e^{-x}dx.$$

Increasing the hazard rate (IHR) implies decreasing the mean residual life (DMRL) (Barlow and Proschan, 1975). Therefore, if T is a nonnegative random variable that follows $LBR(\theta)$, then T has the DMRL property.

Another function called the reversed hazard rate function (RHR) has recently been receiving increasing attention in reliability analysis and life testing. Formally, the RHR of a nonnegative random variable X is defined by (cf. Block et al. (1998) and Chandra and Roy (2001))

$$r_X(x) = (d/dx)[\log F(x)], \quad \text{for all } x > 0.$$

The RHR of the LBR distribution takes the form

$$r_T(t,\theta) = \frac{\theta^{\frac{3}{2}} t^2 e^{-\frac{\theta}{2}t^2}}{\sqrt{2}\gamma\left(\frac{3}{2}, \frac{\theta}{2}t^2\right)}.$$

Again, since $f(t,\theta)$ is log-concave, then $r_T(t,\theta)$ is decreasing. Therefore, we conclude that T also has the increasing mean inactivity time (IMIT) property (Kayid and Ahmad, 2004).

Another important tool for judging comparative behavior is the stochastic ordering of positive continuous random variables. A nonnegative random variable X is said to be smaller than a random variable Y in the following:

i. stochastic order (denoted by $X \le_{st} Y$) if $\overline{F}_X(x) \le \overline{F}_Y(x)$ for all x;
ii. hazard rate order (denoted by $X \le_{hr} Y$) if $h_X(x) \ge h_Y(x)$ for all x;
iii. reversed hazard rate order (denoted by $X \le_{rhr} Y$) if $r_X(x) \le r_Y(x)$ for all x;
iv. likelihood ratio order (denoted by $X \le_{lr} Y$) if $f_X(x)/f_Y(x)$ decreases in x.

The following implications between the above stochastic ordering are well known (see Shaked and Shanthikumar (2007)):

$$X \le_{rhr} Y \leftarrow X \le_{lr} Y \rightarrow X \le_{hr} Y \rightarrow X \le_{st} Y. \tag{2.3}$$

LBR distributions are ordered regarding the strongest likelihood ratio ordering as shown in the following result.

Theorem 2.1.

Let $X \sim LBR(t;\theta_1)$ and $Y \sim LBR(t;\theta_2)$. If $\theta_2 < \theta_1$ then

$$X \le_{lr} Y (X \le_{hr} Y, X \le_{rhr} Y, X \le_{st} Y).$$

Proof.

First, let $g(x) = f_X(x)/f_Y(x)$ then

$$g(x) = (\theta_1/\theta_2)^{\frac{3}{2}} e^{-\frac{x^2}{2}(\theta_1-\theta_2)}.$$

Since

$$\frac{d}{dx}[g(x)] = -x(\theta_1-\theta_2)(\theta_1/\theta_2)^{\frac{3}{2}} e^{-\frac{x^2}{2}(\theta_1-\theta_2)},$$

then $g(x)$ is decreasing in x for $\theta_2 < \theta_1$, that is, $X \leq_{lr} Y$. The remaining statements from the implications (2.3).

3. PARAMETER ESTIMATION

In this section, four estimating methods, namely, method of moment, maximum likelihood method, Bayes estimation, and uniformly minimum variance unbiased, are used to estimate the model's parameters.

3.1 Moments estimate

In the method of moments estimator (MME), we have to solve the equation

$$m_r^{'} = \mu_r^{'} \ , \ r = 1, 2, ..., \tag{3.1}$$

where $m_r^{'} = \frac{1}{n}\sum_{i=1}^{n} t_i^r$ the sample moment and $\mu_r^{'} = E(T^r)$, the population moment. Solving (3.1), we have

$$\hat{\theta} = \frac{8}{\pi(\bar{T})^2}, \tag{3.2}$$

3.2 Maximum likelihood estimate

Let $T_1,...,T_n$ be a random sample from the LBR distribution. The likelihood function is given by

$$L(\theta \mid t_1,...,t_n) = \Pi_{i=1}^{n} f(t_i, \theta)$$

$$= \left(\frac{2\theta}{\pi}\right)^{\frac{n}{2}} \theta^n \, \Pi_{i=1}^{n} t_i^2 \exp\left(-\frac{\theta}{2}\sum_{i=1}^{n} t_i^2\right).$$

The log likelihood function is given by

$$\ell(\theta \mid t_1,...,t_n) = \frac{n}{2}\ln 2 + \frac{n}{2}\ln\theta - \frac{n}{2}\ln\pi + n\ln\theta + 2\sum_{i=1}^{n}\ln t_i - \frac{\theta}{2}\sum_{i=1}^{n} t_i^2.$$

The maximum likelihood estimator (MLE) of θ is the solution of the following normal equation

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{2\theta} + \frac{n}{\theta} - \frac{1}{2}\sum_{i=1}^{n} t_i^2 = 0,$$

and thus

$$\theta = \frac{3n}{\sum_{i=1}^{n} t_i^2}. \tag{3.3}$$

The second derivative of the MLE is

$$\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{3n}{2\theta^2}.$$

Then the Fisher information is

$$I(\theta) = -E\left(\frac{\partial^2 \ell}{\partial \theta^2}\right) = \frac{3n}{2\theta^2}.$$

If n is large, MLEs are asymptotically normally distributed with the mean θ and variance of $\theta,$ that is,

$$I(\theta)\left(\hat{\theta} - \theta\right) \xrightarrow{d} N(0,1),\ as\ n \to \infty.$$

A $100(1-\delta)\%$ CI for θ is given by

$$\hat{\theta} \pm Z_{\delta/2}\sqrt{I^{-1}(\theta)},$$

where $Z_{\delta/2}$ is the standard normal variate.

3.3 Bayes estimates

If we have a little information about the parameter, then the appropriate prior for this case when using Bayes estimation is $\pi(\theta) \propto \frac{1}{\theta},$ where $\propto$ is the proportionality sign. The joint posterior will be

$$\pi(\theta \mid \underline{T}) = \frac{\left(\frac{1}{2}\sum_{i=1}^{n} t_i^2\right)^{\frac{3n}{2}}}{\Gamma\left(\frac{3n}{2}\right)} \theta^{\frac{3n}{2}-1} \exp\left(-\frac{\theta}{2}\sum_{i=1}^{n} t_i^2\right),$$

which is a pdf of the Gamma distribution with scale parameter $\frac{1}{2}\sum_{i=1}^{n} t_i^2$ and shape parameter $\frac{3n}{2}$. Using the squared error loss function, the Bayes estimator of θ is given by $\tilde{\theta} = E(\theta \mid \underline{T}).$ Since $\theta \mid \underline{T}$ ~ Gamma $\left(\frac{3n}{2}, \frac{\sum_{i=1}^{n} t_i^2}{2}\right),$ then

$$\tilde{\theta} = E(\theta \mid \underline{T}) = \frac{3n}{\sum_{i=1}^{n} t_i^2}. \tag{3.4}$$

Remark 3.1:

From equation (3.4), it is clear that the MLE and the Bayes estimator coincide.

3.4 Uniformly minimum variance unbiased estimator

Unbiased or asymptotically unbiased estimation plays an important role in point estimation theory. Unbiased estimators can be used as building blocks to construct better estimators. Let $T_1,...,T_n$ be an i.i.d from LBR distribution with an unknown $\theta > 0.$ Then $S = \sum_{i=1}^{n} T_i^2$ is sufficient and complete for $\theta.$ Note that

$$E(S) = E\left(\sum_{i=1}^{n} T_i^2\right) = \sum_{i=1}^{n} E\left(T_i^2\right) \overset{iid}{=} n\,E\left(T_i^2\right),$$

and

$$E\left(T^2\right) = \frac{3}{\theta}.$$

So,

$$E(S) = n\,E\left(T_i^2\right) = \frac{3n}{\theta}.$$

Now,

$$E\left(\frac{1}{S}\right) = \int \frac{1}{s} f_s(s)\ ds = \frac{\theta}{3n-2}$$

Therefore,

$$\frac{3n-2}{2S} = \frac{3n-2}{2\sum_{i=1}^{n} T_i^2},$$

is a function of the complete and sufficient statistic S that is unbiased for θ. Thus, we have

$$\hat{\theta}_{UMVUE} = \frac{3n-2}{\sum_{i=1}^{n} T_i^2}.$$

Remark 3.2.

The MME is not a function of minimal sufficient statistic $S = \sum_{i=1}^{n} T_i^2$, and thus, under the squared error loss function, it is dominated by $\delta_1^* = E(\theta \mid S)$ (the Rao-Blackwell theorem). Using the Basu theorem, it can be concluded that the random variables $S / \overline{T}^2$ and S are independent, and therefore, $\delta_1^* = c^* / S$ for some real constant c^*. It makes sense to restrict to estimators of the form $\delta_c = c / S,\ c > 0$. Using the fact stated in Interpretation 2.2, it follows that $(\theta/2)S \sim Gamma\,(3n/2, 1)$ distribution, and therefore, the bias and mean squared error of δ_c are given, respectively, by:

$$b(\theta, \delta_c) = \left(\frac{c}{3n-2} - 1\right)\theta,$$

and

$$m(\theta, \delta_c) = \frac{\theta^2\left[c^2 - 2(3n-4)c + (3n-2)(3n-4)\right]}{(3n-2)(3n-4)}. \tag{3.5}$$

From (3.5), it is clear that the choice $c = c_0 = 3n-4$ (with corresponding estimator $\delta_{c_0} = 3n - 4 / S$) is the best choice under the squared error loss function.

4. SOME APPLICATIONS

4.1 Numerical example

In this subsection, 1000 different samples are simulated from the LBR with different sizes. We studied the behavior of the estimates of the parameter θ. Table 1 shows the moment estimates while Table 2 shows the maximum likelihood estimates of LBR distribution.

Table 1: *Moments estimate of parameter θ.*

Θ	n	Estimate	Bias	MSE
0.5	20	0.513379	0.0133788	0.000179171
	50	0.508656	0.0086560	0.000075002
	70	0.503849	0.0038492	0.000014831
	150	0.502320	0.0023204	0.000005389
1	20	1.02676	0.0267576	0.000716686
	50	1.017310	0.0173121	0.000300008
	70	1.007700	0.0076985	0.000059326
	150	1.004640	0.0046409	0.000021560
1.5	20	1.537400	0.0373956	0.001399830
	50	1.522760	0.0227579	0.000518442
	70	1.507090	0.0070875	0.000050283
	150	1.502390	0.0023905	0.000005720
2	20	2.048520	0.0485156	0.002356120
	50	2.027020	0.0270195	0.000730784
	70	2.007850	0.0078457	0.000061617
	150	2.001240	0.0012369	0.000001531

Table 2: *Maximum Likelihood estimate of parameter θ.*

θ	n	Estimate	Bias	MSE
0.5	20	0.51650	0.0165077	0.00027278
	50	0.50967	0.0096723	0.00009365
	70	0.50525	0.0052497	0.00002759
	150	0.50270	0.0027021	0.00000731
1	20	1.03302	0.0330153	0.0010911
	50	1.01934	0.0193447	0.00037459
	70	1.01050	0.0104994	0.00011035
	150	1.00540	0.0054042	0.00002923
1.5	20	1.54433	0.044328	0.00196694
	50	1.52157	0.0215688	0.00046568
	70	1.50469	0.0046936	0.00002205
	150	1.49568	-0.0043217	0.00001869
2	20	2.05647	0.0564663	0.00319163
	50	2.01934	0.0193413	0.000374461
	70	2.00057	0.0005669	0.00000032
	150	1.98663	-0.0133735	0.00017903

According to Table 1, the mean square error and the bias of the moment estimate of the parameter θ decrease when the sample size (n) increases.
The asymptotic unbiasedness and consistency of the parameter are clearly shown in Table 2. Comparing Table 1 with Table 2 shows that moment estimation provides much better results than the maximum likelihood estimation in terms of mean square error and bias.

4.2 Real data sets

In this subsection, we provide one data set analysis to see how the new model works in practice. The data set obtained from www.isixigma.com represents the cycle time of a process.

Data set

10	13	13	14	14	15	15	16	17
17	17	17	18	18	18	19	21	21
21	22	22	23	24	25	25	26	26
27	27	27	28	28	30	34	35	35
38	42	42	53					

Some properties of the data set are computed in Table 3.

Table 3: *Properties of the data set.*

E(T)	Var(T)	γ1	γ2
23.8537	84.028	1.06327	1.01897

Table 3 shows that the distribution of the data set is positive skewed right and leptokurtic. The parameter of the sample is estimated numerically. We used Eqs. (3.2) and (3.3) to obtain the MME and the MLE. The results are given in Table 4.

Table 4: *The MM and MLE estimate for the data set.*

Distribution	MM	MLE
RD	0.00276064	0.00307231
LBR	0.00447538	0.00460847

We want to test whether this data fits the RD or the LBR.
We use the Kolmogorov-Smirnov (K-S) distance between the empirical distribution function, the fitted distribution function, and the chi-squared goodness of fit test to determine the appropriateness of the model. The MLE estimate, log-likelihood value, and K-S are presented in Table 5.

Table 5: *The parameter, log-likelihood, and K-S of the data set.*

Distribution	Parameter	Log-likelihood	K-S
RD	0.00307231	-150.875	0.204258
LBR	0.00460847	-146.973	0.121086

By comparing the K-S distance of two distributions for the data set, the K-S of the LBR is smaller than the K-S of the RD. Thus, the data fits the LBR better than the RD. We also present the observed frequencies and the expected frequencies based on the fitted models in Tables 6 and 7. For the RD and the LBR, the chi-square value is equal to 7.815.

Table 6: *Observed frequencies and expected frequencies for the RD.*

Intervals	Observed (Oi)	Expected frequencies (Ei)
0 - 14	5	10.6594
14 - 22	16	10.8472
22 - 31	13	10.1251
31 - ∞	7	9.36825

Table 7: *Observed frequencies and expected frequencies for the LBR.*

Intervals	Observed (Oi)	Expected frequencies (Ei)
0 - 14.5	5	7.84046
14.5 - 20.5	11	9.14804
20.5 - 25.5	10	7.93372
25.5 - 31.5	8	7.63621
31.5 - ∞	7	8.44157

Now, from Tables 6 and 7, the statistics are 6.8674840 and 2.2056456, respectively. In comparing the statistics, χ_0^2 of the LBR is smaller than χ_0^2 of the RD. Therefore, LBR distribution fits better than the RD.

5. CONCLUSION

The proposed length-biased Rayleigh distribution has several desirable properties and nice physical interpretations. This model is useful and practical in areas such as physics, reliability, and life testing. The model has a unimodal pdf and an eventually increasing hazard rate function. Such characteristics are useful for modeling continuous data from life testing experiments. Real data sets are analyzed, and the proposed distribution can provide a better fit than other well-known distributions. Studying length-biased versions of other distributions such as Lomax distribution, log normal, and inverse Gaussian, among others, would be interesting.

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