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Availability equivalence factors of a general repairable series-parallel system

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Abstract. The availability equivalence factors of a general repairable series-parallel system is discussed in this paper based on the availability function of the system. The system components are assumed to be repairable and independent but not identical. The life and repair times of the system components are exponentially distributed with different parameters. Two types of availability equivalent factors of the system are derived. The results derived in this paper generalizes those given in the literature. A case study is introduced to illustrate how the idea of this work can be applied.

Key Words: *Reliability engineering, duplication methods*

1. INTRODUCTION

In non-repairable system reliability analysis, there are two main methods to improve a system design. These two methods are the reduction and redundancy methods, Sarhan (2009). The reduction method assumes that the system design can be improved by reducing the failure rate(s) of a set of system components by a factor ρ, $0 < \rho < 1$, Sarhan (2000, 2002, 2005, 2009), Sarhan et al. (2004), and Råde (1989, 1993). In the redundancy method, it is assumed that the system can be improved by increasing its components, see Meng (1993) and Yun and Bai (1986). There are more than one redundancy method such as hot, warm, cold and cold with imperfect switch redundancy, named respectively as hot, warm, old and cold with imperfect switch duplication methods, Sarhan (2000). The redundancy methods can be applied on repairable systems as well. In addition to the reduction method, the repairable system can be improved by increasing the repair rate of some of the system component(s) by a factor σ, $\sigma > 1$, Hu et al. (2011).

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Using the redundancy method might not be a practical solution for a system in which the minimum size and weight are overriding considerations: for example, in satellites or other space applications, in well-logging equipment, and in pacemakers and similar biomedical applications, Lewis (1996). In such applications space or weight limitations may indicate an increase in component performance rather than redundancy. Then more emphasis must be placed on better design, manufacturing quality control and on controlling the operating environment. Therefore, the concept of reliability/availability equivalence takes place. In such concept, the design of the system that is improved according to reduction or increasing method should be equivalent to the design of the system improved according to one of redundancy methods. That is, in this concept, one may say that the performance of a system can be improved through an alternative design, see Leemis (1996). In this case, different system designs should be comparable based on a performance characteristic such as (i) the reliability function or mean time to failure in the case of no repairs or (ii) the availability in the case of repairable systems.

The concept of comparing different designs is applied in the literature in order to: (i) improve the reliability of a non-repairable system, see Kumar et al. (2007) and Babar et al. (1988); (ii) determine a representative service provider and create equivalent elements, see Billinton and Wang (1999); (iii) derive the reliability equivalence factors of some non-repairable systems, see Sarhan (2009) and the references therein; and (iv) derive the availability equivalence factors of a repairable system, see Hu et al. (2011).

Råde(1993), Sarhan (2000, 2002, 2004, 2005, 2009), Sarhan et al.(2008), Xia and Zhang (2007) and E-Damcese (2009) applied such concept on various non-repairable systems. In that work, the reliability function and mean time to failure are used as characteristic measures to compare different system designs to derive reliability/mean time equivalence factors.

Repairable system indicates a system that can be repaired to operate normally in the event of any failure, such as automobiles, airplanes, computer network, manufacturing system, sewage systems, power plant or fire prevention system. Availability comprises “reliability” and “recovery part of unreliability after repair”, indicating the probability that repairable systems, machines or components maintain the function at a specific moment, Wang (1992). It is generally expressed as the operable time over total time. Series-parallel system indicates sub-systems in which several components are connected in parallel, and then in series, or sub-systems that several components are connected in series, and then in parallel, Juang et al. (2008). The reliability/availability of a series-parallel system has drawn continuous attention in both problem characteristics and solution methodologies, Kolowrocki (1999), Cichocki (2001), Sarhan (2004, 2009), Yalaoui et al. (2005), Juang et al. (2008), and Tavakkoli-Moghaddam et al. (2008). Recently, Hu et al. (2011) discussed the availability equivalence factors of a repairable series-parallel system with independent and identical components.

Our goal in this paper is to derive the availability equivalence factors of a repairable series-parallel system with independent and non-identical components. The availability function of the system will be used as a performance measure to compare different system designs of the original system and other improved systems in order to derive these factors. The results presented here generalize those results available in the literature; for non-repairable systems, see Sarhan et al. (2008) and references therein and for repairable system, see Hu et al. (2011).

The structure of this paper is organized as follows. Section 2 introduces the illustration of the series-parallel system and the system availability. Section 3 presents the availabilities of the systems improved according to five different methods that can be applied to improve the performance of the original system. Two types of availability equivalence factors of the system are discussed in Section 4. A case study is investigated in Sections 5. Finally, Section 6 concludes the paper.

2. A GENERAL REPAIRABLE SERIES-PARALLEL SYSTEM

The system considered here consists of n subsystems connected in series, and with subsystem i consisting of m_i independent, repairable and nonidentical components connected in parallel for i = 1, 2, … , n. We refer to such system as a general repairable series-parallel system. Figure 1 shows the diagram of that system. Let T_{ij} and Y_{ij} be the lifetime and repair time, respectively, of component j in subsystem $i, 1 \leq i \leq n, 1 \leq j \leq m_i$. It is assumed that the life and repair times of component j in subsystem $i, 1 \leq i \leq n, 1 \leq j \leq m_i$, follow exponential distributions with failure rate $\lambda_{ij}$and repair rate μ_{ij}. Let N be the total number of the system components, that is $N = \sum_{i=1}^{n} m_i$.

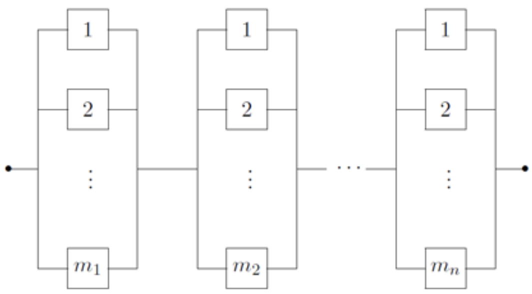

Figure 1. Series-parallel system diagram

Special Cases: This system generalizes the following cases:

1. Repairable series-parallel system with identical components, Hu et al. (2012), when $\lambda_{ij} = \lambda_i$ and$\mu_{ij} = \mu_i$, j = 1, 2,⋯ ,m_i , and i = 1, 2, ⋯, n.
2. Repairable parallel system with non-identical components, when n = 1, j = 1, 2, ⋯,m.
3. Repairable series system with non-identical components, when mi = 1, i = 1, 2, ⋯, n.

Definition 2.1 The time availability of the component j in subsystem i at any given time t, denoted $A^*_{ij}(t)$, is (Ebeling; 2001)

$$A^*_{ij}(t) = \frac{\mu_{ij}}{\mu_{ij} + \lambda_{ij}} + \frac{\lambda_{ij}}{\mu_{ij} + \lambda_{ij}} e^{-(\lambda_{ij}+\mu_{ij})t}.$$

Let A_{ij} , be the steady state availability, for simplicity we say the availability from now on, of the component j in subsystem i and Ai be the availability of the subsystem i, $1 \le i \le n, 1 \le j \le m_i$. One can easily derive A_{ij}andA_i, respectively, as

$$A_{ij} = \lim_{t\to\infty} A^*_{ij}(t) = \frac{\mu_{ij}}{\mu_{ij}+\lambda_{ij}} = \frac{1}{1+\eta_{ij}}, \text{ where } \eta_{ij} = \frac{\lambda_{ij}}{\mu_{ij}}, \tag{2.1}$$

and

$$A_i = 1 - \prod_{j=1}^{m_i}(1 - A_{ij}) = 1 - \prod_{j=1}^{m_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right). \tag{2.2}$$

Therefore, the system availability, denoted AS, can be derived as

$$A_S = \prod_{i=1}^{n} A_i = \prod_{i=1}^{n}\left[1 - \prod_{j=1}^{m_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right)\right]. \tag{2.3}$$

3. DIFFERENT DESIGNS OF IMPROVED SYSTEMS

The system can improved according to one of the following three different methods:

1. Reduction method. In this method it is assumed that the component can be improved by reducing its failure rate by a factor ρ, $0 < \rho < 1$.
2. Increasing method. It is assumed in this method that the component can be improved by increasing its repair rate by a factor σ, $\sigma > 1$.
3. Standby redundancy method:
   (a) Hot duplication method: this method assumes that the component is duplicated by an identical hot standby component.
   (b) Warm duplication method: this method assumes that the component is duplicated by an identical warm standby component.
   (c) Cold duplication method: this method assumes that the component is duplicated by an identical cold standby component.

In the following sections, we derive the availability of the system improved according to the methods mentioned above.

3.1 The reduction method

It is assumed in the reduction method that the system can be improved by reducing the failure rates of a set R components by a factor ρ, $0 < \rho < 1$. We assume that $R = \bigcup_{i=1}^{n} R_i$where R_i is a set of the subsystem i components, $1 \le i \le n$. Also, we assume that $|R_i| = r_i$, $0 \le r_i \le m_i$, and $|R| = r = \sum_{i=1}^{n} r_i$ $(1 \le r \le N)$.

Let $A_{ij,\rho}$ be the availability of the component j in subsystem i, improved by reducing its failure rate λ_{ij} by the factor ρ. One can easily derive

$$A_{ij,\rho} = \frac{1}{1+\rho\eta_{ij}}, \text{ where } \eta_{ij} = \frac{\lambda_{ij}}{\mu_{ij}}. \tag{3.1}$$

Therefore, the availability of subsystem *i* improved by reducing the failure rates of a set R_i components by the factor ρ, denoted$A_{R_{i,\rho}}$, can be written as

$$A_{R_{i,\rho}} = 1 - \prod_{j\in R_i}(1 - A_{ij,\rho}) \prod_{j\in R_i}(1 - A_{ij})$$

$$= 1 - \prod_{j \in R_i} \left(\frac{\rho \eta_{ij}}{1 + \rho \eta_{ij}} \right) \prod_{j \in \bar{R}_i} \left(\frac{\eta_{ij}}{1 + \eta_{ij}} \right)$$

where $\bar{R}_i = M_i / R_i$,M_i is the set of all subsystem *i* components, $M_i = \{1,2,\cdots,m_i\}$ and $1 \le i \le n$. Finally, the availability of the system improved by reducing the failure rates of a set *R* components by the same factor ρ, denoted $A_{R,\rho}$, can be derived as

$$A_{R,\rho} = \prod_{i=1}^{n} \left[1 - \prod_{j \in R_i} \left(\frac{\rho \eta_{ij}}{1+\rho \eta_{ij}} \right) \prod_{j \in \bar{R}_i} \left(\frac{\eta_{ij}}{1+\eta_{ij}} \right) \right] \tag{3.3}$$

3.2 The increasing method

It is assumed in the increasing method that the system can be improved by increasing the repair rates of a set *S* components by a factor σ, $\sigma > 1$. We assume that $S = \bigcup_{i=1}^{n} S_i$, where S_i is a set of the subsystem *i* components, $1 \le i \le n$. Also, we assume that $|S_i| = s_i$, $0 \le s_i \le m_i$ and $|S| = s = \sum_{i=1}^{n} s_i$, $1 \le s \le N$.

Let $A_{ij,\sigma}$ be the availability of component *j* in subsystem *i* after increasing its repair rate μ_{ij} by the factor σ, $\sigma > 1$; and $A_{S_{i,\sigma}}$ be the availability of subsystem *i* which is improved by increasing the repair rates of a set S_i components by the same factor σ; and $A_{S,\sigma}$ be the availability of the system improved by increasing the repair rates of a set *S* components by the same factor σ. One can derive these availabilities in the following forms

$$A_{S_{i,\sigma}} = \frac{\sigma \mu_{ij}}{\sigma \mu_{ij} + \lambda_{ij}} = \frac{\sigma}{\sigma + \eta_{ij}}, \tag{3.4}$$

$$A_{S_{i,\sigma}} = 1 - \prod_{j \in S_i} (1 - A_{ij,\sigma}) \prod_{j \in \bar{S}_i} (1 - A_{ij})$$

$$= 1 - \prod_{j \in S_i} \left(\frac{\eta_{ij}}{\sigma + \eta_{ij}} \right) \prod_{j \in \bar{S}_i} \left(\frac{\eta_{ij}}{1+\eta_{ij}} \right), \tag{3.5}$$

And

$$A_{S,\sigma} = \prod_{i=1}^{n} \left[1 - \prod_{j \in S_i} \left(\frac{\eta_{ij}}{\sigma+\eta_{ij}} \right) \prod_{j \in \bar{S}_i} \left(\frac{\eta_{ij}}{1+\eta_{ij}} \right) \right], \tag{3.6}$$

where $\bar{S}_i = M_i / S_i$, for $1 \le i \le n$.

3.3 The hot duplication method

It is assumed in the hot duplication method that the system can be improved by connecting every element is a set *B* components with an identical component in parallel. We assume that $B = \bigcup_{i=1}^{n} B_i$, where B_i is a set of the subsystem *i* components, $1 \le i \le n$. Also, we assume that $|B_i| = h_i, 0 \le h_i \le m_i$, and $|B| = h = \sum_{i=1}^{n} h_i$, $1 \le h \le N$.

Let $A_{B_i}^{H}$ be the availability of the subsystem i which is improved by improving a set $B_i \subseteq M_i$ components, $1 \le i \le n$; and A_B^H be the availability of the system improved by improving a set B components according to the hot duplication method. One can derive

$$A_{B_i}^{H} = 1 - \prod_{j \in B_i} (1 - A_{ij})^2 \prod_{j \in \bar{B}_i} (1 - A_{ij}),$$

$$= 1 - \prod_{j \in B_i} \left(\frac{\eta_{ij}}{1+\eta_{ij}} \right)^2 \prod_{j \in \bar{B}_i} \left(\frac{\eta_{ij}}{1+\eta_{ij}} \right), \tag{3.7}$$

where $\bar{B}_i = M_i / B_i$, for $1 \le i \le n$.

3.4 The warm duplication method

We say that, a component j in subsystem i is warm duplicated if it is connected in parallel with a non-identical component, having a failure rate ν_{ij}, in parallel via a perfect switch. In the warm duplication method, it is assumed that the system can be improved when every component in a set B components is warm duplicated. We assume that $B = \bigcup_{i=1}^{n} B_i$, where B_i is a set of the subsystem i components, $1 \le i \le n$. Also, we assume that $|B_i| = w_i, 0 \le w_i \le m_i$, and $|B| = w = \sum_{i=1}^{n} w_i,\ 1 \le w \le N$.

Let A_{ij}^{W} be the availability of the component j in the subsystem i when it is improved according to the warm duplication method. Using Markov process, A_{ij}^{W} can be obtained as, Liu and Zheng (2010),

$$A_{ij}^{W} = \frac{1+\eta_{ji}+\xi_{i_{ij}}}{1+\eta_{ji}+\xi_{i_{ij}}+\frac{1}{2}\eta_{ij}^{2}+\frac{1}{2}\eta_{ji}\xi_{i_{ij}}}, \quad (3.9)$$

where $\xi_{i_{ij}} = \nu_{ij}/\mu_{ij}$ for $1 \le j \le m_i$ and $1 \le i \le n$.

Let $A_{B_i}^{W}$ be the availability of the subsystem *i* improved by improving B_i subsystem components according to the warm duplication method. Therefore, one can derive

$$A_{B_i}^{W} = 1 - \prod_{j\in B_i}\left(\frac{\frac{1}{2}\eta_{ji}(\eta_{ji}+\xi_{i_{ij}})}{1+\eta_{ji}+\xi_{i_{ij}}+\frac{1}{2}\eta_{ij}^{2}+\frac{1}{2}\eta_{ji}\xi_{i_{ij}}}\right)\prod_{j\in B_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right). \quad (3.10)$$

Finally, let A_{B}^{W} be the availability of the system improved by improving a set *B* components according to the warm duplication methods. Using (3.10), we get

$$A_{B}^{W} = \prod_{i=1}\left[1 - \prod_{j\in B_i}\left(\frac{\frac{1}{2}\eta_{ji}(\eta_{ji}+\xi_{i_{ij}})}{1+\eta_{ji}+\xi_{i_{ij}}+\frac{1}{2}\eta_{ij}^{2}+\frac{1}{2}\eta_{ji}\xi_{i_{ij}}}\right)\prod_{j\in B_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right)\right] \quad (3.11)$$

3.5 The cold duplication method

It is assumed in the cold duplication method, that each component of set B components is connected in parallel with an identical component via a perfect switch. We assume that $B = \bigcup_{i=1}^{n} B_i$, where B_i is a set of the subsystem i components, $1 \le i \le n$. Also, we assume that $|B_i| = c_i, 0 \le c_i \le m_i$, and $|B| = c = \sum_{i=1}^{n} c_i,\ 1 \le c \le N$.

Let A_{ij}^{C} is the availability of the component j in subsystem i when it is improved according to the cold duplication method; $A_{B_i}^{C}$ be the availability of subsystem i, which is improved according to cold duplication method; and A_{B}^{C} be the availability of the system improved by improving set B components according to the cold duplication method. Using Markov process theory, A_{ij}^{C} is, Gu and Wei (2006),

$$A_{ij}^{C} = \frac{\mu_{ij}^{2}+\lambda_{ij}\mu_{ij}}{\mu_{ij}^{2}+\lambda_{ij}\mu_{ij}+\frac{1}{2}\lambda_{ij}^{2}} = \frac{1+\eta_{ij}}{1+\eta_{ij}+\frac{1}{2}\eta_{ij}^{2}}, \quad (3.12)$$

Using (3.12) and the nature of the parallel subsystem *i*, one can derive

$$A_{B_i}^{C} = 1 - \prod_{j\in B_i}\left(\frac{\frac{1}{2}\eta_{ij}^{2}}{1+\eta_{ij}+\frac{1}{2}\eta_{ij}^{2}}\right)\prod_{j\in B_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right), \quad (3.13)$$

Finally, using (3.13) and the nature of the series connection of the subsystems, we get

$$A_{B}^{C} = \prod_{i=1}^{n}\left[1 - \prod_{j\in B_i}\left(\frac{\frac{1}{2}\eta_{ij}^{2}}{1+\eta_{ij}+\frac{1}{2}\eta_{ij}^{2}}\right)\prod_{j\in B_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right)\right]. \quad (3.14)$$

4. AVAILIBILITY EQUIVALENCE FACTORS

In this section, we derive the availability equivalence factors of a repairable series-parallel system with independent, non-identical and repairable components. Two types of availability equivalence factors will be discussed. These two types are referred as availability equivalent reducing factor and availability equivalent increasing factor. Following the definition of reliability equivalence factors introduced by Sarhan (2000), we can introduce the availability equivalence factors.

4.1 Availability Equivalence Reducing Factor

Availability equivalence reducing factor, in short AERF, referred as $\rho = \rho^{D}_{R,B}$, D = H, W,C for hot, warm and cold, respectively, is defined as the factor ρ by which the failure rate of a set R components should be reduced in order to get equality of the availability of another better design which can be obtained from the original system by assuming hot, warm and cold duplications of a set B components. That is, $\rho = \rho^{D}_{R,B}$, for D = H,W,C, is the solution of the following equations in ρ

$$A_{R,\rho} = A^{D}_{B},\ D = H,W,C\,. \tag{4.1}$$

In what follows, we give the non-linear equations needed to be solved to get the three possible AERF's.

(1) Hot availability equivalence reducing factor (HAERF): Substituting (3.3) and (3.8) into (4.1), $\rho = \rho^{H}_{R,B}$, is the solution of the following non-linear equation in ρ

$$\prod_{i=1}^{n}\left[1-\prod_{j\in R_i}\left(\frac{\rho\eta_{ij}}{1+\rho\eta_{ij}}\right)\prod_{j\in R_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right)\right]= \prod_{i=1}^{n}\left[1-\prod_{j\in R_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right)^{2}\prod_{j\in R_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right)\right]. \tag{4.2}$$

(2) Warm availability equivalence reducing factor (WAERF): Substituting (3.3) and (3.11) into (4.1), $\rho = \rho^{D}_{R,B}$, is the solution of the following non-linear equation in ρ

$$\prod_{i=1}^{n}\left[1-\prod_{j\in R_i}\left(\frac{\rho\eta_{ij}}{1+\rho\eta_{ij}}\right)\prod_{j\in R_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right)\right] = \prod_{i=1}^{n}\left[1-\prod_{j\in B_i}\left(\frac{\frac{1}{2}\eta_{ji}(\eta_{ji}+\xi_{i_{ij}})}{1+\eta_{ji}+\xi_{i_{ij}}+\frac{1}{2}\eta_{ij}^{2}+\frac{1}{2}\eta_{ji}\xi_{i_{ij}}}\right)\prod_{j\in B_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right)\right]. \tag{4.3}$$

(3) Cold availability equivalence reducing factor (CAERF): Substituting (3.3) and (3.14) into (4.1), $\rho = \rho^{C}_{R,B}$, satisfies the following non-linear equation

$$\prod_{i=1}^{n}\left[1-\prod_{j\in R_i}\left(\frac{\rho\eta_{ij}}{1+\rho\eta_{ij}}\right)\prod_{j\in R_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right)\right] = \prod_{i=1}^{n}\left[1-\prod_{j\in B_i}\left(\frac{\frac{1}{2}\eta_{ij}^{2}}{1+\eta_{ij}+\frac{1}{2}\eta_{ij}^{2}}\right)\prod_{j\in B_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right)\right]. \tag{4.4}$$

Equations (4.2 - 4.4) have no closed solutions, therefore, a numerical technique method is needed to get their solutions.

4.2 Availability Equivalence Increasing Factor

Availability equivalence increasing factor, in short AEIF, referred as $\sigma = \sigma_{S,B}^{D}$, D = H,W,C for hot, warm and cold, respectively, is defined as the factor ρ by which the failure rate of a set S components should be reduced in order to get equality of the availability of another better design which can be obtained from the original system by assuming hot, warm and cold duplications of a set B components. $\sigma = \sigma_{S,B}^{D}$ is the solution of the following equation. That is, $\sigma = \sigma_{S,B}^{D}$, for D = H,W,C, is the solution of the following equations in σ

$$A_{S,\sigma} = A_B^D,\ D = H,W,C. \tag{4.5}$$

In what follows, we give the non-linear equations needed to be solved to get the three possible AEIF's.

(1) Hot availability equivalence increasing factor (HAEIF): Substituting (3.6) and (3.8) into (4.5), , $\sigma = \sigma_{S,B}^{H}$, is the solution of the following non-linear equation

$$\prod_{i=1}^{n}\left[1-\prod_{j\in S_i}\left(\frac{\eta_{ij}}{\sigma+\eta_{ij}}\right)\prod_{j\in S_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right)\right] = \prod_{i=1}^{n}\left[1-\prod_{j\in B_i}\left(\frac{\eta_{ij}}{1+\rho\eta_{ij}}\right)^2\prod_{j\in B_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right)\right]. \tag{4.6}$$

Table 1. Set values of the system parameters

i	j	λ_{ij}	μ_{ij}	ν_{ij}	η_{ij}	ξ_{ij}
1	1	0.11	1.2	0.1	0.091667	0.083333
2	1	0.13	1.4	0.12	0.092857	0.085714
	2	0.12	1.3	0.11	0.092308	0.084615

(2) Warm availability equivalence increasing factor (WAEIF): Substituting (3.6) and (3.11) into (4.5), $\sigma = \sigma_{S,B}^{W}$ is the solution of the following equation in σ

$$\prod_{i=1}^{n}\left[1-\prod_{j\in S_i}\left(\frac{\eta_{ij}}{\sigma+\eta_{ij}}\right)\prod_{j\in S_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right)\right] =$$
$$\prod_{i=1}^{n}\left[1-\prod_{j\in B_i}\left(\frac{\frac{1}{2}\eta_{ji}(\eta_{ji}+\xi_{ij})}{1+\eta_{ji}+\xi_{ij}+\frac{1}{2}\eta_{ij}^2+\frac{1}{2}\eta_{ji}\xi_{ij}}\right)\prod_{j\in B_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right)\right]. \tag{4.7}$$

(3) Cold availability equivalence increasing factor (CAEIF): Substituting (3.6) and (3.14) into(4.5), $\sigma = \sigma_{S,B}^{C}$ is the solution of the following equation in σ

$$\prod_{i=1}^{n}\left[1-\prod_{j\in S_i}\left(\frac{\eta_{ij}}{\sigma+\eta_{ij}}\right)\prod_{j\in S_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right)\right] =$$
$$\prod_{i=1}^{n}\left[1-\prod_{j\in B_i}\left(\frac{\frac{1}{2}\eta_{ij}^2}{1+\eta_{ij}+\frac{1}{2}\eta_{ij}^2}\right)\prod_{j\in B_i}\left(\frac{\eta_{ij}}{1+\eta_{ij}}\right)\right]. \tag{4.8}$$

The above equations (4.6 - 4.8) have no closed-form solutions in σ, so a numerical technique method to get the value of σ.

5. A CASE STUDY

To explain how one can utilize the theoretical results obtained, we introduce a practical example. Consider a sewage collection and disposal system in a small town consisting of buried pumps and pipelines, which convey the waste water from households. The system consists of three pumps connected in series-parallel, as shown in Figure 1, with $n = 2$, $m_1 = 1$, and $m_2 = 2$. Table 1 gives the values failure and repair rates of the system components $_{ij}$ and μ_{ij} as well as the values of ν_{ij} , η_{ij} and ξ_{ij}, for $i = 1, 2$ and $j = 1, \cdots m_i$.

The main goal here is to improve the sewage system by improving the performance of some pumps instead of increasing the number of the pumps. This can be achieved by obtaining the two types of availability equivalence factors of the sewage system. The availability of the original system is $A_S = 0.909453$. Table 2 shows the availability of the improved system obtained from the original system by applying hot, warm and cold duplications using all possible set B components, where $B = B_1 \quad B_2$ and φ is the empty set.

Table 2: The availability of the improved system, A_B^D, D = H, W, C

$\lvert B \rvert$	$= B_1 \cup B_2$	A_B^H	A_B^W	A_B^C
1	$B_1 = \{1\},\ B_2 = \phi$	0.985819	0.986088	0.989013
	$B_1 = \phi,\ B_2 = \{1\}$	0.915472	0.915490	0.915726
	$B_1 = \phi,\ B_2 = \{2\}$	0.915475	0.915494	0.915728
2	$B_1 = \{1\},\ B_2 = \{1\}$	0.992343	0.992634	0.995835
	$B_1 = \{1\},\ B_2 = \{2\}$	0.992347	0.992639	0.995837
	$B_1 = \phi,\ B_2 = \{1, 2\}$	0.915983	0.915986	0.916017
3	$B_1 = \{1\},\ B_2 = \{1, 2\}$	0.992898	0.993172	0.996151

From the results shown in Table 2, one can easily see that:

1. $A_S < A_B^H < A_B^W < A_B^C$, for all possible set B components;
2. Improving the only pump of subsystem 1, according to a duplication method, provides a better design of the sewage system than that can be achieved by duplicating one pump of subsystem 2, according to the same method;
3. Duplicating two pumps, one from each subsystem, produces a better sewage design than that can be obtained by duplicating the two pumps in subsystem 2, according to the same method; and
4. Cold duplicating all pumps in the sewage system provides the best design, in the sense of having the highest availability.

We used Mathematica Program System to calculate all possible availability equivalence factors of the sewage system. Tables 3 and 4 give the hot, warm and cold (D = H,W,C) availability equivalence reducing factors, $\rho = \rho_{R,B}^D$, and the hot, warm and cold availability equivalence increasing factors, $\sigma = \sigma_{S,B}^D$, respectively, for all possible sets R, S and B.

Table 3: The availability equivalence reducing factors $(\rho^{D}_{R,B},\ D = H, W, C)$ for different R, B.

$\lvert R\rvert$	$= R_1 \cup R_2$	$\lvert B\rvert = 1$			$\lvert B\rvert = 2$			$\lvert B\rvert = 3$
		$B_1 = \{1\}$, $B_2 = \phi$	$B_1 = \phi$, $B_2 = \{1\}$	$B_1 = \phi$, $B_2 = \{2\}$	$B_1 = \{1\}$, $B_2 = \{1\}$	$B_1 = \{1\}$, $B_2 = \{2\}$	$B_1 = \phi$, $B_2 = \{1,2\}$	$B_1 = \{1\}$, $B_2 = \{1,2\}$
$\rho^{H}_{R,B}$								
1	$R_1 = \{1\}$, $R_2 = \phi$	0.077465	0.921707	0.921667	0.005236	0.00520	0.915098	NA
	$R_1 = \phi$, $R_2 = \{1\}$	NA	0.078313	0.077886	NA	NA	0.006574	NA
	$R_1 = \phi$, $R_2 = \{2\}$	NA	0.078350	0.077922	NA	NA	0.006578	NA
2	$R_1 = \{1\}$, $R_2 = \{1\}$	0.144538	0.927424	0.927388	0.077531	0.077498	0.921298	0.071876
	$R_1 = \{1\}$, $R_2 = \{2\}$	0.144543	0.927427	0.927390	0.077534	0.077501	0.921301	0.071878
	$R_1 = \phi$, $R_2 = \{1,2\}$	NA	0.273548	0.272787	0.000046	0.000026	0.078117	NA
3	$R_1 = \{1\}$, $R_2 = \{1,2\}$	0.154716	0.932073	0.932039	0.083524	0.083489	0.926312	0.077474
$\rho^{W}_{R,B}$								
1	$R_1 = \{1\}$, $R_2 = \phi$	0.074468	0.921473	0.921415	0.002044	0.001991	0.915058	NA
	$R_1 = \phi$, $R_2 = \{1\}$	NA	0.075758	0.075128	NA	NA	0.006137	NA
	$R_1 = \phi$, $R_2 = \{2\}$	NA	0.075793	0.075163	NA	NA	0.006140	NA
2	$R_1 = \{1\}$, $R_2 = \{1\}$	0.141758	0.927207	0.927154	0.074569	0.074520	0.921261	0.069081
	$R_1 = \{1\}$, $R_2 = \{2\}$	0.141762	0.927210	0.927156	0.074572	0.074523	0.921263	0.069083
	$R_1 = \phi$, $R_2 = \{1,2\}$	0.000005	0.268968	0.267829	0.000034	0.000006	0.075459	0.000039
3	$R_1 = \{1\}$, $R_2 = \{1,2\}$	0.151780	0.931869	0.931819	0.080356	0.080304	0.926277	0.074482
$\rho^{C}_{R,B}$								
1	$R_1 = \{1\}$, $R_2 = \phi$	0.041985	0.918416	0.918393	NA	NA	0.914669	NA
	$R_1 = \phi$, $R_2 = \{1\}$	NA	0.042484	0.042233	NA	NA	0.001946	NA
	$R_1 = \phi$, $R_2 = \{2\}$	NA	0.042504	0.042254	NA	NA	0.001947	NA
2	$R_1 = \{1\}$, $R_2 = \{1\}$	0.111623	0.924374	0.924352	0.042024	0.042004	0.920900	0.038827
	$R_1 = \{1\}$, $R_2 = \{2\}$	0.111627	0.924376	0.924355	0.042026	0.042006	0.920903	0.038829
	$R_1 = \phi$, $R_2 = \{1,2\}$	NA	0.200480	0.199879	0.000047	0.000036	0.042368	0.000051
3	$R_1 = \{1\}$, $R_2 = \{1,2\}$	0.119857	0.929205	0.929185	0.045430	0.045409	0.925938	0.041988

Table 4: The availability equivalence increasing factors $\left(\sigma^D_{S,B},\ D = H, W, C\right)$ for different S, B.

$\|S\|$	$= S_1 \cup S_2$	$\|B\|=1$: $B_1=\{1\}$, $B_2=\phi$	$\|B\|=1$: $B_1=\phi$, $B_2=\{1\}$	$\|B\|=1$: $B_1=\phi$, $B_2=\{2\}$	$\|B\|=2$: $B_1=\{1\}$, $B_2=\{1\}$	$\|B\|=2$: $B_1=\{1\}$, $B_2=\{2\}$	$\|B\|=2$: $B_1=\phi$, $B_2=\{1,2\}$	$\|B\|=3$: $B_1=\{1\}$, $B_2=\{1,2\}$
	$\sigma^H_{S,B}$							
1	$S_1=\{1\},\ S_2=\phi$	12.9091	1.08494	1.08499	NA	NA	1.09278	NA
	$S_1=\phi,\ S_2=\{1\}$	NA	12.7692	12.8393	NA	NA	NA	NA
	$S_1=\phi,\ S_2=\{2\}$	NA	12.7633	12.8333	NA	NA	NA	NA
2	$S_1=\{1\},\ S_2=\{1\}$	6.91861	1.07826	1.07830	12.8980	12.9036	1.08542	13.913
	$S_1=\{1\},\ S_2=\{2\}$	6.91838	1.07825	1.07829	12.8976	12.9031	1.08542	13.9124
	$S_1=\phi,\ S_2=\{1,2\}$	NA	3.65566	3.66586	145.594	NA	12.8012	NA
3	$S_1=\{1\}, S_2=\{1,2\}$	6.46347	1.07288	1.07292	11.9725	11.9777	1.07955	12.9076
	$\sigma^W_{S,B}$							
1	$S_1=\{1\},\ S_2=\phi$	13.4286	1.08522	1.08529	NA	NA	1.09283	NA
	$S_1=\phi,\ S_2=\{1\}$	NA	13.2000	13.3105	NA	NA	NA	NA
	$S_1=\phi,\ S_2=\{2\}$	NA	13.1939	13.3043	NA	NA	NA	NA
2	$S_1=\{1\}, S_2=\{1\}$	7.05429	1.07851	1.07857	13.4104	13.4192	1.08547	14.4759
	$S_1=\{1\}, S_2=\{2\}$	7.05405	1.07850	1.07857	13.4099	13.4188	1.08547	14.4753
	$S_1=\phi,\ S_2=\{1,2\}$	NA	3.71791	3.73373	NA	NA	13.2521	NA
3	$S_1=\{1\}, S_2=\{1,2\}$	6.58849	1.07311	1.07317	12.4446	12.4527	1.07959	13.4261
	$\sigma^C_{S,B}$							
1	$S_1=\{1\},\ S_2=\phi$	23.8182	1.08883	1.08886	NA	NA	1.09329	NA
	$S_2=\{1\},\ S_2=\phi$	NA	23.5385	23.6781	NA	NA	NA	NA
	$S_1=\phi,\ S_2=\{2\}$	NA	23.5271	23.6667	NA	NA	NA	NA
2	$S_1=\{1\}, S_2=\{1\}$	8.95870	1.08181	1.08184	23.796	23.8071	1.08589	25.7551
	$S_1=\{1\}, S_2=\{2\}$	8.95839	1.08181	1.08184	23.7951	23.8062	1.08589	25.7541
	$S_1=\phi,\ S_2=\{1,2\}$	NA	4.98803	5.00302	NA	NA	23.6025	NA
3	$S_1=\{1\}, S_2=\{1,2\}$	8.34326	1.07619	1.07621	22.0118	22.0221	1.07999	23.8165

From the results presented in Tables 3 and 4, we can immediately conclude that:

1. Hot duplication of the only pump in subsystem 1, B_1 = {1}and $B_2 = \phi$, increases the system availability from A_S= 0.909453 to A_B^H= 0.985819, $B = B_1 \cup B_2$, see Table 2. The improved sewage system with A_B^H= 0.985819 can be achieved by performing one of the following:

   (1.1) reducing the failure rate(s) of (see Table 3): (i) the only pump in subsystem 1, $R = R_1 \cup R_2$ where R_1= *{1}* and $R_2 = \varphi$, by the HAERF $\rho_{R,B}^H$ = 0.077465; (ii) the only pump in subsystem 1 and the first pump in subsystem 2, R_1= {1},R_2 = *{1}*, by the HAERF $\rho_{R,B}^H$ = 0.144538; (iii) the only pump in subsystem 1 and the second pump of subsystem 2, R_1= {1},R_2 = *{1}*, by the HAERF $\rho_{R,B}^H$ = 0.144543; (iv) all the three pumps , R_1= {1},R_2 = *{2}*, by the HAERF $\rho_{R,B}^H$ = 0.154716.

   (1.2) increasing the repair rate(s) of (see Table 4): (i) the only pump in subsystem 1, $S = S_1 \cup S_2$ where S_1= {1} and $S_2 = \varphi$, by the HAEIF $\sigma_{S,B}^H$ = 12.9091; (ii) the only pump in subsystem 1 and first pump in subsystem 2, S_1= {1} and S_2 = {1}, by the HAEIF $\sigma_{S,B}^H$ = 6.91861; (iii) the only pump in subsystem 1 and second pump in subsystem 2, S_1= {1} and S_2 = {1}, by the HAEIF $\sigma_{S,B}^H$ = 6.91838; (iv) all the three pumps, S_1= {1} and S_2 = {1,2}, by the HAEIF $\sigma_{S,B}^H$ = 6.46347.

2. Worm duplication of the only pump in subsystem 1, B_1= {1} and $B_2 = \varphi$, increases the system availability from A_S = 0.909453 to A_B^W= 0.986088, $B = B_1 \cup B_2$, see Table 2. The improved sewage system with A_B^W = 0.985819 can be achieved by performing one of the following:

   (2.1) reducing the failure rate(s) of (see Table 3): (i) the only pump in subsystem 1, $R = R_1 \cup R_2$ where R_1= *{1}* and $R_2 = \varphi$, by the WAERF $\rho_{R,B}^W$ = 0.074468; (ii) the only pump in subsystem 1 and the first pump in subsystem 2, R_1= {1},R_2 = *{1}*, by the WAERF $\rho_{R,B}^W$ = 0.141758; (iii) the only pump in subsystem 1 and the second pump of subsystem 2, R_1= {1},R_2 = *{2}*, by the WAERF $\rho_{R,B}^W$= 0.141762; (iv) the two pumps in subsystem 2, $R_1 = \varphi$, R_2 = {1,2}, by the WAERF $\rho_{R,B}^W$ = 0.000005; (v) all the three pumps , $R_1 = \varphi$, R_2 = {1,2}, by the WAERF $\rho_{R,B}^W$ = 0.151780.

   (2.2) increasing the repair rate(s) of (see Table 4): (i) the only pump in subsystem 1, $S = S_1 \cup S_2$ where S_1= {1} and $S_2 = \varphi$, by the WAEIF $\sigma_{S,B}^W$ = 13.4286; (ii) the only pump in subsystem 1 and first pump in subsystem 2, S_1= {1},S_2 = *{1}*, by the WAEIF $\sigma_{S,B}^W$= 7.05429; (iii) the only pump in subsystem 1 and second pump in subsystem 2, S_1= {1},S_2 = *{2}*, by the WAEIF $\sigma_{S,B}^W$ = 7.05405; (iv) all the three pumps, S_1= {1},S_2 = *{1,2}*, by the WAEIF $\sigma_{S,B}^W$ = 6.58849.

3. Cold duplication of the only pump in subsystem 1, B_1= *{1}* and $B_2 = \varphi$, increases the

system availability from A_S= 0.909453 to A_B^C = 0.989013, $B = B_1 \cup B_2$, see Table 2. The improved sewage system with A_B^C = 0.989013 can be achieved by performing one of the following:

(3.1) reducing the failure rate(s) of (see Table 3): (i) the only pump in subsystem 1, $R = R_1 \cup R_2$ where R_1= *{1}* and R_2= φ, by the CAERF $\rho_{R,B}^C$ = 0.041985; (ii) the only pump in subsystem 1 and the first pump in subsystem 2, R_1= {1},R_2 = *{1}*, by the CAERF $\rho_{R,B}^C$= 0.111623, (iii) the only pump in subsystem 1 and the second pump of subsystem 2, R_1= {1},R_2 = *{2}*, by the CAERF $\rho_{R,B}^C$= 0.111627, (iv) all the three pumps , R_1= {1},R_2 = *{1,2}*, by the CAERF $\rho_{R,B}^C$ = 0.119857.

(3.2) increasing the repair rate(s) of (see Table 4): (i) the only pump in subsystem 1, $S = S_1 \cup S_2$ where S_1= {1} and S_2 = φ, by the CAEIF $\sigma_{S,B}^C$ = 23.8182, (ii) the only pump in subsystem 1 and first pump in subsystem 2, S_1= {1},S_2 = *{1}*, by the CAEIF $\sigma_{S,B}^C$ = 8.95870 (iii) the only pump in subsystem 1 and second pump in subsystem 2, S_1= {1},S_2 = *{2}*, by the CAEIF $\sigma_{S,B}^C$ = 8.95839, (iv) all the three pumps, S_1= {1},S_2 = *{1,2}*, by the CAEIF $\sigma_{S,B}^C$ = 8.34326.

4. In the same manner, we can illustrate the rest of results shown in Tables 3 and 4.

5. The notation NA, means that there is no possible equivalence between the two improved systems that can be achieved by reducing (increasing) the failure (repair) rates of the set *R* (*S*) of pumps and that can be achieved by duplicating elements of set *B* of pumps.

6. CONCLUSION

This paper discusses the availability equivalence factors of a general repairable series-parallel system with independent but non-identical components. The system studied here generalizes several well known systems such as a repairable series-parallel system with independent and identical components; repairable series and repairable parallel systems with independent and non-identical or identical components. We derived two types of the availability equivalence factors of the system. We presented a case study to illustrate how the theoretical results derived in the paper can be applied.

Indeed there are several possible extensions of the this work. As an example, the case of a general repairable series-parallel system with non constant failure rates can be studied.

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