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Testing unknown age classes of life distributions based on TTT-transform

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Abstract. A nonparametric procedure for testing exponentially against used better than aged in expectation (UBAE) class of life distributions is presented. We construct a test statistics based on scaled total time on test (TTT)-transformation, to test exponentiality against UBAE class of life distributions. The distribution of the statistic is investigated via simulation. Practical applications of the proposed test are presented.

Key Words: *UBAE class of life distributions, survival function, exponentiality, total time on test transform*

1. INTRODUCTION

Let X be a non-negative continuous random variable with distribution function F; survival function $\overline{F} = 1 - F$ and finite mean $\mu = E[X] = \int_0^\infty \overline{F}(x)dx$. At age t, the randomresidual life is defined by X_t with survival function $\overline{F}_t = \frac{\overline{F}(t+x)}{\overline{F}(t)}$, x, t ≥ 0. The mean residual life of X_t is given by

$$\mu(t) = E[X_t] = \int_t^\infty \overline{F}(u)du/\overline{F}(t)\ , t \geq 0, \overline{F}(t) > 0$$

Some properties concerning the asymptotic behavior of $X_t$as $t \to \infty$ will be used.

Definition 1.1. If X is non-negative random variable, its distribution function F is said to be finitely and positively smooth if a number $\gamma \in (0,\infty)$ exists such that

$$\lim_{t\to\infty} \frac{\overline{F}(t+x)}{\overline{F}(t)} = e^{-\gamma x} \quad (1.1)$$

where γ is called the asymptotic decay coefficient of X. See Bhattacharjee (1982).

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Two classes were discussed by Ahmed (1994) who called used better than aged (UBA) and used better than aged in expectation (UBAE).

Definition 1.2. The distribution function F is said to be used better than aged (UBA) if for all x, t ≥ 0

$$\overline{F}(x+t) \geq \overline{F}(t)e^{-\gamma x} \tag{1.2}$$

Definition 1.3. The distribution function F is said to be used better than aged in expectation(UBAE) if for all x, t ≥ 0

$$\int_t^\infty \overline{F}(u)du \geq \frac{\overline{F}(t)}{\gamma} \qquad or \qquad v(t) \geq \frac{\overline{F}(t)}{\gamma}$$

where $v(t) = \int_t^\infty \overline{F}(u)du.$

The equality in (1.3) is achieved when F(x) has an exponential distribution with mean μ equal to the coefficient of asymptotic decay γ, where the exponential distribution is theonly one which has the no aging property.

It was shown that the UBA class is a subclass of UBAE and the IHR (increasing hazard rate) is contained in the UBA class. Similar implications between UBAE, NBUE and HN-BUE were given by Di Crescenzo (1999). Also, Willmot and Cai (2000) showed that the UBA class includes the DMRL class (decreasing mean residual life) while the UBAE includes the DVRL (decreasing variance residual life). Thus we have

$$\begin{array}{c} IHR \subset DMRL \subset UBA \subset UBAE \\ \cup \\ DVRL \end{array}$$

For definition and properties of these classes we refer the readers to the surveys by Deshpand et al (1986) and Deshpande and Purohit (2005).

Testing exponentiality against the classes of life distributions has seen a good deal of attention in the literature. For this literature, we refer the reader to Doksum and Yandel (1984), Barlow and Proschan (1981), Kanjo (1993), Alwasel (1997) and Abu-Youssef (2004) among others. Ahmed (2004) discussed some properties of the UBA and UBAE classes including the moment inequalities and moment generating functions behavior. Also, he discussed the nonparametric estimation and testing of the survival functions of these classes. Abu Youssef (2004, 2009), Al-Zahrany and Stoyanov (2011) and Khoorashadizadeh et all (2011) discussed the properties of the DVRL

The main object in this paper is to deal with the problem of testing H_0: F is exponentialagainst H_1: F is UBAE. The paper is organized as follows: in section 2, we give a brief review of TTT-transform and present a test of H_0. In section 3, we derive the empirical teststatistic for the UBAE class based on the scaled TTT-transforms. In section 4, a study ofthis test statistic is performed through simulation. The power estimates of this statistic are given in section 5, with respect to some commonly used distribution in reliability. Finally, examples using practical data given in Alwasel et al (1997) in medical science are given in section 6.

2. THE CONCEPT OF TTT-TRANSFORMATION

Let $T_1, T_2, \ldots, T_n$ be a random sample of size n from the distribution F whose survival functionis $\overline{F} = 1 - F$ and finite mean $\mu = E[X] = \int_0^\infty \overline{F}(x)\,dx$. We present the following definition of Barlow and Campo (1975).

Definition 2.1. (i) The TTT-transform $H_F^{-1}(t)$ of F where t=F(x) is defined by

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} \overline{F}(u)du \quad \text{for } 0 \le t \le 1$$

where $F^{-1}(t) = \inf\{u : F(u) \ge t\}$, the mean of the distribution F is given by

$$\mu = H_F^{-1}(t) = \int_0^{F^{-1}(1)} \overline{F}(u)du$$

(ii) The function

$$\emptyset_F(t) = \frac{H_F^{-1}(t)}{H_F^{-1}(1)} \tag{2.1}$$

is called the scaled TTT-transform.

Note that if F is the exponential distribution with parameter λ then TTT-transform is given by

$$\emptyset_F(t) = \left(\int_0^{F^{-1}(t)} e^{-\lambda u} du\right) \Big/ \left(\int_0^{F^{-1}(1)} e^{-\lambda u} du\right) = t \; for \; 0 \le t \le 1.$$

Now let

$$D_j = (n - j - 1)\left(t_{(j)} - t_{(j-1)}\right) \quad j = 1,2,\ldots,n$$

and

$$S_j = \sum_{k=1}^{j} D_k = t_{(1)} + \ldots + t_{(j)}$$

D_J denote the sample TTT-transform at $t_{(j)}$, where So = 0. The value S_j/n is an estimateof $H_F^{-1}(t)$, and $W_j = S_j/S_n$ is an estimate of of the scaled TTT-transform. An estimator of $\emptyset_F(t)$ is obtained as

$$\emptyset_F\left(\frac{j-1}{n}\right) = W_{j-1}, \quad j = 1,2,\ldots,n \tag{2.2}$$

The TTT-plot is obtained by plotting W_{j-1} againt $(j-1)/n$ for j = 1, 2, …, n and joining the plotted points by straight lines. It has been shown in Barlow and Doksum (1972) that using Glivenko-Cantelli lemma, that for strictly increasing F, W_j converges to $\emptyset_F(t)$ with probability one and uniformly in [0,1] as $n \to \infty$ and j/n converges to t. Scaled TTT-transforms for some families of life distributions are given by Barlow and Campo (1975), Barlow (1979) and Klefsjo (1982b, 1983).
The following theorem give another definitions of the UBA class in terms of the scaled TTT-transformation.

Theorem 2.1. Let F be a continuous distribution function and $\emptyset_F(t)$ be as in (2.1), then the distribution F is UBAE if

$$\mu\left(1 - \emptyset_F(s)\right) \ge (1 - s)\gamma^{-1} \tag{2.3}$$

Proof. From Eq. (1.3) the life distribution is UBAE distribution if

$$\int_t^\infty \bar{F}(u)du \geq \frac{\bar{F}(t)}{\gamma}$$

since

$$\int_t^\infty \bar{F}(u)du = \mu\left(1 - \int_0^t \bar{F}(u)du\right)$$

Using equation (2.1) and F(x) = s yields

$$\mu\big(1 - \emptyset_F(s)\big) \geq (1 - s)\gamma^{-1}$$

This completes the proof.

3. TESTING UBAE CLASS OF LIFE DISTRIBUTIONS

In this section we present test statistic using the scaled TTT-transform for testing H_0: F is exponentially distributed against H_1: F is UBAE and not exponential based on a sample $T_{(1)}, \ldots, T_{(n)}$ from F. Now, since W_{j-1} converges to $\emptyset_F(t)$ as n→∞ and j-1/n converges tot, then the TTT-plot behaves as $\emptyset_F(t)$ does.This suggests the following test statistic basedon the scaled TTT-transform. Since F is UBAE, then from (2.3) we use the followingmeasure of departure from H_0.

$$\Delta_1 = \int_0^1 \mu\big(1 - \emptyset_F(s)\big) - (1 - s)\gamma^{-1}$$

for $0 \leq S \leq 1$. Integrating both sides of the above equation with respect to s , we get:

$$\Delta_1 = \int_0^1 \mu(1 - \emptyset_F(s) - (1 - s)\gamma^{-1} ds$$

The measure Δ_1 is estimated at a specific time t as follows:

$$\sum_{i=1}^n \bar{x}(1 - W_{i-1}) - \left(1 - \frac{i-1}{n}\right)(t_i - t_{i-1})$$

To reduce the size of the test statistic we use

$$\hat{\Delta}_1 = \frac{\Delta_1}{n}$$

where $t_{(1)}, \ldots, t_{(n)}$ are the ordered statistics of the independent random sample $T_1, \ldots, T_n$ and $T_0 = 0$. Note that $H_0 : \hat{\Delta}_1 = 0$ if F is exponential and $H_1 : \hat{\Delta}_1 > 0$ if F is UBAE andnot exponential.

4. SIMULATION OF SMALL SAMPLE

We have simulated the upper percentile points of $\hat{\Delta}_1$for 90%, 95%, 98% and 99%. The calculations are based on 10,000 simulated samples of size n = 5(1)50. The parameter γ is estimated by $\frac{1}{x}$.

Table 1: Crtical Values of Δ_1

n	90 th	95 th	98th	99 th
5	.457251	.503727	.614752	.592805
6	.473171	.515597	.616949	.596914
7	.451298	.490578	.584411	.565863
8	.427160	.463902	.551675	.534325
9	.402235	.436876	.519629	.503271
10	.381198	.414061	.492568	.477049
11	.361999	.393333	.468186	.453389
12	.344511	.374511	.446178	.432011
13	.329179	.358002	.426857	.413246
14	.315290	.343064	.409415	.396299
15	.301829	.328662	.392762	.380091
16	.291158	.317139	.379204	.366935
17	.281114	.306319	.366531	.354629
18	.270302	.294796	.353312	.341745
19	.260807	.284648	.341603	.330344
20	.253583	.276821	.332334	.321361
21	.246236	.268914	.323089	.312380
22	.238472	.260628	.313558	.303095
23	.232527	.254197	.305963	.295730
24	.225935	.247148	.297824	.287807
25	.220242	.241027	.290679	.280864
26	.214376	.234757	.283445	.273821
27	.210181	.230181	.277959	.268515
28	.205031	.224671	.271588	.262314
29	.200106	.219405	.265505	.256392
30	.195585	.214559	.259885	.250925
31	.191902	.210567	.255156	.246342
32	.187443	.205814	.249701	.241025
33	.183854	.201945	.245161	.236619
34	.180607	.198430	.241006	.232590

Table 2: continue of Crtical Values of $\hat{\Delta}_1$

n	90 th	95 th	98th	99 th
35	.177375	.194941	.236905	.228610
36	.173929	.191250	.232626	.224447
37	.170524	.187609	.228422	.220355
38	.168283	.185141	.225414	.217453
39	.164788	.181429	.221183	.213325
40	.162128	.178560	.217813	.210054
41	.159754	.175984	.214756	.207092
42	.157051	.173086	.211394	.203821
43	.052207	.068055	.105915	.098431
44	.081214	.096881	.134307	.126909
45	.095483	.110975	.147983	.140667
46	.148494	.218297	.254901	.247665
47	.146256	.161415	.266684	.259525
48	.144168	.159168	.195001	.292085
49	.141917	.156763	.192229	.185218
50	.140338	.155034	.190144	.183204

Conclusion:

It is clear from the table that the values of the percentiles decreases when the sample size increases.

5. THE POWER ESTIMATE OF THE UBA TEST STATISTIC

The power estimate of the test statistic $\hat{\Delta}_1$ in (3.1) is considered for the significant level at 95^{th} upper percentile in Table (2) for three of the most commonly used alternatives (seeHollander and Proschan [10]), which are

- (i) Linear failure rate family: $\overline{F}_1(x) = e^{-x-\frac{\theta x^2}{2}}$, $x \geq 0, \theta \geq 0$.
- (ii) Makeham family: $\overline{F}_2(x) = e^{-x-\theta(x-1+e^{-z})}, x \geq 0, \theta \geq 0$
- (iii)Weibullfamily: $\overline{F}_3(x) = e^{-x^{\theta}}, x \geq 0, \theta \geq 0$

These distributions are reduced to exponential distribution for appropriate values of θ.

Table 3: Power Estimates of $\hat{\Delta}_1$

Distribution	θ	$n=10$	$n=20$	$n=30$
Linear failure rate family	1	0.993	0.997	0.998
$\overline{F}_1$	2	0.997	0.999	1
(L.F.R)	3	0.999	1	1
Makeham family	2	0.991	0.998	0.999
$\overline{F}_2$	3	0.995	0.999	0.999
(M.F)	3	0.999	0.999	1
Weibull family: $\overline{F}_3$	2	0.999	1	1
	3	0.999	1	1

Note that; power estimates increase when the parameter $\hat{\theta}$ is far from exponentiality and when the size of the sample n increases.

6. APPLICATION

Example 6.1. The following data represent 40 patients suffering from blood cancer from one of the Ministry of Health Hospitals in Saudi Arabia and the ordered life times (in days) are:

115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1169, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1604, 1696, 1735, 1799, 1815, 1852

Using (3.1), it is found that the value of the test statistic $\hat{\Delta}$=14. Then we reject H_0 which states that the set of data have the exponential property under significant level α =0.05.

Example 6.2. In an experiment at the Florida State University to study the effect of methyl mercury poisoning on the life lengths of goldfish, goldfish were subjected to various dosages of methyl mercury (Kochar 21). At one dosage level, the ordered times in days to death are: 0.86, 0.88, 1.04, 1.24, 1.35, 1.41, 1.45, 1.65, 1.67, 1.67. Using equation (3.1), the values of test statistics based on the above data are

$$\bar{\Delta}= 0:014$$

This value leads to the acceptance of H_0 at the significance level $\alpha = 0.05$, see Table 1. Therefore, the data do not have UBAE property.

7. CONCLUSION

Testing exponentiality against the classes of life distributions has a good deal of attention. In this study, we derive a new test statistic based on a total time on test transform for testing the exponentiality against the UBAE class of life distributions which are not exponential. This test is simple and its power is estimated for some commonly used alternatives. Critical values are tabulated for sample sizes 5(1)50. Two sets of real data are used as examples to elucidate the use of the proposed test statistic for practical reliability analysis.

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