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THE BRAIDINGS IN THE MAPPING CLASS GROUPS OF SURFACES

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Abstract. The disjoint union of mapping class groups of surfaces forms a braided monoidal category $\mathcal{M}$, as the disjoint union of the braid groups $\mathcal{B}$ does. We give a concrete and geometric meaning of the braidings $\beta_{r,s}$ in $\mathcal{M}$. Moreover, we find a set of elements in the mapping class groups which correspond to the standard generators of the braid groups. Using this, we can define an obvious map $\phi : B_g \to \Gamma_{g,1}$. We show that this map ϕ is injective and nongeometric in the sense of Wajnryb. Since this map extends to a braided monoidal functor $\Phi : \mathcal{B} \to \mathcal{M}$, the integral homology homomorphism induced by ϕ is trivial in the stable range.

1. Introduction

Let $\Gamma_{g,1}$ be the mapping class group of the surface $S_{g,1}$, the compact orientable surface of genus g with one boundary component obtained by deleting a disk from the closed surface S_g. Let $\mathcal{M} = \coprod_{g\geq 0} \Gamma_{g,1}$ be the disjoint union of $\Gamma_{g,1}$'s. Then it is regarded as the category whose objects are nonnegative integers. Here, $\hom_{\mathcal{M}}(g,g) = \Gamma_{g,1}$ and there is no morphism between distinct integers. The monoid structure, called the F-product, in $\mathcal{M}$ is induced by the pair-of-pants connected sum. The braid structure of $\mathcal{M}$ was given in [8] in terms of the actions of braidings on the fundamental group of $S_{g,1}$. Recall that $\Gamma_{g,1}$ may be identified with the subgroup of the automorphism group of $\pi_1 S_{g,1}$ that consists of the automorphisms fixing the fundamental relator $R = [y_1, x_1] \cdots [y_g, x_g]$. Here, $\pi_1 S_{g,1}$ is a free group on $2g$ generators $x_1, y_1, x_2, y_2, \ldots, x_g, y_g$ and the fundamental relator represents a loop along the boundary of $S_{g,1}$. The F-product on $\mathcal{M}$ may be identified with the operation taking the free product of automorphisms.

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The action of the braiding $\beta_{r,s} : r \otimes s \to s \otimes r$, which is an element of $\Gamma_{r+s,1}$, on the free group on $x_1, y_1, x_2, y_2, \ldots, x_{r+s}, y_{r+s}$ was given in [8]. The braiding could also be expressed as a product of the standard Dehn twists, the generators of $\Gamma_{g,1}$.

The disjoint union of braid groups $\mathcal{B} = \coprod_{g\geq 0} B_g$ forms a braided monoidal category. Let $\sigma_1, \ldots, \sigma_{g-1}$ be the standard generators of B_g, where σ_i crosses the i-th and $(i+1)$-st strings. The monoid structure is induced by the juxtaposition and the braiding $(\sigma_{r,s} : r \otimes s \to s \otimes r) \in B_{r+s}$ is the crossing of the front r strings and the rear s strings. We can express $\sigma_{r,s}$ in terms of σ_i's as follows:

$$\sigma_{r,s} = (\sigma_r \sigma_{r-1} \cdots \sigma_1)(\sigma_{r+1}\sigma_r \cdots \sigma_2) \cdots (\sigma_{r+s-1}\sigma_{r+s-2} \cdots \sigma_s).$$

In this paper, we give a concrete and geometric meaning of the braiding $\beta_{r,s}$ in the braided monoidal category $\mathcal{M}$. Moreover, we find a set of elements $\beta_1, \ldots, \beta_{g-1}$ of $\Gamma_{g,1}$ which are analogous to the standard generators of B_g. Using these generators, we can define an obvious map $\phi : B_g \to \Gamma_{g,1}, \sigma_i \mapsto \beta_i$. An embedding of a braid group into a mapping class group is said to be *geometric* by Wajnryb ([13, 14]) if it maps the standard generators of the braid group to Dehn twists. We show that the map $\phi : B_g \to \Gamma_{g,1}$ is injective and is, moreover, *nongeometric*.

The natural embedding $\phi : B_g \to \Gamma_{g,1}$ can be extended to a braided monoidal functor $\Phi : \mathcal{B} \to \mathcal{M}$. Namely, Φ preserves the braidings. As a consequence, we show that $\phi : B_g \to \Gamma_{g,1}$ induces a map of double loop spaces $B\phi^+ : BB_\infty^+ \to B\Gamma_\infty^+$ and hence the homology homomorphism $\phi_* : H_*(B_\infty; \mathbb{Z}) \to H_*(\Gamma_\infty; \mathbb{Z})$ induced by ϕ is trivial, where $B_\infty = \varinjlim B_g$ and $\Gamma_\infty = \varinjlim \Gamma_{g,1}$.

The classifying space of a braided monoidal category naturally gives rise to a double loop space ([5]). More precisely speaking, the group completion of the classifying space of a braided monoidal category is homotopy equivalent to a double loop space. For example, for $\mathcal{B} = \coprod_{g\geq 0} B_g$, we have

$$\overline{B\mathcal{B}} = \Omega B(\coprod_{g\geq 0} BB_g) \simeq \mathbb{Z}_+ \times BB_\infty^+ \simeq \Omega^2 S^2,$$

where $\overline{B\mathcal{B}}$ means the group completion of the classifying space of $\mathcal{B}$, and $+$ means the Quillen's plus construction.

The braided monoidal functor $\Phi : \mathcal{B} \to \mathcal{M}$ induces a map of double loop spaces $\overline{B\mathcal{B}} \to \overline{B\mathcal{M}}$. In other words, $B\phi^+ : BB_\infty^+ \to B\Gamma_\infty^+$ is a map of double loop spaces. Since every map $\Psi : BB_\infty^+ \to B\Gamma_\infty^+$ of double loop spaces is null-homotopic ([10, Lemma 5.3]), so is $B\phi^+$. This proves that the homomorphism $\phi_* : H_*(B_\infty; \mathbb{Z}) \to H_*(\Gamma_\infty; \mathbb{Z})$ is trivial since the plus construction does not change the homology.

When we first think of the geometric meaning of the braiding $\beta_{r,s}$, since it interchanges the front r genus holes and the rear s genus holes of $S_{r+s,1}$, we might conjecture that it would be a 180° rotation around the boundary of the surface. Here, we consider the boundary as the waist of the surface with two arms; the left arm is the front r genus holes and the right arm is the rear s genus holes. However, just a 180° rotation around the waist is not sufficient. We also need two half-twists of the arms. Here, it is important to determine the directions of these half-twists. There are four choices of directions: two choices on the waist and two choices on the arms. By analyzing the simplest case, $\beta_{1,1}$ for $S_{2,1}$, we show that the braiding $\beta_{r,s}$ is the composite of two half-twists, the *reverse* half Dehn twist along the boundary and half Dehn twists along the shoulders of two arms. The key part of this paper is to figure out a correct geometric meaning of $\beta_{1,1}$ as a self-homeomorphism of $S_{2,1}$ and to extend this to the general case, $\beta_{r,s}$ in $\Gamma_{r+s,1}$. The half Dehn twists are regarded as elements of the automorphism group of $\pi_1 S_{2,1} = F_{\{x_1,y_1,x_2,y_2\}}$. The concrete expression of the actions of these on the fundamental group can be obtained by series of geometric calculations, which are given in Lemma 3.2 through Lemma 3.5. These four lemmas play a key role for the proof of Theorem 3.6. We believe that these lemmas also have a great interest in their own right.

2. Braided monoidal category

Let $S_{g,1}$ be a compact connected orientable surface of genus g with one boundary component. The mapping class group $\Gamma_{g,1}$ is the group of isotopy classes of orientation-preserving self-homeomorphisms of $S_{g,1}$ fixing the boundary of $S_{g,1}$ pointwise. $\Gamma_{g,1}$ is generated by the standard Dehn twists $a_1, \ldots, a_g, b_1, \ldots, b_g, w_1, \ldots, w_{g-1}$ (Figure 1; see, for example, [12]).

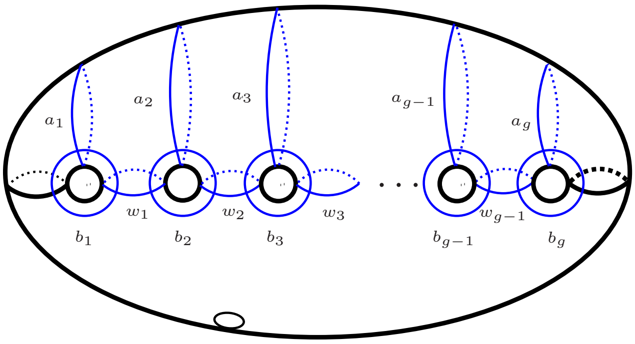

FIGURE 1. Standard Dehn twists of $\Gamma_{g,1}$

Definition 2.1. A *braided monoidal category* is a monoidal category $(\mathcal{C}, \otimes, I)$ together with a family of natural commutativity isomorphisms

$$\beta_{A,B} : A \otimes B \to B \otimes A$$

called *braidings*, natural in both variables, such that for all objects A, B, C in $\mathcal{C}$,

(a) $\beta_{A,I} = \beta_{I,A} = 1_A$,
(b) $\beta_{A\otimes B,C} = (\beta_{A,C} \otimes 1_B) \circ (1_A \circ \beta_{B,C})$,
(c) $\beta_{A,B\otimes C} = (1_B \otimes \beta_{A,C}) \circ (\beta_{A,B} \otimes 1_C)$.

The equations in (b), (c) imply the Yang-Baxter equation:

$$(1_C \otimes \beta_{A,B})(\beta_{A,C} \otimes 1_B)(1_A \otimes \beta_{B,C}) = (\beta_{B,C} \otimes 1_A)(1_B \otimes \beta_{A,C})(\beta_{A,C} \otimes 1_C).$$

The disjoint union $\mathcal{M} = \coprod_{g\geq 0} \Gamma_{g,1}$ forms a category whose objects are nonnegative integers and morphisms satisfy

$$\hom(g,h) = \begin{cases} \Gamma_{g,1} & \text{if } g = h, \\ \phi & \text{if } g \neq h. \end{cases}$$

$\mathcal{M}$ has a monoid structure. The F-product

$$\Gamma_{g,1} \times \Gamma_{h,1} \to \Gamma_{g+h,1}$$

is induced by extending two self-homeomorphisms on $S_{g,1}$ and $S_{h,1}$ to the surface $S_{g+h,1}$ obtained by attaching a pair of pants (a sphere with three boundary components) to the surfaces $S_{g,1}$ and $S_{h,1}$ along the fixed boundary circles. We extend the identity map on the boundary to the whole pants.

It is shown in [8] that $\mathcal{M}$ is a braided monoidal category whose braidings are defined as follows. Let $x_1, y_1, x_2, y_2, \dots, x_g, y_g$ be the generators of $\pi_1 S_{g,1}$ which are represented by loops parallel to the Dehn twist loops $a_1, b_1, a_2, b_2, \dots, a_g, b_g$, respectively, in Figure 1. $\Gamma_{g,1}$ is identified with the subgroup of the automorphism group of $\pi_1 S_{g,1}$ fixing the fundamental relator

$$R = [y_1, x_1][y_2, x_2] \cdots [y_g, x_g].$$

The (r,s)-braiding $\beta_{r,s} : r \otimes s \to s \otimes r$, which is an element of $\Gamma_{r+s,1}$, acts on the free group on $\{x_1, y_1, x_2, y_2, \dots, x_g, y_g\}$ as follows:

$$\begin{gathered} x_1 \mapsto R_s x_{s+1} R_s^{-1},\ y_1 \mapsto R_s y_{s+1} R_s^{-1} \\ \vdots \\ x_r \mapsto R_s x_{s+r} R_s^{-1},\ y_r \mapsto R_s y_{s+r} R_s^{-1} \\ x_{r+1} \mapsto x_1,\ y_{r+1} \mapsto y_1 \\ x_{r+2} \mapsto x_2,\ y_{r+1} \mapsto y_2 \\ \vdots \\ x_{r+s} \mapsto x_s,\ y_{r+s} \mapsto y_s \end{gathered}$$

where $R_s = [y_1, x_1] \cdots [y_s, x_s]$. We can easily check that the (r,s)-braidings fix the fundamental relator R and satisfy the braid equations (b), (c) of Definition 2.1.

Since the group completion of the classifying space of a braided monoidal category is homotopy equivalent to a double loop space, we have:

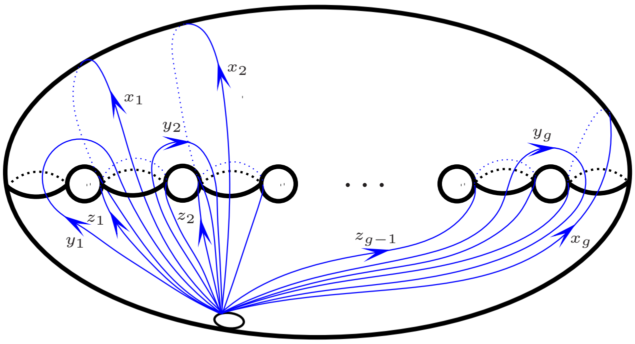

FIGURE 2. The generators $x_1, y_1, \ldots, x_g, y_g$ of $\pi_1 S_{g,1}$

Theorem 2.2. $\overline{B\mathcal{M}} = \Omega B(\coprod_{g\geq 0} B\Gamma_{g,1})$ *is homotopy equivalent to a double loop space. Here,* $\overline{B\mathcal{M}}$ *denotes the group completion of* $B\mathcal{M}$.

3. The geometric analysis of braidings

In this section we analyze the geometric meanings of braidings in the mapping class group. As an application, we have a natural embedding $\phi : B_g \hookrightarrow \Gamma_{g,1}$, which is nongeometric and induces the trivial homology homomorphism in the stable range in the integral coefficient.

The braiding $\beta_{r,s} \in \Gamma_{r+s,1}$ can be expressed as a product of the standard Dehn twists. First, the braiding $\beta_{1,1}$ was expressed in [8] in terms of the Dehn twists a_1, a_2, b_1, b_2, w_1:

Lemma 3.1. *The* $(1,1)$*-braiding* $\beta_{1,1}$ *in* $\Gamma_{2,1}$ *equals*

$$\beta_{1,1} = (a_1 b_1 a_1)^4 (a_2 b_2 (a_1 b_1 a_1)^{-1} w_1 a_1 b_1 a_1^2 b_1)^{-3}. \tag{3.1}$$

The standard Dehn twists $a_1, \ldots, a_g, b_1, \ldots, b_g, w_1, \ldots, w_{g-1}$ act on $\pi_1 S_{g,1}$, which is the free group generated by $\{x_1, y_1, x_2, y_2, \ldots, x_g, y_g\}$, as follows:

$$\begin{aligned} a_i :& y_i \mapsto y_i x_i^{-1} \\ b_i :& x_i \mapsto x_i y_i \\ w_i :& x_i \mapsto z_i^{-1} y_{i+1} x_{i+1} y_{i+1}^{-1} \\ & y_i \mapsto y_i z_i \\ & y_{i+1} \mapsto z_i^{-1} y_{i+1} \end{aligned}$$

where $z_i = x_i^{-1} y_{i+1} x_{i+1} y_{i+1}^{-1}$ and these automorphisms fix the generators that do not appear in the list. By using these actions, we can check that the equation given in Lemma 3.1 holds, that is, $\beta_{1,1}$ acts on the free group on $\{x_1, y_1, x_2, y_2\}$

as

$$x_1 \mapsto [y_1, x_1]x_2[x_1, y_1],\ y_1 \mapsto [y_1, x_1]y_2[x_1, y_1]$$
$$x_2 \mapsto x_1,\ y_2 \mapsto y_1$$

as desired. Here, the Dehn twists act on the right on $\pi_1 S_{g,1}$.

Now we construct the (r, s)-braiding $\beta_{r,s}$ by extending $\beta_{1,1}$. Let

$$\beta_i = (a_i b_i a_i)^4 (a_{i+1} b_{i+1} (a_i b_i a_i)^{-1} w_i a_i b_i a_i^2 b_i)^{-3}$$

for $i = 1, \ldots, g-1$. Then these satisfy the braid relations. The (r, s)-braiding $\beta_{r,s}$ in the group $\Gamma_{g,1}$ defined by

$$\beta_{r,s} = (\beta_r \beta_{r-1} \cdots \beta_1)(\beta_{r+1} \beta_r \cdots \beta_2) \cdots (\beta_{r+s-1} \beta_{r+s-2} \cdots \beta_s)$$

is analogous to the (r, s)-braiding in the braid group B_{r+s}. Here, we may think that the composition of two braids is a stacking (from top to bottom) as morphisms map from top to bottom. By simple calculations we can check that this (r, s)-braiding $\beta_{r,s}$ acts on $\pi_1 S_{g,1}$ as described in Section 2.

We now investigate the geometric meaning of $\beta_{r,s}$. We first figure out what $\beta_{1,1}$ actually is as a self-homeomorphism of $S_{2,1}$.

Let C be a *separating* simple closed curve on $S_{g,1}$ with $S_{g,1} - C = \Sigma_1 \coprod \Sigma_2$. The *half* Dehn twist h_C is defined to be the self-homeomorphism of $S_{g,1}$ which is 180° twist along a tubular neighborhood of C in the same direction as the usual Dehn twist, as shown in Figure 3. h'_C denotes the reverse half Dehn twist along C, that is, $h'_C = h_C^{-1}$ in $\Gamma_{g,1}$.

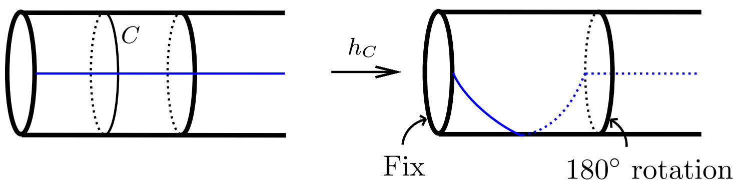

FIGURE 3. The half Dehn twist along C

For a separating simple closed curve C on $S_{g,1}$ with $S_{g,1} - C = \Sigma_1 \coprod \Sigma_2$, we may think that out of two separated surfaces Σ_1, Σ_2, h_C fixes (pointwise) one of them and rotates the other by 180° along a tubular neighborhood of C. Since h_C may be regarded as an element of the automorphism group of $\pi_1 S_{g,1}$, in dealing with its action on the fundamental group, we may think, as a matter of convenience, that the base point is located in the fixed one out of Σ_1, Σ_2. This half Dehn twist may also be defined for a *nonseparating* simple closed curve in a natural way, but it is of no interest because it is supposed to be trivial, namely, homotopic to the identity map.

In this paper a closed curve C on the surface, by abusing notations, may also stand for an element of the fundamental group of the surface. For example, let

R be the closed curve along (parallel to) the boundary of $S_{g,1}$. Then it could also mean the element $R = [y_1, x_1][y_2, x_2] \cdots [y_g, x_g] \in \pi_1 S_{g,1}$.

Let R_1, R_2 be the closed curves on $S_{2,1}$ as given in Figure 4. The arrows on the curves indicate their directions as elements of $\pi_1 S_{2,1}$. Note that R_1 stands for $[y_1, x_1] \in \pi_1 S_{2,1} = F_{\{x_1,y_1,x_2,y_2\}}$, and R_2 stands for $[y_2, x_2]$.

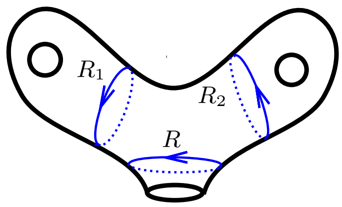

Figure 4.

For the closed curve R on $S_{2,1}$, the half Dehn twist h_R and the reverse half Dehn twist h'_R are illustrated in Figure 5. We may think of them as $180°$ rotations of the whole surface around the axis l, rotating the waist part (the tubular neighborhood of R) and fixing the boundary of the surface (see Figure 3).

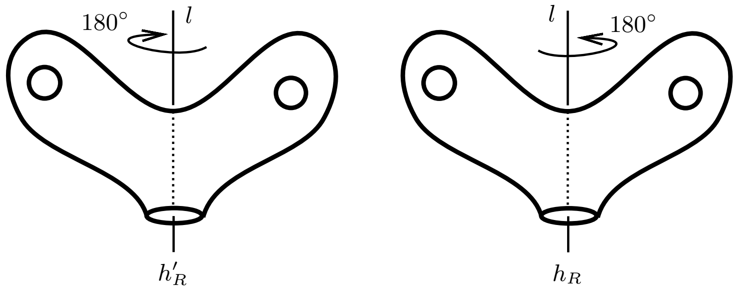

Figure 5. Two half Dehn twists along the boundary

The half Dehn twists are regarded as elements of the automorphism group of $\pi_1 S_{2,1} = F_{\{x_1,y_1,x_2,y_2\}}$. The concrete expression of their actions on the fundamental group can be obtained by series of geometric calculations. The result of these calculations are given in Lemma 3.2 through Lemma 3.5. These four lemmas not only play key roles in analyzing the geometric meaning of the braidings in $\Gamma_{g,1}$, but also are of interest in their own right. The proofs of these lemmas are obtained by tracing the series of geometric transforms generated by the actions of half Dehn twists on the generators of $\pi_1 S_{2,1}$. Here, we may think that the base point of the fundamental group is located on the boundary of the surface $S_{2,1}$ (see Figure 2).

Lemma 3.2. *The half Dehn twist h_R acts on $\pi_1 S_{2,1}$ as follows*:

$$x_1 \mapsto R_2^{-1} x_2^{-1},\ y_1 \mapsto y_2^{-1} R_2,$$
$$x_2 \mapsto R^{-1} x_1^{-1} R_2,\ y_2 \mapsto R_2^{-1} y_1 R.$$

Note that h_R maps R_1 to R_2 and R_2 to $R_2^{-1}R$.

Lemma 3.3. *The reverse half Dehn twist h'_R acts on $\pi_1 S_{2,1}$ as follows*:

$$\begin{aligned}
&x_1 \mapsto R_1 x_2^{-1} R^{-1} = (R_1 y_2) x_2^{-1} (R_1 y_2)^{-1}\\
&y_1 \mapsto R y_2^{-1} R_1^{-1} = (R_1 y_2 x_2) y_2^{-1} (R_1 y_2 x_2)^{-1}\\
&x_2 \mapsto y_1 x_1^{-1} y_1^{-1}\\
&y_2 \mapsto R_1 y_1^{-1} = (y_1 x_1) y_1^{-1} (y_1 x_1)^{-1}.
\end{aligned}$$

Note that h'_R maps R_1 to RR_1^{-1} and R_2 to R_1.

Since the two loops R_1 and R_2 are disjoint, the half Dehn twists along them are commutative. The product (composite) of these two are called the half Dehn twists on *two arms.*

Lemma 3.4. *For the half Dehn twists h_{R_1} and h_{R_2}, let $h_A = h_{R_1} \circ h_{R_2}$, called the half Dehn twists on two arms. Then h_A acts on $\pi_1 S_{2,1}$ as follows*:

$$\begin{aligned}
&x_1 \mapsto R_1^{-1} x_1^{-1} = (x_1 y_1) x_1^{-1} (x_1 y_1)^{-1}\\
&y_1 \mapsto y_1^{-1} R_1 = x_1 y_1^{-1} x_1^{-1}\\
&x_2 \mapsto R_2^{-1} x_2^{-1} = (x_2 y_2) x_2^{-1} (x_2 y_2)^{-1}\\
&y_2 \mapsto y_2^{-1} R_2 = x_2 y_2^{-1} x_2^{-1}.
\end{aligned}$$

Lemma 3.5. *For the reverse half Dehn twists on two arms, $h'_A = h'_{R_1} \circ h'_{R_2}$ acts on $\pi_1 S_{2,1}$ as follows*:

$$\begin{aligned}
&x_1 \mapsto x_1^{-1} R_1^{-1} = y_1 x_1^{-1} y_1^{-1}\\
&y_1 \mapsto R_1 y_1^{-1} = (y_1 x_1) y_1^{-1} (y_1 x_1)^{-1}\\
&x_2 \mapsto x_2^{-1} R_2^{-1} = y_2 x_2^{-1} y_2^{-1}\\
&y_2 \mapsto R_2 y_2^{-1} = (y_2 x_2) y_2^{-1} (y_2 x_2)^{-1}.
\end{aligned}$$

Each of the half Dehn twists, the half Dehn twist along the boundary and the half Dehn twists on two arms, has two directions. The braiding $\beta_{1,1}$ may be expressed as the composition of two half Dehn twists, but we have to determine which choice of the directions is correct. The correct choice of the directions is h'_R and h_A.

Theorem 3.6. *The* $(1,1)$*-braiding $\beta_{1,1} \in \Gamma_{2,1}$ is the product* (*composite*) *of h'_R and h_A in $\Gamma_{2,1}$, that is, $\beta_{1,1} = h'_R \circ h_A$.*

Proof. Since the three closed curves R, R_1, R_2 are mutually disjoint, h_R, h_{R_1}, h_{R_2} are all commutative. We have

$$
\begin{aligned}
x_1 &\xmapsto{h'_R} R_1 x_2 R^{-1} \xmapsto{h_A} R_1 x_2 R_1^{-1}\\
y_1 &\xmapsto{h'_R} R y_2^{-1} R_1^{-1} \xmapsto{h_A} R_1 y_2 R_1^{-1}\\
x_2 &\xmapsto{h'_R} y_1 x^{-1} y_1^{-1} \xmapsto{h_A} x_1\\
y_2 &\xmapsto{h'_R} R_1 y_1^{-1} \xmapsto{h_A} y_1
\end{aligned}
$$

as desired. □

We have found that $\beta_{1,1}$ is the composition of the reverse half Dehn twist along the boundary and the half Dehn twists on two arms. This is partly illustrated in Figures 6 and 7.

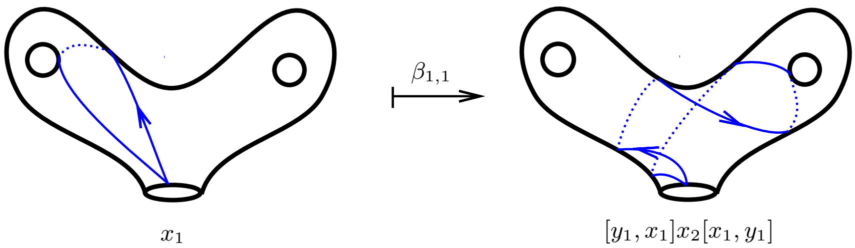

FIGURE 6.

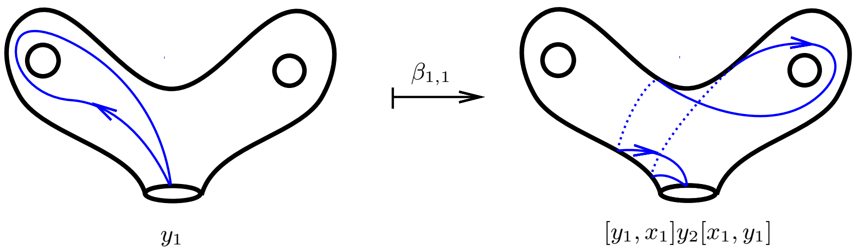

FIGURE 7.

Recall that the (r, s)-braiding $\beta_{r,s} \in \Gamma_{r+s,1}$ equals

$$(\beta_r \beta_{r-1} \cdots \beta_1)(\beta_{r+1}\beta_r \cdots \beta_2) \cdots (\beta_{r+s-1}\beta_{r+s-2} \cdots \beta_s),$$

where $\beta_i = (a_ib_ia_i)^4(a_{i+1}b_{i+1}(a_ib_ia_i)^{-1}w_ia_ib_ia_i^2b_i)^{-3}$. Let $R_i = [y_i, x_i]$. Then $\beta_i \in \Gamma_{r+s,1}$ acts on $\pi_1 S_{r+s,1}$ as follows:

$$\beta_i: \quad \begin{array}{ll} x_i \mapsto R_i x_{i+1} R_i^{-1}, & y_i \mapsto R_i y_{i+1} R_i^{-1}, \\ x_{i+1} \mapsto x_i, & y_{i+1} \mapsto y_i. \end{array}$$

Let $R_{i,j} = [y_i, x_i][y_{i+1}, x_{i+1}] \cdots [y_j, x_j]$ for $j > i$. For a closed curve $R_{i,i+1}$ on $S_{r+s,1}$ (see Figure 8), imagine that we cut out the surface along $R_{i,i+1}$. Then we get a (small) surface $S_{2,1}$.

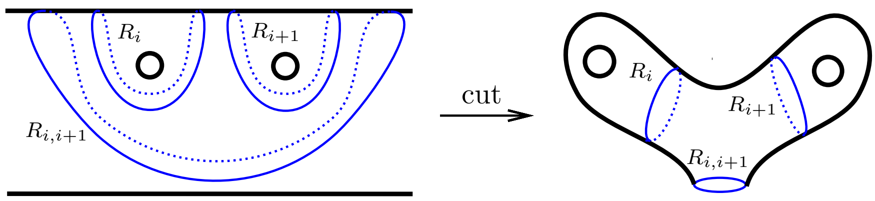

FIGURE 8.

Then β_i is a self-homeomorphism of $S_{r+s,1}$ which is the composition of the reverse half Dehn twist along $h'_{R_{i,i+1}}$ and the half Dehn twists on two arms. That is, we have

$$\beta_i = h'_{R_{i,i+1}} \circ h_{R_i} \circ h_{R_{i+1}}.$$

We can see, by checking the actions on the fundamental group, that these β_i's satisfy the braid relations:

$$\begin{aligned} \beta_i\beta_{i+1}\beta_i &= \beta_{i+1}\beta_i\beta_{i+1}, \\ \beta_i\beta_j &= \beta_j\beta_i \text{ for } |i-j| \geq 2. \end{aligned}$$

The (r,s)-braiding $\beta_{r,s}$ is a composition of a series of local braidings $\beta_i's$. It may be regarded as the composition of the reverse half Dehn twist along the boundary R of $S_{r+s,1}$ and a series of some local half Dehn twists in the upper part of the whole surface. Figure 9 shows how $\beta_{2,3}$ acts on y_1.

Let $\operatorname{Aut} F_n$ be the automorphism of the free group F_n on n generators $x_1, \ldots, x_n$. Then there is an injection $j : B_n \to \operatorname{Aut} F_n$, called the Artin map ([1, 11]), defined by

$$j(\sigma_i): x_i \mapsto x_i x_{i+1} x_i^{-1}, \; x_{i+1} \mapsto x_i,$$

where $\sigma_1, \ldots, \sigma_{n-1}$ are the standard generators of B_n.

There exists an obvious group homomorphism

$$\begin{aligned} \phi: & B_g \to \Gamma_{g,1} \subset \operatorname{Aut} S_{g,1} \\ & \sigma_i \mapsto \beta_i \end{aligned}$$

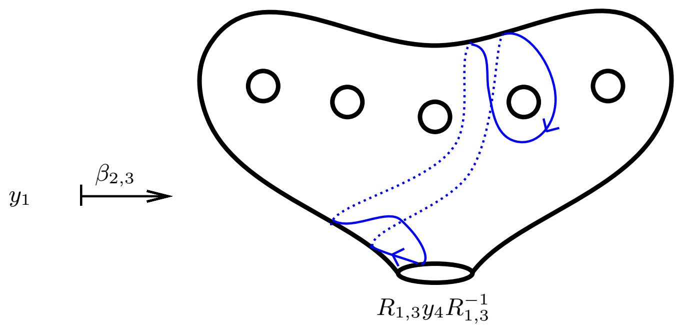

FIGURE 9.

where $\phi(\sigma_i) = \beta_i$ acts on $\pi_1 S_{g,1} = F_{\{x_1,y_1,x_2,y_2,\ldots,x_g,y_g\}}$ as

$$x_i \mapsto R_i x_{i+1} R_i^{-1}, \qquad y_i \mapsto R_i y_{i+1} R_i^{-1},$$
$$x_{i+1} \mapsto x_i, \qquad y_{i+1} \mapsto y_i,$$

where $R_i = [y_i, x_i]$. The map ϕ is injective as the Artin map $j : B_n \to \operatorname{Aut} F_n$ is ([1, 11]).

An embedding of a braid group into a mapping class group is called a geometric embedding if it takes the standard generators of the braid group onto the Dehn twists in the mapping class group ([13]). In [14] Wajnryb raised a question whether there can be a nongeometric embedding. We now show that the map $\phi : B_g \to \Gamma_{g,1}$ is a nongeometric embedding.

Lemma 3.7. *The map $\phi : B_g \to \Gamma_{g,1}$ is a nongeometric embedding.*

Proof. For a standard generator σ_i of B_g, $\phi(\sigma_i) = \beta_i$ is a composite of three half Dehn twists along three disjoint simple closed curves in the local surface $S_{2,1}$. We may think that this self-homeomorphism is supported in the inner surface $S_{0,3}$ obtained by removing the outer dotted parts of the surface $S_{2,1}$ as given in Figure 10. β_i cannot be equal to a full Dehn twist along a simple closed curve in $S_{0,3}$ because there are only three simple closed curves in $S_{0,3}$ and none of the Dehn twists along these three closed curves is equal to $\phi(\sigma_i)$. We can see this by comparing their actions on the fundamental group of $S_{2,1}$. □

From the geometric construction of $\beta_i's$ and the actions of them on the fundamental group of the surface, we have the following theorem.

Theorem 3.8. *Let $\mathcal{B} = \coprod_{g\geq 0} B_g$ and $\mathcal{M} = \coprod_{g\geq 0} \Gamma_{g,1}$. The nongeometric embedding $\phi : B_g \to \Gamma_{g,1}$, $\sigma_i \mapsto \beta_i$ extends to a braided monoidal functor $\Phi : \mathcal{B} \to \mathcal{M}$.*

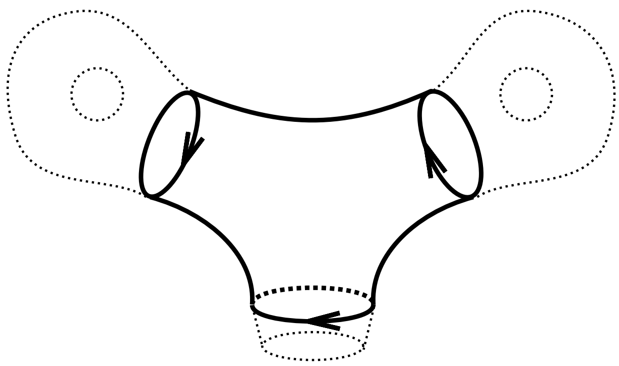

FIGURE 10. 180° twists along the boundary of $S_{0,3}$

The map $\phi : B_g \to \Gamma_{g,1}$ is a more natural embedding than the Harer map, which is defined by

$$h(\sigma_i) = \begin{cases} b_{\frac{i+1}{2}} & \text{if } i \text{ is odd} \\ w_{\frac{i}{2}} & \text{if } i \text{ is even} \end{cases}$$

(see Figure 1 to recall the definitions of b_i's and w_i's) because the map h does not directly give rise to a braided monoidal functor for $\mathcal{B}$ to $\mathcal{M}$ (see [9, Theorem 3.3]).

The group completion of the classifying space of a braided monoidal category is naturally homotopy equivalent to a double loop space ([2, 3, 4, 5]). Therefore, Φ induces a map $\overline{B\mathcal{B}} \to \overline{B\mathcal{M}}$ of double loop spaces, where $\overline{B\mathcal{B}}$ and $\overline{B\mathcal{M}}$ denote the group completions of $B\mathcal{B}$ and $B\mathcal{M}$, respectively.

Since $\overline{B\mathcal{B}} \simeq \mathbb{Z}_+ \times BB_\infty^+$ and $\overline{B\mathcal{M}} \simeq \mathbb{Z}_+ \times B\Gamma_\infty^+$, we have:

Lemma 3.9. $B\phi^+ : BB_\infty^+ \to B\Gamma_\infty^+$ *is a map of double loop spaces and is null-homotopic.*

It is known ([10, Lemma 5.3]) that every double loop map $f : BB_\infty^+ \to B\Gamma_\infty^+$ is null-homotopic.

Since the plus construction does not change homology groups, Lemma 3.9 implies that the homology homomorphism induced by ϕ is trivial in the stable range.

Theorem 3.10. *The homomorphism* $\phi_* : H_*(B_\infty;\mathbb{Z}) \longrightarrow H_*(\Gamma_\infty;\mathbb{Z})$ *induced by* ϕ *is trivial.*

By the homology stability theorem ([6, 7]), we have that $\phi_* : H_i(B_g;\mathbb{Z}) \to H_i(\Gamma_{g,1};\mathbb{Z})$ is trivial for $0 < i < \frac{g}{2}$.

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