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CONJUGACY SEPARABILITY OF CERTAIN GENERALIZED FREE PRODUCTS OF NILPOTENT GROUPS

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Abstract. It is known that generalized free products of finitely generated nilpotent groups are conjugacy separable when the amalgamated subgroups are cyclic or central in both factor groups. However, those generalized free products may not be conjugacy separable when the amalgamated subgroup is a direct product of two infinite cycles. In this paper we show that generalized free products of finitely generated nilpotent groups are conjugacy separable when the amalgamated subgroup is $\langle h\rangle \times D$, where D is in the center of both factors.

1. Introduction

Let S be a subset of a group G. Then G is said to be *S-separable* if for each $x \in G\backslash S$, there exists a normal subgroup N_x of finite index in G such that $x \notin N_xS$. Equivalently, S is a closed subset of G in the profinite topology of G. If $S = \{1\}$, then G is *residually finite* ($\mathcal{R}F$). If for each $x \in G$, G is $\{x\}^G$-separable, where $\{x\}^G$ is the conjugacy class of x in G, then G is called *conjugacy separable* (c.s.). Residual and separability properties are of interest to both group theorists and topologists. They are related to the solvability of the word problem, the conjugacy problem and the generalized word problem (Mal'cev [11] and Mostowski [12]). Topologically, they are related to problems on the embeddability of equivariant subspaces in their regular covering spaces (Scott [18], Niblo [13]).

Blackburn [2] first proved that finitely generated nilpotent groups are conjugacy separable. Stebe showed that free products of conjugacy separable groups are conjugacy separable [19], hence free groups are conjugacy separable. Formanek [6] (also Remeslennikov [14]) showed that polycyclic-by-finite groups

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are also conjugacy separable. Dyer [3] showed that free-by-finite groups are conjugacy separable. There are a number of other results on conjugacy separability. In particular, Fine and Rosenberger [5] proved that Fuchsian groups are conjugacy separable. Wilson and Zalesskii [21] showed that the Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers of $\mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 7, 11$, are conjugacy separable.

In general, the generalized free product of two conjugacy separable groups is not conjugacy separable. Baumslag [1] constructed an example of a generalized free product of two finitely generated nilpotent groups amalgamating a direct product of two cycles (see Example 2.1 below) which is not even Hopfian, whence not conjugacy separable. However, Dyer [4] showed that the free product of two free groups –or two finitely generated nilpotent groups– amalgamating a cyclic subgroup is conjugacy separable. Tang [20] showed that generalized free products of surface groups amalgamating a cyclic subgroup are conjugacy separable. Ribes, Segal and Zalesskii [16] showed that generalized free products of polycyclic groups amalgamating cyclic subgroups are conjugacy separable.

As mentioned above, generalized free products of finitely generated nilpotent groups amalgamating a cyclic subgroup are conjugacy separable. But those generalized free products amalgamating a direct product of two cyclic subgroups may not be conjugacy separable. However, those generalized free products amalgamating a central subgroup are conjugacy separable [8]. In this paper, we find some conditions to derive that generalized free products of conjugacy separable groups amalgamating $\langle h\rangle \times D$, where D is in the center of both factors, are conjugacy separable. Using this, we show that generalized free products of finitely generated nilpotent groups amalgamating $\langle h\rangle \times D$, where D is in the center of both factors, are conjugacy separable.

2. Preliminaries

Throughout this paper we use standard notation and terminology.

If A, B are groups, $G = A *_H B$ denotes the generalized free product of A and B amalgamating the subgroup H. If $x \in G = A *_H B$, then $\|x\|$ denotes the free product length of x in G.

$x \sim_G y$ means that x and y are conjugate in G.

$Z(G)$ is the center of G.

$C_H(u) = \{h \in H \mid hu = uh\}$ denotes the centralizer of u in H.

$\mathcal{RF}$ is an abbreviation for "residually finite".

The following example shows that the generalized free product of two finitely generated free nilpotent groups amalgamating a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$ may not be residually finite.

Example 2.1 ([1])**.** Let $A = \langle a, b; [a, b, b] = [a, b, a] = 1\rangle$ and $B = \langle c, d; [c, d, d] = [c, d, c] = 1\rangle$. Let $H = \langle a, [a^2, b]\rangle$ and $K = \langle [c, d], c\rangle$. Clearly, A, B are free-nilpotent groups of class 2, $[a^2, b] \in Z(A)$, and $[c, d] \in Z(B)$. Let G be the generalized free product of A and B amalgamating H and K by letting $a = [c, d]$ and $[a^2, b] = c$. Then G is non-Hopfian, whence G is not conjugacy separable.

Theorem 2.2 ([10, Theorem 4.6]). *Let $G = A *_H B$ and let $x \in G$ be of minimal length in its conjugacy class. Suppose that $y \in G$ is cyclically reduced and that $x \sim_G y$.*

(1) *If $||x|| = 0$, then $||y|| \leq 1$ and if $y \in A$, then there is a sequence $h_1, h_2, \ldots, h_r$ of elements in H such that $y \sim_A h_1 \sim_B h_2 \sim_A \cdots \sim_{A(B)} h_r = x$.*

(2) *If $||x|| = 1$, then $||y|| = 1$ and either $x, y \in A$ and $x \sim_A y$ or $x, y \in B$ and $x \sim_B y$.*

(3) *If $||x|| \geq 2$, then $||x|| = ||y||$ and $y \sim_H x^*$, where x^* is a cyclic permutation of x.*

3. Double coset separability

In this section we prove some results on double coset separability.

Definition 3.1. Let G be a group and H, K be subgroups of G. Then G is said to be HK-double coset separable if for each $x \in G$, G is HxK-separable. In particular, we say that G is H*-double coset separable* if G is HxH-separable for all $x \in G$.

Clearly, if G is H-double coset separable, then G is H-separable. In particular, free groups [17] and finitely generated nilpotent groups [9] are H-double coset separable for each finitely generated subgroup H.

Lemma 3.2. *Let A be a group and $h \in A$ with $|h| = \infty$. Suppose $a \in A$ and $a \neq h^i a h^j$ for all integers i, j except $i = j = 0$. If A is $\langle h^n \rangle$-double coset separable, then there exists $M \lhd_f A$ such that in $\overline{A} = A/M$, if $\overline{a} = \overline{h}^i \overline{a} \overline{h}^j$, then $n \mid i, j$.*

Proof. We note that for $0 \leq i_1, j_1 \leq n-1$, if $h^{-i_1} a h^{-j_1} \in \langle h^n \rangle a \langle h^n \rangle$, then $i_1 = j_1 = 0$. Hence $h^{-i_1} a h^{-j_1} \notin \langle h^n \rangle a \langle h^n \rangle$ for all $0 \leq i_1, j_1 \leq n-1$ except $i_1 = j_1 = 0$. Since A is $\langle h^n \rangle$-double coset separable, there exists $M \lhd_f A$ such that $h^{-i_1} a h^{-j_1} \notin M \langle h^n \rangle a \langle h^n \rangle$ for all $0 \leq i_1, j_1 \leq n-1$ except $i_1 = j_1 = 0$. Then in $\overline{A} = A/M$, if $\overline{a} = \overline{h}^i \overline{a} \overline{h}^j$, then $n \mid i, j$. □

Definition 3.3. Let G be a group and $h \in G$. Then G is called $\langle h \rangle$*-self-conjugate* if $h^i \sim_G h^j$ implies $i = j$.

For example, free groups and finitely generated nilpotent groups are $\langle h \rangle$-self-conjugate for each element h of infinite order (see [4]).

Definition 3.4. Let G be a group and $h \in G$ with $|h| = \infty$. Then we say that G satisfies (C') for $\langle h \rangle$ if there exists an integer $\delta > 0$ such that for each $n > 0$, there exists $M \lhd_f G$ depending on n such that $M \cap \langle h \rangle = \langle h^{n\delta} \rangle$ and if $Mh^i \sim_{G/M} Mh^j$, then $Mh^i = Mh^j$.

Theorem 3.5. *Let $G = A *_{\langle h \rangle} B$, where $|h| = \infty$. Let A, B be $\langle h \rangle$-self-conjugate and $\langle h^n \rangle$-double coset separable for each integer $n > 0$. Suppose A, B satisfy the condition (C') for $\langle h \rangle$. Then G is $\langle h \rangle$-double coset separable.*

Proof. Let $x, y \in G$ and $y \notin \langle h \rangle x \langle h \rangle$.

Case 1. $\|x\| \neq \|y\|$. Suppose $x = a_1 b_1 \cdots a_n b_n$ and $y = c_1 d_1 \cdots c_m d_m$, where $a_i, c_j \in A \backslash \langle h \rangle$ and $b_i, d_j \in B \backslash \langle h \rangle$. Since A, B are $\langle h \rangle$-double coset separable, A, B are $\langle h \rangle$-separable. Hence there exist $M_1 \lhd_f A$ and $N_1 \lhd_f B$ such that $a_i, c_j \notin M_1 \langle h \rangle$ and $b_i, d_j \notin N_1 \langle h \rangle$ for all i, j. Let $M_1 \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $N_1 \cap \langle h \rangle = \langle h^{s_2} \rangle$ for some integers s_1, s_2. By (C'), there exists an integer $\delta_1 > 0$ such that for each $n > 0$, there exists $M' \lhd_f A$ such that $M' \cap \langle h \rangle = \langle h^{n\delta_1} \rangle$ and if $M'h^i \sim_{A/M'} M'h^j$, then $M'h^i = M'h^j$. Similarly, there exists an integer $\delta_2 > 0$ such that for each $n > 0$, there exists $N' \lhd_f B$ such that $N' \cap \langle h \rangle = \langle h^{n\delta_2} \rangle$ and if $N'h^i \sim_{B/N'} N'h^j$, then $N'h^i = N'h^j$. Hence considering $n = s_1 s_2 \delta_2$, there exists $M_2 \lhd_f A$ such that $M_2 \cap \langle h \rangle = \langle h^{s_1 s_2 \delta_1 \delta_2} \rangle$ and if $M_2 h^i \sim_{A/M_2} M_2 h^j$, then $M_2 h^i = M_2 h^j$. Similarly, there exists $N_2 \lhd_f B$ such that $N_2 \cap \langle h \rangle = \langle h^{s_1 s_2 \delta_1 \delta_2} \rangle$ and if $N_2 h^i \sim_{B/N_2} N_2 h^j$, then $N_2 h^i = N_2 h^j$. Let $M = M_1 \cap M_2$ and $N = N_1 \cap N_2$. Clearly, $M \cap \langle h \rangle = \langle h^{s_1 s_2 \delta_1 \delta_2} \rangle = N \cap \langle h \rangle$. Then in $\overline{G} = \overline{A} *_{\langle \overline{h} \rangle} \overline{B}$, where $\overline{A} = A/M$ and $\overline{B} = B/N$, we have $\|\overline{x}\| = \|x\|$ and $\|\overline{y}\| = \|y\|$. Since $\|x\| \neq \|y\|$, $\|\overline{x}\| \neq \|\overline{y}\|$. This implies that $\overline{y} \notin \langle \overline{h} \rangle \overline{x} \langle \overline{h} \rangle$. Since $\overline{G}$ is free-by-finite, it is residually finite. We can find $\overline{P} \lhd_f \overline{G}$ such that $\overline{y} \notin \overline{P} \langle \overline{h} \rangle \overline{x} \langle \overline{h} \rangle$. Let P be the preimage of $\overline{P}$ in G. Then $P \lhd_f G$ and $y \notin P \langle h \rangle x \langle h \rangle$.

The case when $\|x\| = 0$ and $\|y\| \neq 0$ (or $\|x\| \neq 0$ and $\|y\| = 0$) also can be similarly considered.

Case 2. $\|x\| = \|y\| \leq 1$. If $x \in A \backslash \langle h \rangle$ and $y \in B \backslash \langle h \rangle$, then the above method can be applied. So we suppose $x, y \in A$. Since A is $\langle h \rangle$-double coset separable, there exists $M_1 \lhd_f A$ such that $y \notin M_1 \langle h \rangle x \langle h \rangle$. Let $M_1 \cap \langle h \rangle = \langle h^{s_1} \rangle$ for some integer s_1. By (C'), there exist $M_2 \lhd_f A$ and $N_2 \lhd_f B$ such that $M_2 \cap \langle h \rangle = \langle h^{s_1 \delta_1 \delta_2} \rangle = N_2 \cap \langle h \rangle$. Let $M = M_1 \cap M_2$ and $N = N_2$. Then in $\overline{G} = A/M *_{\langle \overline{h} \rangle} B/N$, $\overline{y} \notin \langle \overline{h} \rangle \overline{x} \langle \overline{h} \rangle$. Hence, as before, we can find $\overline{P} \lhd_f \overline{G}$ such that $\overline{y} \notin \overline{P} \langle \overline{h} \rangle \overline{x} \langle \overline{h} \rangle$. Let P be the preimage of $\overline{P}$ in G. Then $P \lhd_f G$ and $y \notin P \langle h \rangle x \langle h \rangle$.

Case 3. $\|x\| = \|y\| \geq 2$. Suppose $x = a_1 b_1 \cdots a_n b_n$ and $y = d_1 c_1 \cdots d_n c_n$, where $a_i, c_i \in A \backslash \langle h \rangle$ and $b_i, d_j \in B \backslash \langle h \rangle$. This case can be similarly handled as in Case 1.

Suppose $x = a_1 b_1 \cdots a_n b_n$ and $y = c_1 d_1 \cdots c_n d_n$, where $a_i, c_i \in A \backslash \langle h \rangle$ and $b_i, d_i \in B \backslash \langle h \rangle$.

(1) Suppose there exists i such that $c_i \notin \langle h \rangle a_i \langle h \rangle$ (or $d_i \notin \langle h \rangle b_i \langle h \rangle$). As in Case 1 above, we can find $M \lhd_f A$ and $N \lhd_f B$ such that in $\overline{G} = A/M *_{\langle \overline{h} \rangle} B/N$, $\overline{c}_i \notin \langle \overline{h} \rangle \overline{a}_i \langle \overline{h} \rangle$, $\|\overline{x}\| = \|x\|$ and $\|\overline{y}\| = \|y\|$. Then $\overline{y} \notin \langle \overline{h} \rangle \overline{x} \langle \overline{h} \rangle$. For, if $\overline{y} \in \langle \overline{h} \rangle \overline{x} \langle \overline{h} \rangle$, then there exist integers α_i, μ_i such that

$$\overline{c}_1 = \overline{h}^{-\alpha_1} \overline{a}_1 \overline{h}^{\mu_1}$$

$$\overline{d}_1 = \overline{h}^{-\mu_1}\overline{b}_1\overline{h}^{\alpha_2}$$

$$\vdots$$

$$\overline{c}_i = \overline{h}^{-\alpha_i}\overline{a}_i\overline{h}^{\mu_i}$$

$$\vdots$$

$$\overline{d}_n = \overline{h}^{-\mu_n}\overline{b}_n\overline{h}^{\alpha_n}.$$

Hence $\overline{c}_i \in \langle\overline{h}\rangle\overline{a}_i\langle\overline{h}\rangle$, a contradiction. Therefore, $\overline{y} \notin \langle\overline{h}\rangle\overline{x}\langle\overline{h}\rangle$. As before, we can find $P \lhd_f G$ such that $y \notin P\langle h\rangle x\langle h\rangle$.

(2) Suppose $c_k \in \langle h\rangle a_k\langle h\rangle$ and $d_k \in \langle h\rangle b_k\langle h\rangle$ for all k. Then there exists i such that $c_1d_1\cdots c_i \in \langle h\rangle a_1b_1\cdots a_i\langle h\rangle$ and $c_1d_1\cdots c_id_i \notin \langle h\rangle a_1b_1\cdots a_ib_i\langle h\rangle$ (or, similarly, $c_1d_1\cdots d_i \in \langle h\rangle a_1b_1\cdots b_i\langle h\rangle$ and $c_1d_1\cdots d_ic_{i+1} \notin \langle h\rangle a_1b_1\cdots b_ia_{i+1}\langle h\rangle$). Let $c_1d_1\cdots c_i = h^{\lambda_1}a_1b_1\cdots a_ih^{\lambda_2}$ and $d_i = h^{\beta_1}b_ih^{\beta_2}$ for some integers $\lambda_1,\lambda_2,\beta_1,\beta_2$. Hence $a_1b_1\cdots a_ih^{\lambda_2+\beta_1}b_i \notin \langle h\rangle a_1b_1\cdots a_ib_i\langle h\rangle$. For convenience, let $w = a_1b_1\cdots a_i$. Then $wh^{\lambda_2+\beta_1}b_i \notin \langle h\rangle wb_i\langle h\rangle$. Hence $h^{\lambda_2+\beta_1} \notin C_{\langle h\rangle}(w)\cdot C_{\langle h\rangle}(b_i)$. Let $C_{\langle h\rangle}(w) = \langle h^s\rangle$ and $C_{\langle h\rangle}(b_i) = \langle h^t\rangle$. Then $h^{\lambda_2+\beta_1} \notin \langle h^s\rangle\langle h^t\rangle$.

(a) $s = 0$ and $t \neq 0$ (or $s \neq 0$ and $t = 0$). Clearly, $h^{\lambda_2+\beta_1} \notin \langle h^t\rangle$. Since A is $\langle h\rangle$-self-conjugate and $C_{\langle h\rangle}(b_i) = \langle h^t\rangle$, we have $b_i^{-1}h^{j_1}b_i \notin \langle h\rangle$ for all $0 \leq j_1 < t$. Since $C_{\langle h\rangle}(w) = 1$, there exists r such that $C_{\langle h\rangle}(a_r) = 1$ for some $1 \leq r \leq i$ (or similarly, $C_{\langle h\rangle}(b_r) = 1$ for some $1 \leq r < i$). This implies that $a_r \neq h^ja_rh^{j'}$ for all j,j' except $j = j' = 0$. Since A is $\langle h^n\rangle$-double coset separable for each $n > 0$, A is $\langle h^n\rangle$-separable for each $n > 0$. Hence there exists $M_1 \lhd_f A$ such that $h^{\lambda_2+\beta_1} \notin M_1\langle h^t\rangle$, $a_k \notin M_1\langle h\rangle$ for all $1 \leq k \leq n$ and if $M_1a_r = M_1h^ja_rh^{j'}$, then $t \mid j,j'$ (Lemma 3.2). Since B is $\langle h\rangle$-self-conjugate and $C_{\langle h\rangle}(b_i) = \langle h^t\rangle$, we have $b_i^{-1}h^{j_1}b_i \notin \langle h\rangle$ for all $0 \leq j_1 < t$. Hence there exists $N_1 \lhd_f B$ such that $b_k \notin N_1\langle h\rangle$ for all $1 \leq k \leq n$ and $b_i^{-1}h^{j_1}b_i \notin N_1\langle h\rangle$ for all $0 \leq j_1 < t$. Let $M_1 \cap \langle h\rangle = \langle h^{s_1}\rangle$ and $N_1 \cap \langle h\rangle = \langle h^{s_2}\rangle$ for some integers s_1, s_2. By (C'), there exists $M_2 \lhd_f A$ such that $M_2 \cap \langle h\rangle = \langle h^{s_1s_2\delta_1\delta_2}\rangle$ and if $M_2h^j \sim_{A/M_2} M_2h^{j'}$, then $M_2h^j = M_2h^{j'}$. Similarly, there exists $N_2 \lhd_f B$ such that $N_2 \cap \langle h\rangle = \langle h^{s_1s_2\delta_1\delta_2}\rangle$, and if $N_2h^j \sim_{B/N_2} N_2h^{j'}$, then $N_2h^j = N_2h^{j'}$. Let $M = M_1 \cap M_2$ and $N = N_1 \cap N_2$. Then in $\overline{G} = A/M *_{\langle\overline{h}\rangle} B/N$, we have $\|\overline{x}\| = \|x\|$ and $\overline{w}\overline{h}^{\lambda_2+\beta_1}\overline{b}_i \notin \langle\overline{h}\rangle\overline{w}\overline{b}_i\langle\overline{h}\rangle$. For, if $\overline{w}\overline{h}^{\lambda_2+\beta_1}\overline{b}_i \in \langle\overline{h}\rangle\overline{w}\overline{b}_i\langle\overline{h}\rangle$, then there exist integers α_i, μ_i such that

$$\overline{a}_1 = \overline{h}^{-\alpha_1}\overline{a}_1\overline{h}^{\mu_1}$$

$$\overline{b}_1 = \overline{h}^{-\mu_1}\overline{b}_1\overline{h}^{\alpha_2}$$

$$\vdots$$

$$\overline{a}_r = \overline{h}^{-\alpha_r}\overline{a}_r\overline{h}^{\mu_r} \tag{3.1}$$

$$\begin{aligned} &\vdots \\ \overline{a}_i &= \overline{h}^{-\alpha_i}\overline{a}_i\overline{h}^{\mu_i} \\ \overline{h}^{\lambda_2+\beta_1}\overline{b}_i &= \overline{h}^{-\mu_i}\overline{b}_i\overline{h}^{\alpha_i}. \end{aligned}$$

From the first equation in (3.1), we have $\overline{h}^{\alpha_1} \sim_{\overline{A}} \overline{h}^{\mu_1}$. By the choice of M_2, $M_2h^{\alpha_1} = M_2h^{\mu_1}$. Hence $h^{\alpha_1-\mu_1} \in M_2 \cap \langle h\rangle \subset M_1 \cap \langle h\rangle$. Thus $h^{\alpha_1-\mu_1} \in M_1 \cap M_2 = M$, whence $\overline{h}^{\alpha_1} = \overline{h}^{\mu_1}$. Similarly, $\overline{h}^{\mu_1} = \overline{h}^{\alpha_2}, \ldots, \overline{h}^{\alpha_i} = \overline{h}^{\mu_i}$. Hence, $\overline{h}^{\alpha_1} = \overline{h}^{\mu_1} = \cdots = \overline{h}^{\mu_i}$. By the choice of r, we have $t \mid \alpha_r, \mu_r$. Hence $\overline{h}^{\mu_i} = \overline{h}^{\mu_r} \in \langle \overline{h}^t\rangle$. From the last equation of (3.1), we have $\overline{b}_i^{-1}\overline{h}^{\lambda_2+\beta_1+\mu_i}\overline{b}_i = \overline{h}^{\alpha_i}$. By the choice of N_1, $t \mid \lambda_2+\beta_1+\mu_i$. Hence $\overline{h}^{\lambda_2+\beta_1+\mu_i} = \overline{h}^{\alpha_i} \in \langle \overline{h}^t\rangle$. Thus $\overline{h}^{\lambda_2+\beta_1} = \overline{h}^{\alpha_i-\mu_i} \in \langle \overline{h}^t\rangle$, a contradiction by the choice of M_1. Thus $\overline{w}\overline{h}^{\lambda_2+\beta_1}\overline{b}_i \notin \langle\overline{h}\rangle\overline{w}\overline{b}_i\langle\overline{h}\rangle$, and hence $\overline{c}_1\overline{d}_1\cdots\overline{c}_i\overline{d}_i \notin \langle\overline{h}\rangle\overline{a}_1\overline{b}_1\cdots\overline{a}_i\overline{b}_i\langle\overline{h}\rangle$. Therefore, $\overline{y} \notin \langle\overline{h}\rangle\overline{x}\langle\overline{h}\rangle$. Since $\overline{G} = A/M *_{\langle\overline{h}\rangle} B/N$ is residually finite, we can find $P \lhd_f G$ such that $y \notin P\langle h\rangle x\langle h\rangle$.

(b) $s = 0$ and $t = 0$. Clearly, $h^{\lambda_2+\beta_1} \neq 1$. Since $C_{\langle h\rangle}(w) = 1$, there exists r such that $C_{\langle h\rangle}(a_r) = 1$ for some $1 \le r \le i$ (or $C_{\langle h\rangle}(b_r) = 1$ for some $1 \le r < i$). Clearly, $h^{\lambda_2+\beta_1} \notin h^{2(\lambda_2+\beta_1)}$. As in (a) above, $a_r \neq h^j a_r h^{j'}$ for all j, j' except $j = j' = 0$. Since A is $\langle h^n\rangle$-double coset separable, A is $\langle h^{2(\lambda_2+\beta_1)}\rangle$-separable and $\langle h\rangle$-separable. Hence there exists $M_1 \lhd_f A$ such that $h^{\lambda_2+\beta_1} \notin M_1\langle h^{2(\lambda_2+\beta_1)}\rangle$, $a_k \notin M_1\langle h\rangle$ for all $1 \le k \le n$ and if $M_1a_r = M_1h^ja_rh^{j'}$, then $2(\lambda_2+\beta_1) \mid j, j'$ (Lemma 3.2). Now, since $t = 0$, $C_{\langle h\rangle}(b_i) = 1$. Hence $b_i \neq h^jb_rh^{j'}$ for all j, j' except $j = j' = 0$. As before, there exists $N_1 \lhd_f B$ such that $b_k \notin N_1\langle h\rangle$ for all $1 \le k \le n$ and $N_1b_i = N_1h^jb_ih^{j'}$. Then $2(\lambda_2+\beta_1) \mid j, j'$. Let $M_1 \cap \langle h\rangle = \langle h^{s_1}\rangle$ and $N_1 \cap \langle h\rangle = \langle h^{s_2}\rangle$ for some integers s_1, s_2. By (C'), there exists $M_2 \lhd_f A$ such that $M_2 \cap \langle h\rangle = \langle h^{s_1s_2\delta_1\delta_2}\rangle$ and if $M_2h^j \sim_{A/M_2} M_2h^{j'}$, then $M_2h^j = M_2h^{j'}$. Similarly, there exists $N_2 \lhd_f B$ such that $N_2 \cap \langle h\rangle = \langle h^{s_1s_2\delta_1\delta_2}\rangle$, and if $N_2h^j \sim_{B/N_2} N_2h^{j'}$, then $N_2h^j = N_2h^{j'}$. Let $M = M_1 \cap M_2$ and $N = N_1 \cap N_2$. Then in $\overline{G} = A/M *_{\langle\overline{h}\rangle} B/N$, we have $\|\overline{x}\| = \|x\|$ and $\overline{w}\overline{h}^{\lambda_2+\beta_1}\overline{b}_i \notin \langle\overline{h}\rangle\overline{w}\overline{b}_i\langle\overline{h}\rangle$. For, if $\overline{w}\overline{h}^{\lambda_2+\beta_1}\overline{b}_i \in \langle\overline{h}\rangle\overline{w}\overline{b}_i\langle\overline{h}\rangle$, then there exist integers α_i, μ_i such that

$$\begin{aligned} \overline{a}_1 &= \overline{h}^{-\alpha_1}\overline{a}_1\overline{h}^{\mu_1} \\ \overline{b}_1 &= \overline{h}^{-\mu_1}\overline{b}_1\overline{h}^{\alpha_2} \\ &\vdots \\ \overline{a}_r &= \overline{h}^{-\alpha_r}\overline{a}_r\overline{h}^{\mu_r} \qquad (3.2)\\ &\vdots \\ \overline{h}^{\lambda_2+\beta_1}\overline{b}_i &= \overline{h}^{-\mu_i}\overline{b}_i\overline{h}^{\alpha_i}. \end{aligned}$$

As in (a) above, from the equations in (3.2), we have $\overline{h}^{\alpha_1} = \overline{h}^{\mu_1} = \cdots = \overline{h}^{\mu_i}$. By the choice of r, $2(\lambda_2+\beta_1) \mid \alpha_r, \mu_r$. Hence $\overline{h}^{\mu_i} = \overline{h}^{\mu_r} \in \langle \overline{h}^{2(\lambda_2+\beta_1)} \rangle$. From the last equation of (3.2), we have $\overline{h}^{\lambda_2+\beta_1+\mu_i} = \overline{h}^{\alpha_i}$ and $2(\lambda_2+\beta_1) \mid \alpha_i$. Hence we have $\overline{h}^{\lambda_2+\beta_1} = \overline{h}^{\alpha_i - \mu_i} \in \langle \overline{h}^{2(\lambda_2+\beta_1)} \rangle$, a contradiction. Thus $\overline{w}\overline{h}^{\lambda_2+\beta_1}\overline{b}_i \notin \langle \overline{h} \rangle \overline{w}\overline{b}_i \langle \overline{h} \rangle$, and hence $\overline{c}_1\overline{d}_1 \cdots \overline{c}_i\overline{d}_i \notin \langle \overline{h} \rangle \overline{a}_1\overline{b}_1 \cdots \overline{a}_i\overline{b}_i \langle \overline{h} \rangle$. Therefore, $\overline{y} \notin \langle \overline{h} \rangle \overline{x} \langle \overline{h} \rangle$. Since $\overline{G} = A/M *_{\langle \overline{h} \rangle} B/N$ is residually finite, we can find $P \lhd_f G$ such that $y \notin P\langle h \rangle x \langle h \rangle$.

(c) $s \neq 0$ and $t \neq 0$. Clearly, $w^{-1}h^{i_1}w \notin \langle h \rangle$ for all $1 \leq i_1 < s$ and $b_i^{-1}h^{j_1}b_i \notin \langle h \rangle$ for all $1 \leq j_1 < t$. Let $m = \gcd\{s,t\}$. Then $h^{\lambda_2+\beta_1} \notin \langle h^s \rangle \langle h^t \rangle = \langle h^m \rangle$. As before, we can find $M \lhd_f A$ and $N \lhd_f B$ such that in $\overline{G} = A/M *_{\langle \overline{h} \rangle} B/N$,

(1) $\|\overline{x}\| = \|x\|$ and $\overline{h}^{\lambda_2+\beta_1} \notin \langle \overline{h}^m \rangle = \langle \overline{h}^s \rangle \langle \overline{h}^t \rangle$,

(2) for each $1 \leq k \leq i$, $\overline{a}_k^{-1}\overline{h}^{i_k}\overline{a}_k \notin \langle \overline{h} \rangle$ for all $1 \leq i_k < n_k$, where $C_{\langle h \rangle}(a_k) = \langle h^{n_k} \rangle$,

(3) for each $1 \leq k \leq i$, $\overline{b}_k^{-1}\overline{h}^{j_k}\overline{b}_k \notin \langle \overline{h} \rangle$ for all $1 \leq j_k < m_k$, where $C_{\langle h \rangle}(b_k) = \langle h^{m_k} \rangle$, and

(4) if $\overline{h}^i \sim_{\overline{A}(\overline{B})} \overline{h}^j$, then $\overline{h}^i = \overline{h}^j$.

Then we have $\overline{w}\overline{h}^{\lambda_2+\beta_1}\overline{b}_i \notin \langle \overline{h} \rangle \overline{w}\overline{b}_i \langle \overline{h} \rangle$. For, if $\overline{w}\overline{h}^{\lambda_2+\beta_1}\overline{b}_i \in \langle \overline{h} \rangle \overline{w}\overline{b}_i \langle \overline{h} \rangle$, then there exist integers α_i, μ_i such that the equations in (3.2) hold. As before, we have $\overline{h}^{\alpha_1} = \overline{h}^{\mu_1} \in C_{\langle \overline{h} \rangle}(\overline{a}_1)$. Similarly, $\overline{h}^{\mu_1} = \overline{h}^{\alpha_2} \in C_{\langle \overline{h} \rangle}(\overline{b}_1)$, $\ldots$, $\overline{h}^{\alpha_i} = \overline{h}^{\mu_i} \in C_{\langle \overline{h} \rangle}(\overline{a}_i)$. Hence $\overline{h}^{\alpha_1} = \overline{h}^{\mu_1} = \cdots = \overline{h}^{\mu_i} \in C_{\langle \overline{h} \rangle}(\overline{a}_1\overline{b}_1 \cdots \overline{a}_i) = \langle \overline{h^s} \rangle$. From the last equation of (3.2), $\overline{h}^{\lambda_2+\beta_1+\mu_i} = \overline{h}^{\alpha_i} \in C_{\langle \overline{h} \rangle}(\overline{b}_i) = \langle \overline{h^t} \rangle$. Hence $\overline{h}^{\lambda_2+\beta_1} = \overline{h}^{\alpha_i-\mu_i} \in \langle \overline{h}^s \rangle \langle \overline{h}^t \rangle$, a contradiction. Thus $\overline{w}\overline{h}^{\lambda_2+\beta_1}\overline{b}_i \notin \langle \overline{h} \rangle \overline{w}\overline{b}_i \langle \overline{h} \rangle$, and hence $\overline{c}_1\overline{d}_1 \cdots \overline{c}_i\overline{d}_i \notin \langle \overline{h} \rangle \overline{a}_1\overline{b}_1 \cdots \overline{a}_i\overline{b}_i \langle \overline{h} \rangle$. Therefore, $\overline{y} \notin \langle \overline{h} \rangle \overline{x} \langle \overline{h} \rangle$. Since $\overline{G} = A/M *_{\langle \overline{h} \rangle} B/N$ is residually finite, we can find $P \lhd_f G$ such that $y \notin P\langle h \rangle x \langle h \rangle$. □

Lemma 3.6. *Let G be a group and N be a finite normal subgroup of G. Let S be a subset of G. Suppose that G is S-separable. Then the group G/N is S/N-separable, where $S/N = \{sN \mid s \in S\}$.*

Proof. Let $xN \notin S/N$. Then $xn \notin S$ for every $n \in N$. Therefore, since G is S-separable, there exists $M_n \lhd_f G$ such that $xn \notin M_n S$. Let $M = \cap_{n \in N} M_n$. Then $M \lhd_f G$ and $xn \notin MS$ for all $n \in N$. Thus $xN \notin (MN/N)(S/N)$ and $MN/N \lhd_f G/N$. □

Definition 3.7. Let G be a group and $\langle h \rangle \times D$ be a subgroup of G with $|h| = \infty$. Then we say that G satisfies (C) for $\langle h \rangle \times D$ if there exists an integer $\delta > 0$ such that for each $n > 0$, there exists $M \lhd_f G$ depending on n such that $D \subset M$, $M \cap \langle h \rangle = \langle h^{n\delta} \rangle$ and if $Mh^i \sim_{G/M} Mh^j$, then $Mh^i = Mh^j$.

It is easy to see that if G satisfies (C) for $\langle h \rangle \times D$, then G satisfies (C') for $\langle h \rangle$.

Corollary 3.8. *Let $G = A *_H B$ with $H = \langle h \rangle \times D$, where $|h| = \infty$ and $D \subset Z(A) \cap Z(B)$ is finite. Suppose A, B are $\langle h \rangle$-self-conjugate, $\langle h^n \rangle$-double coset separable for each $n > 0$. Let A, B satisfy the condition (C) for $\langle h \rangle \times D$. Then G is H-double coset separable.*

Proof. Let $y, x \in G$ and $y \notin HxH$. Let $\overline{G} = G/D$. Then $\overline{G} = \overline{A} *_{\langle \overline{h} \rangle} \overline{B}$, where $\overline{A} = A/D$ and $\overline{B} = B/D$. Since $D \subset H$, $\overline{y} \notin \overline{H}\overline{x}\overline{H} = \langle \overline{h} \rangle \overline{x} \langle \overline{h} \rangle$ in $\overline{G}$. By Lemma 3.6, $\overline{A}, \overline{B}$ are $\langle \overline{h}^n \rangle$-double coset separable for each $n > 0$. To show that $\overline{A}$ is $\langle \overline{h} \rangle$-self-conjugate, suppose $\overline{h}^i \sim_{\overline{A}} \overline{h}^j$. Then $h^i \sim_A h^j d$ for some $d \in D$. Let $|d| = \alpha$. Then $h^{i\alpha} \sim_A h^{j\alpha} d^\alpha = h^{j\alpha}$. Since A is $\langle h \rangle$-self-conjugate, we have $i\alpha = j\alpha$, that is, $i = j$. Hence $\overline{A}$ is $\langle \overline{h} \rangle$-self-conjugate. Similarly, B is also $\langle \overline{h} \rangle$-self-conjugate. By Theorem 3.5, $\overline{G} = \overline{A} *_{\langle \overline{h} \rangle} \overline{B}$ is $\langle \overline{h} \rangle \overline{x} \langle \overline{h} \rangle$-separable for all $\overline{x} \in \overline{G}$. There exists $\overline{P} \lhd_f \overline{G}$ such that $\overline{y} \notin \overline{P} \langle \overline{h} \rangle \overline{x} \langle \overline{h} \rangle = \overline{PH}\overline{x}\overline{H}$. Let P be the preimage of $\overline{P}$. Then $P \lhd_f G$ and $y \notin PHxH$. Hence G is HxH-separable for all $x \in G$, that is, G is H-double coset separable. □

4. Criterion

Theorem 4.1. *Let $G = A *_H B$ with $H = \langle h \rangle \times D$, where $|h| = \infty$ and D is a finite normal subgroup of both A and B. Suppose there exists an integer $\delta_1 > 0$ such that for each $n > 0$, there exists $M \lhd_f A$ such that $D \subset M$, $M \cap \langle h \rangle = \langle h^{n\delta_1} \rangle$. Similarly, suppose there exists an integer $\delta_2 > 0$ such that for each $n > 0$, there exists $N \lhd_f B$ such that $D \subset N$, $N \cap \langle h \rangle = \langle h^{n\delta_2} \rangle$. If A, B and $\tilde{G} = G/D$ are residually finite, then G is residually finite.*

Proof. Let $1 \neq x \in G$. We shall find $P \lhd_f G$ such that $x \notin P$.

(1) Let $x \notin D$. Then $1 \neq \tilde{x} \in \tilde{G} = G/D$. Since $\tilde{G}$ is residually finite, there exists $\tilde{P} \lhd_f \tilde{G}$ such that $\tilde{x} \notin \tilde{P}$. Let P be the preimage of $\tilde{P}$. Then $P \lhd_f G$ and $x \notin P$.

(2) Let $x \in D$. Since A, B are residually finite, there exist $M_1 \lhd_f A$ and $N_1 \lhd_f B$ such that $M_1 \cap D = 1 = N_1 \cap D$. Let $M_1 \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $N_1 \cap \langle h \rangle = \langle h^{s_2} \rangle$ for some s_1, s_2. By assumption, there exists $M_2 \lhd_f A$ such that $M_2 \cap \langle h \rangle = \langle h^{s_1 s_2 \delta_1 \delta_2} \rangle$. Similarly, there exists $N_2 \lhd_f B$ such that $N_2 \cap \langle h \rangle = \langle h^{s_1 s_2 \delta_1 \delta_2} \rangle$. Let $M = M_1 \cap M_2$ and $N = N_1 \cap N_2$. Clearly, $M \lhd_f A$, $N \lhd_f B$ and $M \cap H = \langle h^{s_1 s_2 \delta_1 \delta_2} \rangle = N \cap H$. Let $\overline{G} = A/M *_{\overline{H}} B/N$, where $\overline{H} = \langle \overline{h} \rangle \times \overline{D}$. Then $\overline{G}$ is residually finite. Since $1 \neq \overline{x} \in \overline{D}$, there exists $\overline{P} \lhd_f \overline{G}$ such that $\overline{x} \notin \overline{P}$. Let P be the preimage of $\overline{P}$. Then $P \lhd_f G$ and $x \notin P$. □

In the above result, we note that the same conclusion can be drawn by assuming that A, B are H-separable instead of the residual finiteness of $\tilde{G} = G/D$.

Lemma 4.2. *Let A be a group and D be a finite subgroup of $Z(A)$. Suppose A is $\langle h \rangle$-self-conjugate. Let $a \in A$ and $a^{-1} h^{-i} a h^i \notin D$ for all $i \neq 0$. Then for* *each positive integer m, $h^{-i_1}ah^{j_1}d \notin \langle h^m\rangle a\langle h^m\rangle$ for all $0 \le i_1, j_1 < m$ and for all $d \in D$ except $i_1 = j_1 = 0$ and $d = 1$.*

Proof. Let $0 \le i_1, j_1 < m$ and $d \in D$ such that $h^{-i_1}ah^{j_1}d = h^{k_1 m}ah^{k_2 m}$ for some k_1, k_2. Then $a^{-1}h^{i_1+k_1 m}a = h^{j_1-k_2 m}d$. Let $|d| = \alpha$. Then

$$a^{-1}h^{(i_1+k_1 m)\alpha}a = h^{(j_1-k_2 m)\alpha}d^\alpha = h^{(j_1-k_2 m)\alpha}.$$

Since A is $\langle h\rangle$-self-conjugate, $(i_1 + k_1 m)\alpha = (j_1 - k_2 m)\alpha$, and hence $i_1 + k_1 m = j_1 - k_2 m$. Thus $a^{-1}h^{i_1+k_1 m}a = h^{j_1-k_2 m}d = h^{i_1+k_1 m}d$. By assumption, $i_1 + k_1 m = 0$. This implies that $i_1 = j_1 = 0$ and $d = 1$. □

Lemma 4.3. *Let $G = A *_H B$ with $H = \langle h\rangle \times D$, where $|h| = \infty$ and $D \subset Z(A) \cap Z(B)$ is finite. Let A, B be $\langle h\rangle$-self-conjugate and $\langle h^n\rangle$-double coset separable for each $n > 0$ and let A, B satisfy (C) above. Let $x = a_1 b_1 \cdots a_n b_n$, where $a_i \in A\backslash H$, $b_i \in B\backslash H$. If $x \not\sim_H xu$, where $u \in D$, then there exist $M \lhd_f A$ and $N \lhd_f B$ such that $M \cap H = N \cap H$ and in $\overline{G} = A/M *_{\overline{H}} B/N$, $\|\overline{x}\| = \|x\|$ and $\overline{x} \not\sim_{\overline{H}} \overline{xu}$.*

Proof. **Case 1.** Suppose there is a positive integer s such that $x^{-1}h^{-s}xh^s \in D$. We can assume that s is the least among such integers and $x^{-1}h^{-s}xh^s = c \in D$. Then $h^{-ks}xh^{ks} = xc^k$ for integer k. Since $x \not\sim_H xu$, we have $u \notin \langle c\rangle$. By the minimality of s, we have $x^{-1}h^{-i_1}xh^{i_1} \notin D$ for all $1 \le i_1 \le s-1$. Note that G is residually finite (Theorem 4.1). Hence there exists $P \lhd_f G$ such that $x^{-1}h^{-i_1}xh^{i_1} \notin PD$ for all $1 \le i_1 \le s-1$ and $u \notin P\langle c\rangle$ (D is finite, so $\langle c\rangle$ is finite). Let $M = P \cap A$ and $N = P \cap B$. Then $M \cap H = N \cap H$. In $\overline{G} = A/M *_{\overline{H}} B/N$, we shall prove that $\overline{x} \not\sim_{\overline{H}} \overline{xu}$.

Suppose $\overline{x} \sim_{\overline{H}} \overline{xu}$. Then $\overline{xu} = \overline{h}^{-i}\overline{x}\overline{h}^i$ for some i. Let $i = ks + i_1$, where $0 \le i_1 < s$. Then $\overline{xu} = \overline{h}^{-i}\overline{x}\overline{h}^i = \overline{h}^{-i_1}\overline{h}^{-ks}\overline{x}\overline{h}^{ks}\overline{h}^{i_1} = \overline{h}^{-i_1}\overline{xc^k}\overline{h}^{i_1}$. Hence $\overline{x}^{-1}\overline{h}^{-i_1}\overline{x}\overline{h}^{i_1} = \overline{uc^{-k}} \in \overline{D}$. By the choice of P, $i_1 = 0$. Then $\overline{u} = \overline{c}^k \in \langle\overline{c}\rangle$, a contradiction. Therefore, we have $\overline{x} \not\sim_{\overline{H}} \overline{xu}$.

Case 2. Suppose there is no positive integer s such that $x^{-1}h^{-s}xh^s \in D$.

In this case, there exists either a_r such that $a_r^{-1}h^{-i}a_r h^i \notin D$ for all $i \neq 0$ or b_r such that $b_r^{-1}h^{-i}b_r h^i \notin D$ for all $i \neq 0$. Note that $x \sim_H xu$ if and only if $x^* \sim_H x^*u$ for any cyclic permutation x^* of x. Hence, without loss of generality, we may assume $a_1^{-1}h^{-i}a_1 h^i \notin D$ for all $i \neq 0$.

Suppose $a_\ell^{-1}h^{-s}a_\ell h^s \in D$ for some integer $s \neq 0$, say $h^{-s}a_\ell h^s = a_\ell w_\ell$ for some $w_\ell \in D$. Since D is finite, $h^{-sk}a_\ell h^{sk} = a_\ell w_\ell^k = a_\ell$ for some k. Thus if $a_\ell^{-1}h^{-s}a_\ell h^s \in D$ for some $s \neq 0$, then there exists smallest positive integer s_ℓ such that $h^{-s_\ell}a_\ell h^{s_\ell} = a_\ell$. Similarly, if $b_\ell^{-1}h^{-s}b_\ell h^s \in D$ for some integer $s \neq 0$, then there exists smallest positive integer t_ℓ such that $h^{-t_\ell}b_\ell h^{t_\ell} = b_\ell$. Let

(4.1) $\quad I = \{\ell \mid a_\ell = h^{-s_\ell}a_\ell h^{s_\ell}$ for some minimal $s_\ell > 0\}$ and

(4.2) $\quad J = \{\ell \mid b_\ell = h^{-t_\ell}b_\ell h^{t_\ell}$ for some minimal $t_\ell > 0\}$.

Of course, I, J can be empty. But $1 \notin I$ by the assumption in the beginning of Case 2. Let $s = \prod_{i\in I} s_i$ and $t = \prod_{i\in J} t_i$, where $s = 1$ if $I = \emptyset$ and $t = 1$ if $J = \emptyset$. Let $m = st$.

Since A is $\langle h\rangle$-separable and D is finite, A is H-separable. Similarly, B is H-separable. There exists $M_1 \lhd_f A$ such that $M_1 \cap D = 1$ and $a_\lambda \notin M_1 H$ for all $1 \le \lambda \le n$. Thus $u \notin M_1$. Similarly, there exists $N_1 \lhd_f B$ such that $N_1 \cap D = 1$ and $b_\lambda \notin N_1 H$ for all $1 \le \lambda \le n$.

For those a_r where $r \notin I$, $a_r^{-1} h^{-i} a_r h^i \notin D$ for all $i \neq 0$. Thus by Lemma 4.2, $h^{-i_1} a_r h^{j_1} d \notin \langle h^m\rangle a_r \langle h^m\rangle$ for all $0 \le i_1, j_1 < m$ and for all $d \in D$ except $i_1 = j_1 = 0$ and $d = 1$. Since A is $\langle h^m\rangle$-double coset separable, there exists $M_2 \lhd_f A$ such that for all possible $r \notin I$, $h^{-i_1} a_r h^{j_1} d \notin M_2 \langle h^m\rangle a_r \langle h^m\rangle$ for all $0 \le i_1, j_1 < m$ and for all $d \in D$ except $i_1 = j_1 = 0$ and $d = 1$. Note that $D \cap M_2 = 1$. Then in $\tilde{A} = A/M_2$, if $\tilde{a}_r = \tilde{h}^i \tilde{a}_r \tilde{h}^j \tilde{d}$ for $d \in D$, then $m \mid i, j$ and $d = 1$. For those b_r where $r \notin J$, we have $h^{-i} b_r^{-1} h^i b_r \notin D$ for all $i \neq 0$. Then, as before, there exists $N_2 \lhd_f B$ such that for all possible $r \notin J$, if $\tilde{b}_r = \tilde{h}^i \tilde{b}_r \tilde{h}^j \tilde{d}$ where $\tilde{B} = B/N_2$ and $d \in D$, then $m \mid i, j$ and $d = 1$.

Let $M_1 \cap M_2 \cap \langle h\rangle = \langle h^{s_1}\rangle$ and $N_1 \cap N_2 \cap \langle h\rangle = \langle h^{s_2}\rangle$. By (C), there exist $M_3 \lhd_f A$ and $N_3 \lhd_f B$ such that $M_3 \cap H = \langle h^{s_1 s_2 \delta_1 \delta_2}\rangle \times D = N_3 \cap H$. Let $M = M_1 \cap M_2 \cap M_3$ and $N = N_1 \cap N_2 \cap N_3$. In $\overline{G} = A/M *_{\overline{H}} B/N$, where $\overline{H} = \langle \overline{h}\rangle \times \overline{D}$, we shall prove that $\overline{x} \not\sim_{\overline{H}} \overline{xu}$.

Suppose $\overline{x} \sim_{\overline{H}} \overline{xu}$. There exist α_i, μ_i and $c_i, d_i \in D$ such that

$$
\begin{aligned}
\overline{a}_1 &= \overline{h}^{-\alpha_1} \overline{d}_1^{-1} \overline{a}_1 \overline{h}^{\mu_1} \overline{c}_1 \\
\overline{b}_1 &= \overline{h}^{-\mu_1} \overline{c}_1^{-1} \overline{b}_1 \overline{h}^{\alpha_2} \overline{d}_2 \\
\overline{a}_2 &= \overline{h}^{-\alpha_2} \overline{d}_2^{-1} \overline{a}_2 \overline{h}^{\mu_2} \overline{c}_2 \\
&\vdots \\
\overline{b}_n \overline{u} &= \overline{h}^{-\mu_n} \overline{c}_n^{-1} \overline{b}_n \overline{h}^{\alpha_1} \overline{d}_1 .
\end{aligned}
\tag{4.3}
$$

From the first equation in (4.3), we have $\overline{a}_1 = \overline{h}^{-\alpha_1} \overline{a}_1 \overline{h}^{\mu_1} \overline{d}_1^{-1} \overline{c}_1$. Since $1 \notin I$, by the choice of M_2 we have $m \mid -\alpha_1, \mu_1$ and $d_1^{-1} c_1 = 1$. If $1 \notin J$ again, we can easily see that $c_1^{-1} d_2 = 1$. So suppose $1 \in J$, that is, there exists a minimal positive integer t_1 such that $b_1 = h^{-t_1} b_1 h^{t_1}$. Thus $h^{-t_1 k} b_1 h^{t_1 k} = b_1$ for all integer k. From the second equation of (4.3), we have $\overline{b}_1 = \overline{h}^{-\mu_1} \overline{b}_1 \overline{h}^{\mu_1} \overline{h}^{-\mu_1+\alpha_2} \overline{a}_1^{-1} \overline{d}_2$. Since $m \mid \mu_1$, let $\mu_1 = mc = (\prod_{i\in I} s_i)(\prod_{i\in J} t_i)c$. Then $\overline{b}_1 = \overline{b}_1 \overline{h}^{-\mu_1+\alpha_2} \overline{a}_1^{-1} \overline{d}_2$. Hence $\overline{h}^{-\mu_1+\alpha_2} \overline{a}_1^{-1} \overline{d}_2 = 1$. Since $\langle \overline{h}\rangle \cap \overline{D} = 1$, we have $\overline{h}^{\mu_1} = \overline{h}^{\alpha_2}$ and $\overline{a}_1^{-1} \overline{d}_2 = 1$.

By a similar way, we can see that all $\overline{d}_i^{-1} \overline{c}_i = 1$ and $\overline{c}_i^{-1} \overline{d}_{i+1} = 1$. From the last equation in (4.3), we have $\overline{b}_n = \overline{h}^{-\mu_n} \overline{b}_n \overline{h}^{\alpha_1} \overline{c}_n^{-1} \overline{d}_1 \overline{u}^{-1}$. As before, we can prove that $\overline{c}_n^{-1} \overline{d}_1 \overline{u}^{-1} = 1$. Then

$$\overline{u}^{-1} = \overline{d}_1^{-1} \overline{c}_1 \cdot \overline{c}_1^{-1} \overline{d}_2 \cdots \overline{d}_i^{-1} \overline{c}_i \cdot \overline{c}_i^{-1} \overline{d}_{i+1} \cdots \overline{c}_n^{-1} \overline{d}_1 \overline{u}^{-1} = 1,$$

a contradiction by the choice of M_1. This proves that $\overline{x} \not\sim_{\overline{H}} \overline{xu}$. $\square$

Definition 4.4. *Let G be a group and let H be a subgroup of G. We say G is H-conjugacy separable if for each $x \in G$ such that $\{x\}^G \cap H = \emptyset$, there exists $N \lhd_f G$ such that in $\overline{G} = G/N$, $\{\overline{x}\}^{\overline{G}} \cap \overline{H} = \emptyset$.*

Dyer [4] noted the importance of the above notion in the study of conjugacy separability of generalized free products, that is, if G has a subgroup H and is not H-conjugacy separable, then $G *_H G$ is not conjugacy separable.

Now we are ready to prove our main result.

Theorem 4.5. *Let $G = A *_H B$ with $H = \langle h \rangle \times D$, where $|h| = \infty$ and $D \subset Z(A) \cap Z(B)$ is finite, such that $\tilde{G} = G/D$ is conjugacy separable. Let A, B be conjugacy separable, $\langle h \rangle$-self-conjugate, and $\langle h^n \rangle$-double coset separable for each $n > 0$ and H-conjugacy separable. If A, B satisfy (C), then G is conjugacy separable.*

Proof. Let $x, y \in G$ be elements of minimal lengths in their conjugate classes and $x \not\sim_G y$. Since $\tilde{G} = G/D = \tilde{A} *_{\langle \tilde{h} \rangle} \tilde{B}$ is conjugacy separable, if $\tilde{x} \not\sim_{\tilde{G}} \tilde{y}$, then we can find $\tilde{P} \lhd_f \tilde{G}$ such that $\tilde{P}\tilde{x} \not\sim_{\tilde{G}/\tilde{P}} \tilde{P}\tilde{y}$. Let P be the preimage of $\tilde{P}$. Then $P \lhd_f G$ and $Px \not\sim_{G/P} Py$. Hence we assume that $\tilde{x} \sim_{\tilde{G}} \tilde{y}$. Then $y \sim_G xu$ for some $u \in D$. Hence we can take $y = xu$ and $x \not\sim_G xu$. We shall find $M \lhd_f A$ and $N \lhd_f B$ such that in $\overline{G} = A/M *_{\overline{H}} B/N$, $\overline{x} \not\sim_{\overline{G}} \overline{xu}$. Since $\overline{G}$ is conjugacy separable, there exists $\overline{P} \lhd_f \overline{G}$ such that $\overline{P}\overline{x} \not\sim_{\overline{G}/\overline{P}} \overline{P}\overline{xu}$. Let P be the preimage of $\overline{P}$. Then $P \lhd_f G$ and $Px \not\sim_{G/P} Py$. Hence G is conjugacy separable.

Case 1. $x \in H$. Without loss of generality, we assume $x = h^\alpha$. Then $h^\alpha \not\sim_A h^\alpha u$ and $h^\alpha \not\sim_B h^\alpha u$. Since A is conjugacy separable and D is finite, there exists $M_1 \lhd_f A$ such that $M_1 \cap D = 1$ and $M_1 h^\alpha \not\sim M_1 h^\alpha d$ for all $d \in D$ such that $h^\alpha \not\sim_A h^\alpha d$. Similarly, there exists $N_1 \lhd_f B$ such that $N_1 \cap D = 1$ and $N_1 h^\alpha \not\sim N_1 h^\alpha d$ for all $d \in D$ such that $h^\alpha \not\sim_B h^\alpha d$. Let $M_1 \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $N_1 \cap \langle h \rangle = \langle h^{s_2} \rangle$ for some integers s_1, s_2. By (C), for $n = s_1 s_2 \delta_2$, there exists $M_2 \lhd_f A$ such that $D \subset M_2$, $M_2 \cap \langle h \rangle = \langle h^{s_1 s_2 \delta_1 \delta_2} \rangle$ and if $M_2 h^i \sim_{A/M_2} M_2 h^j$, then $M_2 h^i = M_2 h^j$. Similarly, for $n = s_1 s_2 \delta_1$, there exists $N_2 \lhd_f B$ such that $D \subset N_2$, $N_2 \cap \langle h \rangle = \langle h^{s_1 s_2 \delta_1 \delta_2} \rangle$ and if $N_2 h^i \sim_{B/N_2} N_2 h^j$, then $N_2 h^i = N_2 h^j$. Let $M = M_1 \cap M_2$ and $N = N_1 \cap N_2$. Clearly, $M \cap H = \langle h^{s_1 s_2 \delta_1 \delta_2} \rangle = N \cap H$. In $\overline{G} = A/M *_{\overline{H}} B/N$, where $\overline{H} = \langle \overline{h} \rangle \times \overline{D}$, we show that $\overline{h}^\alpha \not\sim_{\overline{G}} \overline{h}^\alpha \overline{u}$.

Suppose $\overline{h}^\alpha \sim_{\overline{G}} \overline{h}^\alpha \overline{u}$. Then by Theorem 2.2, there exist integers c_i and $d_i \in D$ such that

$$\overline{h}^\alpha \sim_{\overline{A}} \overline{h}^{c_1} \overline{d}_1 \sim_{\overline{B}} \overline{h}^{c_2} \overline{d}_2 \sim_{\overline{A}} \cdots \sim_{\overline{B}(\overline{A})} \overline{h}^{c_n} \overline{d}_n \sim_{\overline{A}(\overline{B})} \overline{h}^\alpha \overline{u}. \tag{4.4}$$

From the first conjugate relation $\overline{h}^\alpha \sim_{\overline{A}} \overline{h}^{c_1} \overline{d}_1$, we have $M_2 h^\alpha \sim M_2 h^{c_1}$. Hence, by the choice of M_2, we have $M_2 h^\alpha = M_2 h^{c_1}$. Thus $h^{\alpha - c_1} \in M_2 \cap \langle h \rangle = M \cap \langle h \rangle$, whence $\overline{h}^\alpha = \overline{h}^{c_1}$. It follows that $\overline{h}^\alpha \sim_{\overline{A}} \overline{h}^\alpha \overline{d}_1$. Hence $M_1 h^\alpha \sim M_1 h^\alpha d_1$. By the choice of M_1, we have $h^\alpha \sim_A h^\alpha d_1$. From the second conjugate relation of (4.4), we have $\overline{h}^{\alpha} = \overline{h}^{c_1} \sim_{\overline{B}} \overline{h}^{c_2}\overline{d}_2\overline{d}_1^{-1}$. Then, as before, $\overline{h}^{\alpha} = \overline{h}^{c_2}$ and $h^{\alpha} \sim_B h^{\alpha}d_2d_1^{-1}$. Hence $h^{\alpha}d_1 \sim_B h^{\alpha}d_2$. Similarly, from (4.4), we have $\overline{h}^{\alpha} = \overline{h}^{c_n}$ and $h^{\alpha}d_n \sim_{A(B)} h^{\alpha}u$. Thus we have

$$h^{\alpha} \sim_A h^{\alpha}d_1 \sim_B \cdots \sim_{B(A)} h^{\alpha}d_n \sim_{A(B)} h^{\alpha}u.$$

It follows that $h^{\alpha} \sim_G h^{\alpha}u$, a contradiction. Therefore, $\overline{h}^{\alpha} \not\sim_{\overline{G}} \overline{h}^{\alpha}\overline{u}$.

Case 2. $x \in A\backslash H$ (or $x \in B\backslash H$). Let $x \in A$ and x have the minimal length 1 in its conjugacy class in G. Let $x \not\sim_G xu$, where $u \in D$. Since x has the minimal length 1 in its conjugacy class, $x \not\sim_A h^i d$ for all i and $d \in D$. Hence, by H-conjugacy separability, there exists $M_1 \lhd_f A$ such that $M_1x \not\sim_{A/M_1} M_1h^id$ for all i and all $d \in D$. Since A is conjugacy separable, there exists $M_2 \lhd_f A$ such that $M_2 \cap D = 1$ and $M_2x \not\sim_{A/M_2} M_2xd$ for all $d \in D$ such that $x \not\sim_A xd$. Similarly, since B is conjugacy separable, there exists $N_1 \lhd_f B$ such that $N_2 \cap D = 1$. Let $M_1 \cap M_2 \cap \langle h\rangle = \langle h^{s_1}\rangle$ and $N_1 \cap \langle h\rangle = \langle h^{s_2}\rangle$ for some s_1, s_2. By (C), there exist $M_3 \lhd_f A$ and $N_2 \lhd_f B$ such that $D \subset M_3$, $D \subset N_2$ and $M_3 \cap \langle h\rangle = \langle h^{s_1s_2\delta_1\delta_2}\rangle = N_2 \cap \langle h\rangle$.

Let $M = M_1 \cap M_2 \cap M_3$ and $N = N_1 \cap N_2$. Then $M \cap H = \langle h^{s_1s_2\delta_1\delta_2}\rangle = N \cap H$. In $\overline{G} = A/M *_{\overline{H}} B/N$, where $\overline{H} = \langle\overline{h}\rangle \times \overline{D}$, we shall prove that $\overline{x} \not\sim_{\overline{G}} \overline{xu}$.

Suppose $\overline{x} \sim_{\overline{G}} \overline{xu}$. By the choice of M_1, $\overline{x}$ is not conjugate to any element in $\overline{H}$. Hence $\overline{x}$ has the minimal length 1 in its conjugacy class in $\overline{G}$. It follows from Theorem 2.2 that $\overline{x} \sim_{\overline{A}} \overline{xu}$. Then by choice of M_2, $x \sim_A xu$, a contradiction. Therefore, $\overline{x} \not\sim_{\overline{G}} \overline{xu}$.

Case 3. $x = a_1b_1 \cdots a_nb_n$ (or $x = b_1a_1 \cdots b_na_n$), where $a_i \in A\backslash H$ and $b_i \in B\backslash H$. Let $x \not\sim_G xu$, where $u \in D$. Let $x_i = a_ib_i \cdots a_nb_na_1b_1 \cdots a_{i-1}b_{i-1}$ for $1 \le i \le n$. Clearly, $x = x_1$. By Theorem 2.2, $x_i \not\sim_H xu$ for all $1 \le i \le n$. We shall find $M_i \lhd_f A$ and $N_i \lhd_f B$ such that $M_i \cap H = N_i \cap H$ and in $\overline{G} = A/M_i *_{\overline{H}} B/N_i$, $\|\overline{x}_i\| = 2n = \|x\|$ and $\overline{x}_i \not\sim_{\overline{H}} \overline{xu}$ for each $1 \le i \le n$. The case of $i = 1$ is done by Lemma 4.3. Thus let $M_1 \lhd_f A$ and $N_1 \lhd_f B$ such that $M_1 \cap H = N_1 \cap H$ and in $\overline{G} = A/M_1 *_{\overline{H}} B/N_1$, $\|\overline{x}_1\| = 2n = \|x\|$ and $\overline{x}_1 \not\sim_{\overline{H}} \overline{xu}$. We shall consider the case when $i > 1$.

Suppose $x_i \notin HxuH$. Then $x_i \notin HxH$. By Corollary 3.8, there exists $Q \lhd_f G$ such that $x_i \notin QHxH$, $a_i \notin QH$ and $b_i \notin QH$ for all $1 \le i \le n$. Let $M_i = Q \cap A$ and $N_i = Q \cap B$. Then in $\overline{G} = A/M_i *_{\overline{H}} B/N_i$, $\|\overline{x}_i\| = 2n = \|x\|$ and $\overline{x}_i \notin \overline{H}\overline{x}\overline{H} = \overline{H}\overline{xu}\overline{H}$. Hence $\overline{x}_i \not\sim_{\overline{H}} \overline{xu}$ as required.

Suppose $x_i \in HxuH$, say $x_i = h_1xh_2$ for $h_1, h_2 \in H$. Clearly, $x_i \sim_H xh_2h_1$. Hence $xh_2h_1 \not\sim_H xu$. Let $h_2h_1 = h^su_1$, where $u_1 \in D$.

(a) $s = 0$. Then we have $xu_1 \not\sim_H xu$, or equivalently, $x \not\sim_H xuu_1^{-1}$, where $uu_1^{-1} \in D$. By Lemma 4.3, there exist $M_i \lhd_f A$ and $N_i \lhd_f B$ such that $M_i \cap H = N_i \cap H$ and in $\overline{G} = A/M_i *_{\overline{H}} B/N_i$, $\|\overline{x}\| = 2n = \|x\|$ and $\overline{x} \not\sim_{\overline{H}} \overline{xuu_1}^{-1}$. Then $\overline{x}_i \sim_{\overline{H}} \overline{xu}_1 \not\sim_{\overline{H}} \overline{xu}$, as required.

(b) $s \ne 0$. By Theorem 4.1 and Corollary 3.8, there exists $Q \lhd_f G$ such that $h^s \notin Q$, $a_i \notin QH$ and $b_i \notin QH$ for all $1 \le i \le n$. Let $P \cap \langle h\rangle = \langle h^{s_1}\rangle$. By (C), there exists $M \lhd_f A$ such that $D \subset M$, $M \cap \langle h\rangle = \langle h^{ss_1\delta_1\delta_2}\rangle$ and if $Mh^i \sim_{A/M}$

Mh^j, then $Mh^i = Mh^j$. Similarly, there exists $N \lhd_f B$ such that $D \subset N$, $N \cap \langle h \rangle = \langle h^{ss_1\delta_1\delta_2} \rangle$ and if $Nh^i \sim_{B/N} Nh^j$, then $Nh^i = Nh^j$. Let $M_i = Q \cap M$, $N_i = Q \cap N$ and $\overline{G} = A/M_i *_{\overline{H}} B/N_i$. Clearly, $M_i \cap \langle h \rangle = \langle h^{ss_1\delta_1\delta_2} \rangle = N_i \cap \langle h \rangle$ and $\|\overline{x}_i\| = 2n = \|x\|$. We shall show that $\overline{x}_i \not\sim_{\overline{H}} \overline{xu}$. Suppose $\overline{x}_i \sim_{\overline{H}} \overline{xu}$. Then $\overline{x}\overline{h}^s\overline{u}_1 \sim_{\overline{H}} \overline{xu}$. Hence there exist α_i, μ_i and $d_i, c_i \in D$ such that

$$
\begin{aligned}
\overline{a}_1 &= \overline{h}^{-\alpha_1}\overline{d}_1^{-1}\overline{a}_1\overline{h}^{\mu_1}\overline{c}_1 \\
\overline{b}_1 &= \overline{h}^{-\mu_1}\overline{c}_1^{-1}\overline{b}_1\overline{h}^{\alpha_2}\overline{d}_2 \\
\overline{a}_2 &= \overline{h}^{-\alpha_2}\overline{d}_2^{-1}\overline{a}_2\overline{h}^{\mu_2}\overline{c}_2 \\
&\vdots \\
\overline{b}_n\overline{h}^s\overline{u}_1 &= \overline{h}^{-\mu_n}\overline{c}_n^{-1}\overline{b}_n\overline{u}\overline{h}^{\alpha_1}\overline{d}_1.
\end{aligned} \tag{4.5}
$$

From the first equation in (4.5), we have $Ma_1 = Mh^{-\alpha_1}a_1h^{\mu_1}$, that is, $Mh^{\alpha_1} \sim Mh^{\mu_1}$. By the choice of M, we have $Mh^{\alpha_1} = Mh^{\mu_1}$. Thus $h^{\mu-\alpha_1} \in M \cap \langle h \rangle \subset P$. Therefore, $\overline{h}^{\alpha_1} = \overline{h}^{\mu_1}$. Similarly, from the second equation of (4.3), we have $\overline{h}^{\mu_1} = \overline{h}^{\alpha_2}$. Moreover, we have $\overline{h}^{\alpha_2} = \overline{h}^{\mu_2}$, $\overline{h}^{\mu_2} = \overline{h}^{\alpha_3}$, $\ldots$, $\overline{h}^{\alpha_n} = \overline{h}^{\mu_n}$, and $\overline{h}^{\mu_n} = \overline{h}^{\alpha_1 - s}$. Therefore, $\overline{h}^{\alpha_1} = \overline{h}^{\alpha_1 - s}$, which implies $\overline{h}^s = 1$, a contradiction. Hence $\overline{x}_i \not\sim_{\overline{H}} \overline{xu}$ as required. □

5. Generalized free products of nilpotent groups

In this section we apply the main result to finitely generated nilpotent groups.

Lemma 5.1. *Let A be finitely generated nilpotent and $h \in A$ with $|h| = \infty$. Then there exists a positive integer δ such that for each $n > 0$, there exists $M \lhd_f A$ such that $M \cap \langle h \rangle = \langle h^{n\delta} \rangle$ and if $Mh^i \sim_{A/M} Mh^j$, then $Mh^i = Mh^j$.*

Proof. Since A is finitely generated nilpotent, there exists an integer $i \geq 0$ such that $Z_i(A) \cap \langle h \rangle = 1$ and $Z_{i+1}(A) \cap \langle h \rangle = \langle h^\delta \rangle$ for some $\delta > 0$. In $\hat{A} = A/Z_i(A)$, $|\hat{h}| = \infty$ and $\hat{h}^\delta \in Z(\hat{A})$. For a given integer $n > 0$, consider $\overline{A} = \hat{A}/\langle \hat{h}^{n\delta} \rangle$. Clearly, $|\overline{h}| = n\delta$. If $\overline{h}^i \sim_{\overline{A}} \overline{h}^j$, then $\hat{h}^j = \hat{g}^{-1}\hat{h}^i\hat{g}\hat{h}^{kn\delta} = \hat{g}^{-1}\hat{h}^{i+kn\delta}\hat{g}$ for some $g \in A$ and for some integer k. Since $\hat{A}$ is finitely generated nilpotent and $|\hat{h}| = \infty$, we have $j = i + kn\delta$. Thus $\overline{h}^i = \overline{h}^j$. This shows that if $\overline{h}^i \sim_{\overline{A}} \overline{h}^j$, then $\overline{h}^i = \overline{h}^j$. Since $\overline{A}$ is finitely generated nilpotent, $\overline{A}$ is conjugacy separable. There exists $\overline{M} \lhd_f \overline{A}$ such that $\overline{Mh}^i \not\sim_{\overline{A}/\overline{M}} \overline{Mh}^j$ for all $\overline{h}^i \not\sim_{\overline{A}} \overline{h}^j$ (there are only finitely many). Considering $j = 0$, we have $1 \neq \overline{h}^i \notin \overline{M}$. Let M be the preimage of $\overline{M}$. Then $M \lhd_f A$, $M \cap \langle h \rangle = \langle h^{n\delta} \rangle$, and if $Mh^i \sim_{A/M} Mh^j$, then $Mh^i = Mh^j$. □

The above result proves that finitely nilpotent groups satisfy (C') in Theorem 3.5. The next shows that those nilpotent groups satisfy (C) in Definition 3.7.

Lemma 5.2. *Let A be finitely generated nilpotent and $D \subset Z(A)$. Let $h \in A$ with $|h| = \infty$ and $D \cap \langle h \rangle = 1$. Then there exists a positive integer δ such that for each $n > 0$, there exists $M \lhd_f A$ such that $D \subset M$, $M \cap \langle h \rangle = \langle h^{n\delta} \rangle$, and if $Mh^i \sim_{A/M} Mh^j$, then $Mh^i = Mh^j$.*

Proof. Since $\overline{A} = A/D$ is finitely generated nilpotent and $|\overline{h}| = \infty$, by Lemma 5.1, there exists a positive integer δ such that for each $n > 0$, there exists $\overline{M} \lhd_f \overline{A}$ such that $\overline{M} \cap \langle \overline{h} \rangle = \langle \overline{h}^{n\delta} \rangle$ and if $\overline{M}\overline{h}^i \sim_{\overline{A}/\overline{M}} \overline{M}\overline{h}^j$, then $\overline{M}\overline{h}^i = \overline{M}\overline{h}^j$. Let M be the preimage of $\overline{M}$. Then $M \lhd_f A$, $D \subset M$, $M \cap \langle h \rangle = \langle h^{n\delta} \rangle$, and if $Mh^i \sim_{A/M} Mh^j$, then $Mh^i = Mh^j$. □

Theorem 5.3. *Let $G = A *_H B$ with $H = \langle h \rangle \times D$, where $|h| = \infty$ and $D \subset Z(A) \cap Z(B)$ is finite. If A, B are finitely generated nilpotent, then G is conjugacy separable.*

Proof. Finitely generated nilpotent groups are conjugacy separable [6, 15] and cyclic conjugacy separable [4]. Hence A, B are conjugacy separable and $\tilde{A} = A/D, \tilde{B} = B/D$ are $\langle \tilde{h} \rangle$-conjugacy separable. Thus A, B are H-conjugacy separable. Moreover, they are double coset separable [9], and hence A, B are $\langle h^n \rangle$-double coset separable for each $n > 0$. Since G/D is a generalized free product of finitely generated nilpotent groups amalgamating a cyclic subgroup, G/D is conjugacy separable [4]. By Lemma 5.2, (C) holds for A, B. Since finitely generated nilpotent groups are $\langle h \rangle$-self-conjugate, G is conjugacy separable by Theorem 4.5. □

Theorem 5.4. *Let $G = A *_H B$ with $H = \langle h \rangle \times D$, where $D \subset Z(A) \cap Z(B)$. If A, B are finitely generated nilpotent, then G is conjugacy separable.*

Proof. Let $x, y \in G$ be elements of minimal lengths in their conjugate classes and $x \not\sim_G y$. Since $\tilde{G} = G/D = \tilde{A} *_{\langle \tilde{h} \rangle} \tilde{B}$ is conjugacy separable by [4], if $\tilde{x} \not\sim_{\tilde{G}} \tilde{y}$, then we can find $\tilde{P} \lhd_f \tilde{G}$ such that $\tilde{P}\tilde{x} \not\sim_{\tilde{G}/\tilde{P}} \tilde{P}\tilde{y}$. Let P be the preimage of $\tilde{P}$. Then $P \lhd_f G$ and $Px \not\sim_{G/P} Py$. Hence we assume that $\tilde{x} \sim_{\tilde{G}} \tilde{y}$. Then $y \sim_G xu$ for some $u \in D$. Hence we can take $y = xu$ and $x \not\sim_G xu$, where $u \in D$. Let $D_1 = \{c \in D \mid x \sim_G xc\}$. Then D_1 is a subgroup of D. Clearly, $u \notin D_1$. Since D is finitely generated abelian, there exists $D_2 \lhd_f D$ such that $u \notin D_2 D_1$. Let $\overline{G} = G/D_2$. Then $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$, where $\overline{H} = \langle \overline{h} \rangle \times \overline{D}$. It is easy to see that $\overline{x} \not\sim_{\overline{G}} \overline{xu}$. Clearly, $|h| = |\overline{h}|$. Since $\overline{D}$ is finite, if $\langle \overline{h} \rangle$ is finite, then $\overline{H}$ is finite and $\overline{G}$ is conjugacy separable by [4]. On the other hand, if $\langle \overline{h} \rangle$ is infinite, then $\overline{G}$ is conjugacy separable by Theorem 5.3. Thus there exists $\overline{P} \lhd_f \overline{G}$ such that $\overline{P}\overline{x} \not\sim_{\overline{G}/\overline{P}} \overline{P}\overline{xu}$. Let P be the preimage of $\overline{P}$. Then $P \lhd_f G$ and $Px \not\sim_{G/P} Py$. Hence G is conjugacy separable. □

Note that free groups satisfy most of the conditions in Theorem 4.5 except that they only have trivial centers. Thus we can apply Theorem 4.5 to show conjugacy separability of the following example:

Example 5.5. Let $G = G_1 *_H G_2$, where $G_i = F_i \times S_i$ $(i = 1, 2)$, with F_i free and S_i finitely generated nilpotent groups and $H = \langle h \rangle \times D$, where $\langle h \rangle = F_1 \cap F_2$ and $D \subset Z(S_1) \cap Z(S_2)$. Then G is conjugacy separable.

Proof. It is well-known that free groups F_i are conjugacy separable, $\langle h \rangle$-self-conjugate, and $\langle h^n \rangle$-double coset separable for each $n > 0$. Hence $G_i = F_i \times S_i$ are also conjugacy separable, $\langle h \rangle$-self-conjugate, and $\langle h^n \rangle$-double coset separable for each $n > 0$. It is also well-known that free groups F_i satisfy (C') in Theorem 3.5. Hence G_i satisfy (C). Note that $G_i/D \cong F_i \times \overline{S}_i$, where $\overline{S}_i = S_i/D$ is finitely generated nilpotent. Every finitely generated nilpotent group is a finite extension of a finitely generated torsion-free nilpotent group. Hence each $\overline{S}_i$ is a finite extension of a finitely generated torsion-free nilpotent group $\overline{T}_i$. Then G_i/D is isomorphic to a finite extension of $F_i \times \overline{T}_i$. We note that the group $F_i \times \overline{T}_i$ is a residually finitely generated torsion-free nilpotent group. Hence $G/D = G_1/D *_{\langle \overline{h} \rangle} G_2/D$ is conjugacy separable [7]. Since free groups are cyclic conjugacy separable, $G_i/D \cong F_i \times \overline{S}_i$ are $\langle h \rangle$-conjugacy separable. Hence each G_i is H-conjugacy separable. By Theorem 4.5, G is conjugacy separable when D is finite. Then as in the proof of Theorem 5.4, we can show that G is conjugacy separable when D is infinite. □

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