Honam Mathematical J. **35** (2013), No. 2, pp. 251–302
http://dx.doi.org/10.5831/HMJ.2013.35.2.251

# ON THE CONVOLUTION SUMS OF CERTAIN RESTRICTED DIVISOR FUNCTIONS

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**Abstract.** We study convolution sums of certain restricted divisor functions in detail and present explicit evaluations in terms of usual divisor functions for some specific situations.

## 1. Introduction

For $a, b, n \in \mathbb{N}$, let us define the convolution sum

$$S_{a,b}(n) := \sum_{m=1}^{n-1} \sigma_a(m)\sigma_b(n-m).$$

Ramanujan showed that the sum $S_{a,b}(n)$ can be evaluated in terms of $\sigma_{a+b+1}(n)$, $\sigma_{a+b-1}(n)$, ..., $\sigma_3(n)$, $\sigma_1(n)$ for the nine pairs $(a, b) \in \mathbb{N}^2$ satisfying $a + b = 2, 4, 6, 8, 12$, $a \le b$, $a \equiv b \equiv 1 \pmod 2$. For example, explicitly, we know (see [23]) that

$$S_{1,11}(n) = \frac{691}{65520}\sigma_{13}(n) + \left(\frac{1}{24} - \frac{1}{24}n\right)\sigma_{11}(n) - \frac{691}{65520}\sigma_1(n) \tag{1}$$

and (see [12, p. 35])

$$S_{3,9}(n) = \frac{1}{2640}\sigma_{13}(n) - \frac{1}{240}\sigma_9(n) + \frac{1}{264}\sigma_3(n). \tag{2}$$

From [16], we note that for any integer $n \ge 3$, we have

Received March 28, 2013. Accepted May 4, 2013.
2010 Mathematics Subject Classification. Primary 33E05, 11A25, 11B65, Secondary 11Y35, 11Y70.
Key words and phrases. Divisor functions, restricted divisor functions, convolution sums.
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$$\sum_{\substack{(m_1,m_2,m_3)\in\mathbb{N}^3 \\ m_1+m_2+m_3=n}} m_1 m_2 \sigma_1(m_1)\sigma_1(m_2)\sigma_1(m_3) = \frac{1}{288}\left(n^2\sigma_5(n) + (n^2-4n^3)\sigma_3(n) - (n^3-3n^4)\sigma_1(n)\right). \tag{3}$$

For an elementary proof of (1) and (2), we refer to [27]. An another interesting arithmetical identity (which was stated by Ramanujan) see [23, p. 146], for some analytical proofs of this identity, one may refer to [2, p. 329], [3, p. 136] and [21], also [27] is ; for $n \in \mathbb{N}$, we have

$$\sum_{k=0}^{n-1} \sigma_1(2k+1)\sigma_3(n-k) = \frac{1}{240}\sigma_5(2n+1) - \frac{1}{240}\sigma_3(2n+1). \tag{4}$$

In [19, Theorem 2], G. Melfi established seven identities of the type

$$\sum_{k=0}^{\left[\frac{n}{m}\right]} \sigma_r(k)\sigma_s(n-mk) = A\sigma_{r+s+1}(n) + Bn\sigma_{r+s-1}(n) \tag{5}$$

using the theory of modular forms. The (5) holds for every $n$ satisfying some suitable congruences for integers $m \geq 2$ and $r$, $s = 1$ or 3. The coefficients $A$ and $B$ are rational numbers. Convolution sums and theory of modular forms have lot of interconnections in number theory (for example see [18], [19]).

We define the four restricted divisor functions :

$$\sigma_{s,ee}(n) := \sum_{\substack{d|n \\ d \text{ even} \\ \frac{n}{d} \text{ even}}} d^s, \qquad \sigma_{s,oe}(n) := \sum_{\substack{d|n \\ d \text{ odd} \\ \frac{n}{d} \text{ even}}} d^s,$$

$$\sigma_{s,eo}(n) := \sum_{\substack{d|n \\ d \text{ even} \\ \frac{n}{d} \text{ odd}}} d^s, \qquad \sigma_{s,oo}(n) := \sum_{\substack{d|n \\ d \text{ odd} \\ \frac{n}{d} \text{ odd}}} d^s.$$

The aim of this article is to study the possible ten *simple convolution sums* of the type :

$$\sum_{k=1}^{N-1} \sigma_{s,ee}(k)\sigma_{s',ee}(N-k)$$

and its cognated *all possible convolution sums with arguments being multiple of a prime power,* namely sums of the type :

$$\sum_{k=1}^{N-1} \sigma_{s,ee}(p^m k)\sigma_{s',ee}(p^n(N-k)).$$

We also present some explicit evaluations for some specific situations for each of the above convolution sums.

## 2. Preliminaries

We have :

| Convolution sum | Convolution formulas | Reference |
|---|---|---|
| $\sum_{m<n/2} \sigma_1(m)\sigma_1(n-2m)$ | $\frac{1}{24}\{2\sigma_3(n)+(1-3n)\sigma_1(n)+8\sigma_3(\frac{n}{2})$ $+(1-6n)\sigma_1(\frac{n}{2})\}$ | [13, (4.4)] |
| $\sum_{m=1}^{n-1} \sigma_1(m)\sigma_1(n-m)$ | $\frac{1}{12}\{5\sigma_3(n)+(1-6n)\sigma_1(n)\}$ | [13, (3.10)] |
| $\sum_{m<n/2} \sigma_1(m)\sigma_3(n-2m)$ | $\frac{1}{240}\{5\sigma_5(n)+(10-15n)\sigma_3(n)+16\sigma_5(\frac{n}{2})$ $-\sigma_1(\frac{n}{2})\}$ | [13, Theorem 6] |
| $\sum_{m<n/2} \sigma_3(m)\sigma_1(n-2m)$ | $\frac{1}{240}\{\sigma_5(n)-\sigma_1(n)+20\sigma_5(\frac{n}{2})$ $+(10-30n)\sigma_3(\frac{n}{2})\}$ | [13, Theorem 6] |
| $\sum_{m=1}^{n-1} \sigma_1(m)\sigma_3(n-m)$ | $\frac{1}{240}\{21\sigma_5(n)+(10-30n)\sigma_3(n)-\sigma_1(n)\}$ | [13, (3.12)] |
| $\sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m)$ | $\frac{1}{120}\{\sigma_7(n)-\sigma_3(n)\}$ | [13, (3.17)] |
| $\sum_{m=1}^{n-1} \sigma_1(m)\sigma_5(n-m)$ | $\frac{1}{504}\{20\sigma_7(n)+(21-42n)\sigma_5(n)+\sigma_1(n)\}$ | [13, (3.18)] |
| $\sum_{m=1}^{n-1} \sigma_3(m)\sigma_5(n-m)$ | $\frac{1}{5040}\{11\sigma_9(n)-21\sigma_5(n)+10\sigma_3(n)\}$ | [13, (3.27)] |
| $\sum_{m=1}^{n-1} \sigma_1(m)\sigma_7(n-m)$ | $\frac{1}{480}\{11\sigma_9(n)+(20-30n)\sigma_7(n)-\sigma_1(n)\}$ | [13, (3.28)] |
| $\sum_{m=1}^{n-1} \sigma_5(m)\sigma_7(n-m)$ | $\frac{1}{10080}\{\sigma_{13}(n)+20\sigma_7(n)-21\sigma_5(n)\}$ | [13, (3.29)] |
| $\sum_{m=1}^{n-1} \sigma_3(m)\sigma_9(n-m)$ | $\frac{1}{2640}\{\sigma_{13}(n)-11\sigma_9(n)+10\sigma_3(n)\}$ | [13, (3.30)] |
| $\sum_{m=1}^{n-1} \sigma_1(m)\sigma_{11}(n-m)$ | $\frac{1}{65520}\{691\sigma_{13}(n)+2730(1-n)\sigma_{11}(n)$ $-691\sigma_1(n)\}$ | [13, (3.31)] |
| $\sum_{m<n/4} \sigma(m)\sigma(n-4m)$ | $\frac{1}{48}\{\sigma_3(n)+(2n-3)\sigma(n)+3\sigma_3(\frac{n}{2})$ $+16\sigma_3(\frac{n}{4})+(2-12n)\sigma_1(\frac{n}{4})\}$ | [13, Theorem 4] |

TABLE 1. Convolution formulas

**Proposition 2.1.** ([27, p. 35]) *We have*

(a)
$$\sigma_{s,ee}(n) = 2^s\sigma_s\left(\frac{n}{4}\right).$$

(b)
$$\sigma_{s,oe}(n) = \sigma_s\left(\frac{n}{2}\right) - 2^s\sigma_s\left(\frac{n}{4}\right).$$

(c)
$$\sigma_{s,eo}(n) = 2^s\sigma_s\left(\frac{n}{2}\right) - 2^s\sigma_s\left(\frac{n}{4}\right).$$

(d)
$$\sigma_{s,oo}(n) = \sigma_s(n) - (2^s+1)\sigma_s\left(\frac{n}{2}\right) + 2^s\sigma_s\left(\frac{n}{4}\right).$$

## 3. Simple convolution sums of restricted divisor functions

**Theorem 3.1.** *We have*

$$\sum_{k=1}^{N-1} \sigma_{s,ee}(k)\sigma_{s',ee}(N-k) = 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right).$$

*Proof.* From the definition of $\sigma_{s,ee}(k)$, we obtain

$$\begin{aligned}
\sum_{k=1}^{N-1} \sigma_{s,ee}(k)\sigma_{s',ee}(N-k) &= \sum_{k=1}^{N-1} 2^s\sigma_s(\frac{k}{4}) \cdot 2^{s'}\sigma_{s'}(\frac{N-k}{4}) \\
&= 2^{s+s'} \sum_{k=1}^{N-1} \sigma_s(\frac{k}{4})\sigma_{s'}(\frac{N-k}{4}) \\
&= 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}(\frac{N}{4}-k).
\end{aligned}$$

Hence the theorem follows. □

**Example 3.2.** *We have*

(a) $$\sum_{k=1}^{N-1} \sigma_{1,ee}(k)\sigma_{1,ee}(N-k) = \frac{1}{6}\left\{10\sigma_3\left(\frac{N}{4}\right) - (3N-2)\sigma_1(\frac{N}{4})\right\}.$$

(b) $$\begin{aligned}&\sum_{k=1}^{N-1} \sigma_{1,ee}(k)\sigma_{3,ee}(N-k) \\ &= \frac{1}{30}\left\{42\sigma_5\left(\frac{N}{4}\right) - 5(3N-4)\sigma_3\left(\frac{N}{4}\right) - 2\sigma_1\left(\frac{N}{4}\right)\right\}.\end{aligned}$$

(c) $$\sum_{k=1}^{N-1} \sigma_{3,ee}(k)\sigma_{3,ee}(N-k) = \frac{8}{15}\left\{\sigma_7\left(\frac{N}{4}\right) - \sigma_3\left(\frac{N}{4}\right)\right\}.$$

(d) $$\begin{aligned}&\sum_{k=1}^{N-1} \sigma_{1,ee}(k)\sigma_{5,ee}(N-k) \\ &= \frac{4}{63}\left\{40\sigma_7\left(\frac{N}{4}\right) - 21(N-2)\sigma_5\left(\frac{N}{4}\right) + 2\sigma_1\left(\frac{N}{4}\right)\right\}.\end{aligned}$$

(e)
$$\sum_{k=1}^{N-1} \sigma_{3,ee}(k)\sigma_{5,ee}(N-k)$$
$$= \frac{16}{315}\left\{11\sigma_9\left(\frac{N}{4}\right) - 21\sigma_5\left(\frac{N}{4}\right) + 10\sigma_3\left(\frac{N}{4}\right)\right\}.$$

(f)
$$\sum_{k=1}^{N-1} \sigma_{1,ee}(k)\sigma_{7,ee}(N-k)$$
$$= \frac{1}{15}\left\{88\sigma_9\left(\frac{N}{4}\right) - 20(3N-8)\sigma_7\left(\frac{N}{4}\right) - 8\sigma_1\left(\frac{N}{4}\right)\right\}.$$

(g)
$$\sum_{k=1}^{N-1} \sigma_{5,ee}(k)\sigma_{7,ee}(N-k)$$
$$= \frac{128}{315}\left\{\sigma_{13}\left(\frac{N}{4}\right) + 20\sigma_7\left(\frac{N}{4}\right) - 21\sigma_5\left(\frac{N}{4}\right)\right\}.$$

(h)
$$\sum_{k=1}^{N-1} \sigma_{3,ee}(k)\sigma_{9,ee}(N-k)$$
$$= \frac{256}{165}\left\{\sigma_{13}\left(\frac{N}{4}\right) - 11\sigma_9\left(\frac{N}{4}\right) + 10\sigma_3\left(\frac{N}{4}\right)\right\}.$$

(i)
$$\sum_{k=1}^{N-1} \sigma_{1,ee}(k)\sigma_{11,ee}(N-k)$$
$$= \frac{128}{4095}\left\{1382\sigma_{13}\left(\frac{N}{4}\right) - 1365(N-4)\sigma_{11}\left(\frac{N}{4}\right) - 1382\sigma_1\left(\frac{N}{4}\right)\right\}.$$

(j)
$$\sum_{k=1}^{N-1} \sigma_{5,ee}(k)\sigma_{5,ee}(N-k)$$
$$= \frac{1024}{174132}\left\{65\sigma_{11}\left(\frac{N}{4}\right) + 691\sigma_5\left(\frac{N}{4}\right) - 756\tau\left(\frac{N}{4}\right)\right\},$$

*where*

$$\sum_{n=1}^{\infty} \tau(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

**Theorem 3.3.** *We have*

$$\sum_{k=1}^{N-1} \sigma_{s,ee}(k)\sigma_{s',oe}(N-k)$$
$$= 2^s \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right) - 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right).$$

*Proof.* We observe that

$$\sum_{k=1}^{N-1} \sigma_{s,ee}(k)\sigma_{s',oe}(N-k)$$
$$= \sum_{k=1}^{N-1} 2^s\sigma_s\left(\frac{k}{4}\right)\left\{\sigma_{s'}\left(\frac{N-k}{2}\right) - 2^{s'}\sigma_{s'}\left(\frac{N-k}{4}\right)\right\}$$
$$= 2^s \sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{4}\right)\sigma_{s'}\left(\frac{N-k}{2}\right) - 2^{s+s'}\sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{4}\right)\sigma_{s'}\left(\frac{N-k}{4}\right)$$
$$= 2^s \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right) - 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right).$$

This completes the proof. □

**Example 3.4.** *We have*

(a)
$$\sum_{k=1}^{N-1} \sigma_{1,ee}(k)\sigma_{1,oe}(N-k)$$
$$= \frac{1}{24}\left[4\left\{\sigma_3\left(\frac{N}{2}\right) - 6\sigma_3\left(\frac{N}{4}\right)\right\} - (3N-2)\sigma_1\left(\frac{N}{2}\right) + 6(N-1)\sigma_1\left(\frac{N}{4}\right)\right].$$

(b)
$$\sum_{k=1}^{N-1} \sigma_{1,ee}(k)\sigma_{3,oe}(N-k)$$
$$= \frac{1}{240}\left[10\sigma_5\left(\frac{N}{2}\right) - 304\sigma_5\left(\frac{N}{4}\right) - (15N-20)\sigma_3\left(\frac{N}{2}\right) + 40(3N-4)\sigma_3\left(\frac{N}{4}\right) + 14\sigma_1\left(\frac{N}{4}\right)\right].$$

(c)
$$\sum_{k=1}^{N-1} \sigma_{3,ee}(k)\sigma_{1,oe}(N-k)$$
$$= \frac{1}{30}\left\{\sigma_5\left(\frac{N}{2}\right) - 22\sigma_5\left(\frac{N}{4}\right) - 10\sigma_3\left(\frac{N}{4}\right) - \sigma_1\left(\frac{N}{2}\right) + 2\sigma_1\left(\frac{N}{4}\right)\right\}.$$

**Theorem 3.5.** *We have*
$$\sum_{k=1}^{N-1} \sigma_{s,ee}(k)\sigma_{s',eo}(N-k)$$
$$= 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right) - 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right).$$

*Proof.* We find that
$$\sum_{k=1}^{N-1} \sigma_{s,ee}(k)\sigma_{s',eo}(N-k)$$
$$= \sum_{k=1}^{N-1} 2^s\sigma_s\left(\frac{k}{4}\right)\left\{2^{s'}\sigma_{s'}\left(\frac{N-k}{2}\right) - 2^{s'}\sigma_{s'}\left(\frac{N-k}{4}\right)\right\}$$
$$= 2^{s+s'}\sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{4}\right)\sigma_{s'}\left(\frac{N-k}{2}\right) - 2^{s+s'}\sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{4}\right)\sigma_{s'}\left(\frac{N-k}{4}\right)$$
$$= 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right) - 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right).$$

Thus the theorem follows. □

Let $\sigma_s^*(n) := \sum_{\substack{d|n \\ \frac{n}{d} \text{ odd}}} d^s = \sigma_s(n) - \sigma_s(\frac{n}{2})$.

**Example 3.6.** *We have*

(a)
$$\sum_{k=1}^{N-1} \sigma_{1,ee}(k)\sigma_{1,eo}(N-k)$$
$$= \frac{1}{12}\left[4\sigma_3^*\left(\frac{N}{2}\right) - 3N\sigma_1\left(\frac{N}{2}\right) + 2\sigma_1^*\left(\frac{N}{2}\right)\right].$$

(b)
$$\sum_{k=1}^{N-1} \sigma_{1,ee}(k)\sigma_{3,eo}(N-k) = \frac{1}{6}\left[2\sigma_5^*\left(\frac{N}{2}\right) - (3N-4)\sigma_3^*\left(\frac{N}{2}\right)\right].$$

(c)
$$\sum_{k=1}^{N-1} \sigma_{3,ee}(k)\sigma_{1,eo}(N-k) = \frac{1}{30}\left[2\sigma_5^*\left(\frac{N}{2}\right) - 15N\sigma_3\left(\frac{N}{4}\right) - 2\sigma_1^*\left(\frac{N}{2}\right)\right].$$

**Theorem 3.7.** *We have*

$$\sum_{k=1}^{N-1} \sigma_{s,ee}(k)\sigma_{s',oo}(N-k) = 2^s \sum_{k<N/4} \sigma_s(k)\sigma_{s'}(N-4k) - 2^s\left(2^{s'}+1\right)\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right) + 2^{s+s'}\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right).$$

*Proof.* We notice that

$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{s,ee}(k)\sigma_{s',oo}(N-k)\\
&= \sum_{k=1}^{N-1} 2^s\sigma_s\left(\frac{k}{4}\right)\left\{\sigma_{s'}(N-k) - \left(2^{s'}+1\right)\sigma_{s'}\left(\frac{N-k}{2}\right) + 2^{s'}\sigma_{s'}\left(\frac{N-k}{4}\right)\right\}\\
&= 2^s \sum_{k<N/4} \sigma_s(k)\sigma_{s'}(N-4k) - 2^s\left(2^{s'}+1\right)\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right)\\
&\quad + 2^{s+s'}\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right).
\end{aligned}$$

Thus the theorem follows. □

**Example 3.8.** *We have*

$$\sum_{k=1}^{N-1} \sigma_{1,ee}(k)\sigma_{1,oo}(N-k)$$
$$= \frac{1}{24}\left\{\sigma_3(N) - 9\sigma_3\left(\frac{N}{2}\right) + 8\sigma_3\left(\frac{N}{4}\right) - (3N-2)\sigma_1(N) \right.$$
$$\left. +3(3N-2)\sigma_1\left(\frac{N}{2}\right) - 2(3N-2)\sigma_1\left(\frac{N}{4}\right)\right\}.$$

**Remark 3.9.** In [27, Theorem 3.1(ii)] we can see that

$$\sigma_k(pN) - \left(p^k+1\right)\sigma_k(N) + p^k\sigma_k\left(\frac{N}{p}\right) = 0$$

for a prime $p$ an $k, N \in \mathbb{N}$. Using this fact, for even $N$ we can deduce that

$$\sigma_3(N) - 9\sigma_3\left(\frac{N}{2}\right) + 8\sigma_3\left(\frac{N}{4}\right) = 0 \quad \text{and}$$
$$\sigma_1(N) - 3\sigma_1\left(\frac{N}{2}\right) + 2\sigma_1\left(\frac{N}{4}\right) = 0.$$

So from Example 3.8 we obtain

$$\sum_{k=1}^{N-1} \sigma_{1,ee}(k)\sigma_{1,oo}(N-k) = 0$$

for even $N$.

**Theorem 3.10.** *We have*

$$\sum_{k=1}^{N-1} \sigma_{s,oe}(k)\sigma_{s',oe}(N-k)$$
$$= \sum_{k<N/2} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-k\right) - 2^{s'}\sum_{k<N/4}\sigma_s\left(\frac{N}{2}-2k\right)\sigma_{s'}(k)$$
$$- 2^{s}\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right) + 2^{s+s'}\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right).$$

*Proof.* Since

$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{s,oe}(k)\sigma_{s',oe}(N-k)\\
&= \sum_{k=1}^{N-1} \left\{\sigma_s\left(\frac{k}{2}\right) - 2^s\sigma_s\left(\frac{k}{4}\right)\right\}\left\{\sigma_{s'}\left(\frac{N-k}{2}\right) - 2^{s'}\sigma_{s'}\left(\frac{N-k}{4}\right)\right\}\\
&= \sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{2}\right)\sigma_{s'}\left(\frac{N-k}{2}\right) - 2^{s'}\sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{2}\right)\sigma_{s'}\left(\frac{N-k}{4}\right)\\
&\qquad - 2^s\sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{4}\right)\sigma_{s'}\left(\frac{N-k}{2}\right) + 2^{s+s'}\sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{4}\right)\sigma_{s'}\left(\frac{N-k}{4}\right)\\
&= \sum_{k<N/2} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-k\right) - 2^{s'}\sum_{k<N/4} \sigma_s\left(\frac{N}{2}-2k\right)\sigma_{s'}(k)\\
&\qquad - 2^s\sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right) + 2^{s+s'}\sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right).
\end{aligned}$$

Hence the proof is complete. ☐

**Example 3.11.** *We have*

(a)
$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{1,oe}(k)\sigma_{1,oe}(N-k)\\
&= \frac{1}{12}\left[\sigma_3\left(\frac{N}{2}\right) + 4\sigma_3\left(\frac{N}{4}\right) - \sigma_1\left(\frac{N}{2}\right) + 2\sigma_1\left(\frac{N}{4}\right)\right].
\end{aligned}$$

(b)
$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{1,oe}(k)\sigma_{3,oe}(N-k)\\
&= \frac{1}{240}\left[3\left\{\sigma_5\left(\frac{N}{2}\right) + 48\sigma_5\left(\frac{N}{4}\right)\right\} - 10\left\{\sigma_3\left(\frac{N}{2}\right) - 8\sigma_3\left(\frac{N}{4}\right)\right\}\right.\\
&\qquad \left. +7\left\{\sigma_1\left(\frac{N}{2}\right) - 2\sigma_1\left(\frac{N}{4}\right)\right\}\right].
\end{aligned}$$

**Theorem 3.12.** *We have*

$$\sum_{k=1}^{N-1} \sigma_{s,oe}(k)\sigma_{s',eo}(N-k)$$
$$= 2^{s'} \sum_{k<N/2} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-k\right) - 2^{s'} \sum_{k<N/4} \sigma_s\left(\frac{N}{2}-2k\right)\sigma_{s'}(k)$$
$$- 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right) + 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right).$$

*Proof.* We note that

$$\sum_{k=1}^{N-1} \sigma_{s,oe}(k)\sigma_{s',eo}(N-k)$$
$$= \sum_{k=1}^{N-1} \left\{\sigma_s\left(\frac{k}{2}\right) - 2^s\sigma_s\left(\frac{k}{4}\right)\right\}\left\{2^{s'}\sigma_{s'}\left(\frac{N-k}{2}\right) - 2^{s'}\sigma_{s'}\left(\frac{N-k}{4}\right)\right\}$$
$$= 2^{s'}\sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{2}\right)\sigma_{s'}\left(\frac{N-k}{2}\right) - 2^{s'}\sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{2}\right)\sigma_{s'}\left(\frac{N-k}{4}\right)$$
$$- 2^{s+s'}\sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{4}\right)\sigma_{s'}\left(\frac{N-k}{2}\right) + 2^{s+s'}\sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{4}\right)\sigma_{s'}\left(\frac{N-k}{4}\right)$$
$$= 2^{s'} \sum_{k<N/2} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-k\right) - 2^{s'} \sum_{k<N/4} \sigma_s\left(\frac{N}{2}-2k\right)\sigma_{s'}(k)$$
$$- 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right) + 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right).$$

Thus the theorem follows. □

**Example 3.13.** *We have*

(a)
$$\sum_{k=1}^{N-1} \sigma_{1,oe}(k)\sigma_{1,eo}(N-k)$$
$$= \frac{1}{24}\left[8\sigma_3^*\left(\frac{N}{2}\right) - (3N+2)\,\sigma_1^*\left(\frac{N}{2}\right) + 3N\sigma_1\left(\frac{N}{4}\right)\right].$$

(b)
$$\sum_{k=1}^{N-1} \sigma_{1,oe}(k)\sigma_{3,eo}(N-k) = \frac{1}{3}\left\{\sigma_5^*\left(\frac{N}{2}\right) - \sigma_3^*\left(\frac{N}{2}\right)\right\}.$$

(c)
$$\sum_{k=1}^{N-1}\sigma_{3,oe}(k)\sigma_{1,eo}(N-k) = \frac{1}{240}\left[16\sigma_5^*\left(\frac{N}{2}\right) - 15N\left\{\sigma_3\left(\frac{N}{2}\right) - 8\sigma_3\left(\frac{N}{4}\right)\right\} + 14\sigma_1^*\left(\frac{N}{2}\right)\right].$$

**Theorem 3.14.** *We have*

$$\begin{aligned}
&\sum_{k=1}^{N-1}\sigma_{s,oe}(k)\sigma_{s',oo}(N-k)\\
&= \sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k) - \left(2^{s'}+1\right)\sum_{k<N/2}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-k\right)\\
&\quad + 2^{s'}\sum_{k<N/4}\sigma_s\left(\frac{N}{2}-2k\right)\sigma_{s'}(k) - 2^s\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(N-4k)\\
&\quad + 2^s\left(2^{s'}+1\right)\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right)\\
&\quad - 2^{s+s'}\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right).
\end{aligned}$$

*Proof.* We find that

$$\begin{aligned}
&\sum_{k=1}^{N-1}\sigma_{s,oe}(k)\sigma_{s',oo}(N-k)\\
&= \sum_{k=1}^{N-1}\left\{\sigma_s\left(\frac{k}{2}\right) - 2^s\sigma_s\left(\frac{k}{4}\right)\right\}\left\{\sigma_{s'}(N-k) - \left(2^{s'}+1\right)\sigma_{s'}\left(\frac{N-k}{2}\right)\right.\\
&\qquad \left. +2^{s'}\sigma_{s'}\left(\frac{N-k}{4}\right)\right\}\\
&= \sum_{k=1}^{N-1}\sigma_s\left(\frac{k}{2}\right)\sigma_{s'}(N-k) - \left(2^{s'}+1\right)\sum_{k=1}^{N-1}\sigma_s\left(\frac{k}{2}\right)\sigma_{s'}\left(\frac{N-k}{2}\right)\\
&\quad + 2^{s'}\sum_{k=1}^{N-1}\sigma_s\left(\frac{k}{2}\right)\sigma_{s'}\left(\frac{N-k}{4}\right) - 2^s\sum_{k=1}^{N-1}\sigma_s\left(\frac{k}{4}\right)\sigma_{s'}(N-k)\\
&\quad + 2^s\left(2^{s'}+1\right)\sum_{k=1}^{N-1}\sigma_s\left(\frac{k}{4}\right)\sigma_{s'}\left(\frac{N-k}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&- 2^{s+s'} \sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{4}\right) \sigma_{s'}\left(\frac{N-k}{4}\right) \\
= &\sum_{k<N/2} \sigma_s(k)\sigma_{s'}(N-2k) - \left(2^{s'}+1\right) \sum_{k<N/2} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-k\right) \\
&+ 2^{s'} \sum_{k<N/4} \sigma_s\left(\frac{N}{2}-2k\right)\sigma_{s'}(k) - 2^s \sum_{k<N/4} \sigma_s(k)\sigma_{s'}(N-4k) \\
&+ 2^s\left(2^{s'}+1\right) \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right) \\
&- 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right). \qquad \square
\end{aligned}
$$

**Example 3.15.** *We have*

$$
\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{1,oe}(k)\sigma_{1,oo}(N-k) \\
&= \frac{1}{24}\left\{\sigma_3(N) - 9\sigma_3\left(\frac{N}{2}\right) + 8\sigma_3\left(\frac{N}{4}\right) - \sigma_1(N) + 3\sigma_1\left(\frac{N}{2}\right)\right. \\
&\qquad \left. -2\sigma_1\left(\frac{N}{4}\right)\right\}.
\end{aligned}
$$

**Remark 3.16.** Similar manner to Remark 3.9, Example 3.15 shows that

$$\sum_{k=1}^{N-1} \sigma_{1,oe}(k)\sigma_{1,oo}(N-k) = 0$$

when $N$ is even.

**Theorem 3.17.** *We have*

$$
\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{s,eo}(k)\sigma_{s',eo}(N-k) \\
&= 2^{s+s'} \sum_{k<N/2} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-k\right) - 2^{s+s'} \sum_{k<N/4} \sigma_s\left(\frac{N}{2}-2k\right)\sigma_{s'}(k) \\
&\quad - 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right) + 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right).
\end{aligned}
$$

*Proof.* We observe that

$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{s,eo}(k)\sigma_{s',eo}(N-k) \\
&= \sum_{k=1}^{N-1} \left\{ 2^s \sigma_s\left(\frac{k}{2}\right) - 2^s \sigma_s\left(\frac{k}{4}\right) \right\} \left\{ 2^{s'} \sigma_{s'}\left(\frac{N-k}{2}\right) - 2^{s'} \sigma_{s'}\left(\frac{N-k}{4}\right) \right\} \\
&= 2^{s+s'} \sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{2}\right) \sigma_{s'}\left(\frac{N-k}{2}\right) - 2^{s+s'} \sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{2}\right) \sigma_{s'}\left(\frac{N-k}{4}\right) \\
&\quad - 2^{s+s'} \sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{4}\right) \sigma_{s'}\left(\frac{N-k}{2}\right) + 2^{s+s'} \sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{4}\right) \sigma_{s'}\left(\frac{N-k}{4}\right) \\
&= 2^{s+s'} \sum_{k<N/2} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-k\right) - 2^{s+s'} \sum_{k<N/4} \sigma_s\left(\frac{N}{2}-2k\right)\sigma_{s'}(k) \\
&\quad - 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right) + 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right).
\end{aligned}$$

Hence the proof is complete. □

**Example 3.18.** *We have*

(a) $$\sum_{k=1}^{N-1} \sigma_{1,eo}(k)\sigma_{1,eo}(N-k) = \frac{1}{2}\left[2\sigma_3^*\left(\frac{N}{2}\right) - N\sigma_1^*\left(\frac{N}{2}\right)\right].$$

(b) $$\sum_{k=1}^{N-1} \sigma_{1,eo}(k)\sigma_{3,eo}(N-k) = \frac{1}{2}\left[2\sigma_5^*\left(\frac{N}{2}\right) - N\sigma_3^*\left(\frac{N}{2}\right)\right].$$

**Theorem 3.19.** *We have*

$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{s,eo}(k)\sigma_{s',oo}(N-k) \\
&= 2^s \sum_{k<N/2} \sigma_s(k)\sigma_{s'}(N-2k) - 2^s\left(2^{s'}+1\right) \sum_{k<N/2} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-k\right) \\
&\quad + 2^{s+s'} \sum_{k<N/4} \sigma_s\left(\frac{N}{2}-2k\right)\sigma_{s'}(k) - 2^s \sum_{k<N/4} \sigma_s(k)\sigma_{s'}(N-4k) \\
&\quad + 2^s\left(2^{s'}+1\right) \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right) - 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right).
\end{aligned}$$

*Proof.* From the definition we obtain that

$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{s,eo}(k)\sigma_{s',oo}(N-k) \\
&= \sum_{k=1}^{N-1} \left\{ 2^s\sigma_s(\frac{k}{2}) - 2^s\sigma_s(\frac{k}{4}) \right\} \left\{ \sigma_{s'}(N-k) - \left(2^{s'}+1\right)\sigma_{s'}(\frac{N-k}{2}) \right. \\
&\qquad\qquad \left. +2^{s'}\sigma_{s'}(\frac{N-k}{4}) \right\} \\
&= 2^s \sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{2}\right)\sigma_{s'}(N-k) - 2^s\left(2^{s'}+1\right)\sum_{k=1}^{N-1}\sigma_s\left(\frac{k}{2}\right)\sigma_{s'}\left(\frac{N-k}{2}\right) \\
&\quad + 2^{s+s'}\sum_{k=1}^{N-1}\sigma_s\left(\frac{k}{2}\right)\sigma_{s'}\left(\frac{N-k}{4}\right) - 2^s\sum_{k=1}^{N-1}\sigma_s\left(\frac{k}{4}\right)\sigma_{s'}(N-k) \\
&\quad + 2^s\left(2^{s'}+1\right)\sum_{k=1}^{N-1}\sigma_s\left(\frac{k}{4}\right)\sigma_{s'}\left(\frac{N-k}{2}\right) \\
&\quad - 2^{s+s'}\sum_{k=1}^{N-1}\sigma_s\left(\frac{k}{4}\right)\sigma_{s'}\left(\frac{N-k}{4}\right) \\
&= 2^s\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k) - 2^s\left(2^{s'}+1\right)\sum_{k<N/2}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-k\right) \\
&\quad + 2^{s+s'}\sum_{k<N/4}\sigma_s\left(\frac{N}{2}-2k\right)\sigma_{s'}(k) - 2^s\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(N-4k) \\
&\quad + 2^s\left(2^{s'}+1\right)\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right) \\
&\quad - 2^{s+s'}\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right).
\end{aligned}$$

This completes the proof. □

**Example 3.20.** *We have*

$$\begin{aligned}
&\sum_{k=1}^{N-1}\sigma_{1,eo}(k)\sigma_{1,oo}(N-k) \\
&= \frac{1}{8}\left[\sigma_3(N) - 9\sigma_3\left(\frac{N}{2}\right) + 8\sigma_3\left(\frac{N}{4}\right) - N\left\{\sigma_1(N) - 3\sigma_1\left(\frac{N}{2}\right) + 2\sigma_1\left(\frac{N}{4}\right)\right\}\right].
\end{aligned}$$

**Remark 3.21.** Similar manner to Remark 3.9, Example 3.20 tells us that

$$\sum_{k=1}^{N-1} \sigma_{1,eo}(k)\sigma_{1,oo}(N-k) = 0$$

for even $N$.

**Theorem 3.22.** *we have*

$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{s,oo}(k)\sigma_{s',oo}(N-k) \\
&= \sum_{k<N} \sigma_s(k)\sigma_{s'}(N-k) - \left(2^{s'}+1\right) \sum_{k<N/2} \sigma_s(N-2k)\sigma_{s'}(k) \\
&\quad + 2^{s'} \sum_{k<N/4} \sigma_s(N-4k)\sigma_{s'}(k) - (2^s+1) \sum_{k<N/2} \sigma_s(k)\sigma_{s'}(N-2k) \\
&\quad + (2^s+1)\left(2^{s'}+1\right) \sum_{k<N/2} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-k\right) \\
&\quad - 2^{s'}(2^s+1) \sum_{k<N/4} \sigma_s\left(\frac{N}{2}-2k\right)\sigma_{s'}(k) \\
&\quad + 2^s \sum_{k<N/4} \sigma_s(k)\sigma_{s'}(N-4k) - 2^s\left(2^{s'}+1\right) \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right) \\
&\quad + 2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right).
\end{aligned}$$

*Proof.* It is obvious. □

**Example 3.23.** *We have*

$$\sum_{k=1}^{N-1} \sigma_{1,oo}(k)\sigma_{1,oo}(N-k) = \sigma_3^*\left(\frac{N}{2}\right).$$

## 4. Convolution sums with arguments being a multiple of a prime power

Let us find the general formula about the summation $\sum_{k=1}^{N-1} \sigma_{s,ee}(p^m k)$ $\sigma_{s',ee}(p^n(N-k))$ with prime $p$ and $m, n \in \mathbb{N} \cup \{0\}$ in the following theorem.

**Theorem 4.1.** *Let*

$$H_{p^m,p^n}(N) := \sum_{k=1}^{N-1} \sigma_{s,ee}\left(p^m k\right) \sigma_{s',ee}\left(p^n(N-k)\right).$$

*Then we obtain the Table 1.*

| | $m$ | $n$ | $H_{2^m,2^n}(N)$ |
|---|---|---|---|
| (a) | 0 | 0 | $2^{s+s'} \sum_{k<N/4} \sigma_s(\frac{N}{4}-k)\sigma_{s'}(k)$ |
| (b) | 0 | 1 | $2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}(\frac{N}{2}-2k)$ |
| (c) | 0 | $n \geq 2$ | $\frac{2^{s+s'}}{2^{s'}-1}\{(2^{s'(n-1)}-1)\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(N-4k)$ $+(2^{s'}-2^{s'(n-1)})\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-2k)\}$ |
| (d) | 1 | 0 | $2^{s+s'} \sum_{k<N/4} \sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)$ |
| (e) | 1 | 1 | $2^{s+s'} \sum_{k<N/2} \sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)$ |
| (f) | 1 | $n \geq 2$ | $\frac{2^{s+s'}}{2^{s'}-1}\{(2^{s'(n-1)}-1)\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)$ $+(2^{s'}-2^{s'(n-1)})\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)\}$ |
| (g) | $m \geq 2$ | 0 | $\frac{2^{s+s'}}{2^{s}-1}\{(2^{s(m-1)}-1)\sum_{k<N/4}\sigma_s(N-4k)\sigma_{s'}(k)$ $+(2^{s}-2^{s(m-1)})\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)\}$ |
| (h) | $m \geq 2$ | 1 | $\frac{2^{s+s'}}{2^{s}-1}\{(2^{s(m-1)}-1)\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)$ $+(2^{s}-2^{s(m-1)})\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)\}$ |
| (i) | $m \geq 2$ | $n \geq 2$ | $\frac{2^{s+s'}}{(2^s-1)(2^{s'}-1)}\{(2^{s(m-1)}-1)(2^{s'(n-1)}-1)\sum_{k=1}^{N-1}\sigma_s(k)\sigma_{s'}(N-k)$ $+(2^{s(m-1)}-1)(2^{s'}-2^{s'(n-1)})\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)$ $+(2^{s}-2^{s(m-1)})(2^{s'(n-1)}-1)\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)$ $+(2^{s}-2^{s(m-1)})(2^{s'}-2^{s'(n-1)})\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)\}$ |

TABLE 1. *Formulas for* $H_{2^m,2^n}(N)$

(j) *For* $n, m \in \mathbb{N} \cup \{0\}$ *with odd prime* $p$, *we have*

$$\sum_{k=1}^{N-1} \sigma_{s,ee}\left(p^m k\right) \sigma_{s',ee}\left(p^n(N-k)\right)$$

$$= \frac{2^{s+s'}}{(p^s-1)(p^{s'}-1)} \left\{ \left(p^{s(m+1)}-1\right)\left(p^{s'(n+1)}-1\right) \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right) \right.$$

$$+ \left(p^{s(m+1)}-1\right)\left(p^{s'}-p^{s'(n+1)}\right) \sum_{k<N/4p} \sigma_s\left(\frac{N}{4}-pk\right)\sigma_{s'}(k)$$

$$+ \left(p^s - p^{s(m+1)}\right)\left(p^{s'(n+1)} - 1\right) \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4} - pk\right)$$
$$+ \left(p^s - p^{s(m+1)}\right)\left(p^{s'} - p^{s'(n+1)}\right) \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4p} - k\right)\Bigg\}.$$

*Proof.* (i) For $m, n \geq 2$ we note that

$$\sum_{k=1}^{N-1} \sigma_{s,ee}\left(2^m k\right)\sigma_{s',ee}\left(2^n(N-k)\right)$$
$$= 2^{s+s'} \sum_{k=1}^{N-1} \sigma_s\left(2^{m-2}k\right)\sigma_{s'}\left(2^{n-2}(N-k)\right)$$
$$= 2^{s+s'} \sum_{k=1}^{N-1} \left\{ \frac{2^{s(m-1)} - 1}{2^s - 1}\sigma_s(k) + \frac{2^s - 2^{s(m-1)}}{2^s - 1}\sigma_s\left(\frac{k}{2}\right) \right\}$$
$$\times \left\{ \frac{2^{s'(n-1)} - 1}{2^{s'} - 1}\sigma_{s'}(N-k) + \frac{2^{s'} - 2^{s'(n-1)}}{2^{s'} - 1}\sigma_{s'}\left(\frac{N-k}{2}\right) \right\}.$$

We also observe that

$$\sum_{k=1}^{N-1} \sigma_s(k)\sigma_{s'}\left(\frac{N-k}{2}\right) = \sum_{k<N/2} \sigma_s(N-2k)\sigma_{s'}(k),$$
$$\sum_{k=1}^{N-1} \sigma_s\left(\frac{k}{2}\right)\sigma_{s'}(N-k) = \sum_{k<N/2} \sigma_s(k)\sigma_{s'}(N-2k),$$
$$\sum_{k=1}^{N-1} \sigma_s(\frac{k}{2})\sigma_{s'}\left(\frac{N-k}{2}\right) = \sum_{k<N/2} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2} - k\right).$$

Now (i) follows.

(j) For odd prime $p$, the sum in question can be written as

$$\sum_{k=1}^{N-1} \sigma_{s,ee}\left(p^m k\right)\sigma_{s',ee}\left(p^n(N-k)\right)$$
$$= 2^{s+s'} \sum_{k=1}^{N-1} \sigma_s\left(\frac{p^m k}{4}\right)\sigma_{s'}\left(\frac{p^n(N-k)}{4}\right)$$

$$= 2^{s+s'} \sum_{k=1}^{N-1} \left\{ \frac{p^{s(m+1)}-1}{p^s-1} \sigma_s\left(\frac{k}{4}\right) + \frac{p^s - p^{s(m+1)}}{p^s-1} \sigma_s\left(\frac{k}{4p}\right) \right\}$$
$$\times \left\{ \frac{p^{s'(n+1)}-1}{p^{s'}-1} \sigma_{s'}\left(\frac{N-k}{4}\right) + \frac{p^{s'} - p^{s'(n+1)}}{p^{s'}-1} \sigma_{s'}\left(\frac{N-k}{4p}\right) \right\}.$$

Finally we note that

$$\sum_{k=1}^{N-1} \sigma_s(\frac{k}{4})\sigma_{s'}(\frac{N-k}{4}) = \sum_{k<N/4} \sigma_s(k)\sigma_{s'}(\frac{N}{4}-k),$$
$$\sum_{k=1}^{N-1} \sigma_s(\frac{k}{4})\sigma_{s'}(\frac{N-k}{4p}) = \sum_{k<N/4p} \sigma_s(\frac{N}{4}-pk)\sigma_{s'}(k),$$
$$\sum_{k=1}^{N-1} \sigma_s(\frac{k}{4p})\sigma_{s'}(\frac{N-k}{4}) = \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}(\frac{N}{4}-pk),$$
$$\sum_{k=1}^{N-1} \sigma_s(\frac{k}{4p})\sigma_{s'}(\frac{N-k}{4p}) = \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}(\frac{N}{4p}-k).$$

Hence (j) follows. □

**Explicit evaluations on some specific situations :**

**Corollary 4.2.** *We have*
(1) *For* $m, n \geq 2$,

$$\sum_{k=1}^{N-1} \sigma_{1,ee}\left(2^m k\right) \sigma_{1,ee}\left(2^n (N-k)\right)$$
$$= \frac{1}{12}\Big[\left(3\cdot 2^{m+n} - 2^{n+2} - 2^{m+2} + 4\right)\sigma_3(N)$$
$$+ \left(16 + 2^{m+2} + 2^{n+2} - 3\cdot 2^{m+n}\right)\sigma_3\left(\frac{N}{2}\right)$$
$$+ \left\{2^n + 2^m - 4 + 3\left(2^m + 2^n - 2^{m+n}\right)N\right\}\sigma_1(N)$$
$$+ \left\{8 - 2^m - 2^n + 3\left(2^{m+n} - 2^{n+1} - 2^{m+1}\right)N\right\}\sigma_1\left(\frac{N}{2}\right)\Big].$$

(2) *For* $m, n \geq 2$ ,

$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{1,ee}\left(2^m k\right) \sigma_{3,ee}\left(2^n (N-k)\right) \\
&= \frac{1}{1680}\Big[\left(15 \cdot 2^{3n+m} - 5 \cdot 2^{3n+2} - 2^{m+6} + 48\right) \sigma_5(N) \\
&\quad - \left(15 \cdot 2^{3n+m} - 5 \cdot 2^{3n+2} - 2^{m+6} - 2304\right) \sigma_5\left(\frac{N}{2}\right) \\
&\quad - 5\left(2^n - 2\right)\left(2^{n+1} + 2^{2n} + 4\right)\left(3 \cdot 2^m N - 4\right) \sigma_3(N) \\
&\quad + 5\left(2^n - 4\right)\left(2^{n+2} + 2^{2n} + 16\right)\left(3 \cdot 2^m N - 4\right) \sigma_3\left(\frac{N}{2}\right) \\
&\quad -56\left(2^m - 2\right) \sigma_1(N) + 56\left(2^m - 4\right) \sigma_1\left(\frac{N}{2}\right)\Big].
\end{aligned}$$

(3) *For* $m, n \geq 2$,

$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{3,ee}\left(2^m k\right) \sigma_{1,ee}\left(2^n (N-k)\right) \\
&= \frac{1}{1680}\Big[\left(15 \cdot 2^{3m+n} - 5 \cdot 2^{3m+2} - 2^{n+6} + 48\right) \sigma_5(N) \\
&\quad - \left(15 \cdot 2^{3m+n} - 5 \cdot 2^{3m+2} - 2^{n+6} - 2304\right) \sigma_5\left(\frac{N}{2}\right) \\
&\quad - 5\left(2^m - 2\right)\left(2^{m+1} + 2^{2m} + 4\right)\left(3 \cdot 2^n N - 4\right) \sigma_3(N) \\
&\quad + 5\left(2^m - 4\right)\left(2^{m+2} + 2^{2m} + 16\right)\left(3 \cdot 2^n N - 4\right) \sigma_3\left(\frac{N}{2}\right) \\
&\quad -56\left(2^n - 2\right) \sigma_1(N) + 56\left(2^n - 4\right) \sigma_1\left(\frac{N}{2}\right)\Big].
\end{aligned}$$

(4)

$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{1,ee}(k)\sigma_{1,ee}(2(N-k)) = \sum_{k=1}^{N-1} \sigma_{1,ee}(2k)\sigma_{1,ee}(N-k) \\
&= \frac{1}{6}\left\{2\sigma_3\left(\frac{N}{2}\right) + 8\sigma_3\left(\frac{N}{4}\right) - \left(\frac{3N}{2} - 1\right)\sigma_1\left(\frac{N}{2}\right)\right. \\
&\qquad \left. - (3N-1)\,\sigma_1\left(\frac{N}{4}\right)\right\}.
\end{aligned}$$

(5)
$$\sum_{k=1}^{N-1} \sigma_{1,ee}(k)\sigma_{3,ee}(2(N-k)) = \sum_{k=1}^{N-1} \sigma_{3,ee}(2k)\sigma_{1,ee}(N-k)$$
$$= \frac{1}{30}\left\{10\sigma_5\left(\frac{N}{2}\right) + 32\sigma_5\left(\frac{N}{4}\right) - 5\,(3N-4)\,\sigma_3\left(\frac{N}{2}\right) - 2\sigma_1\left(\frac{N}{4}\right)\right\}.$$

(6)
$$\sum_{k=1}^{N-1} \sigma_{3,ee}(k)\sigma_{1,ee}(2(N-k)) = \sum_{k=1}^{N-1} \sigma_{1,ee}(2k)\sigma_{3,ee}(N-k)$$
$$= \frac{1}{15}\left\{\sigma_5\left(\frac{N}{2}\right) + 20\sigma_5\left(\frac{N}{4}\right) - 5\,(3N-2)\,\sigma_3\left(\frac{N}{4}\right) - \sigma_1\left(\frac{N}{2}\right)\right\}.$$

(7) *For $m = 0$ and $n \geq 2$,*
$$\sum_{k=1}^{N-1} \sigma_{1,ee}(k)\sigma_{1,ee}(2^n(N-k))$$
$$= \frac{1}{24}\left\{(2^n-2)\,\sigma_3(N) - (2^n-10)\,\sigma_3\left(\frac{N}{2}\right) + 32\sigma_3\left(\frac{N}{4}\right) - (2^n-2)\,(3N-2)\,\sigma_1(N) + (2^n-4)\,(3N-2)\,\sigma_1\left(\frac{N}{2}\right) - 2\,(3\cdot 2^n N-2)\,\sigma_1\left(\frac{N}{4}\right)\right\}.$$

(8)
$$\sum_{k=1}^{N-1} \sigma_{1,ee}(2k)\sigma_{1,ee}(2(N-k)) = \frac{1}{3}\left\{5\sigma_3\left(\frac{N}{2}\right) - (3N-1)\,\sigma_1\left(\frac{N}{2}\right)\right\}.$$

(9)
$$\sum_{k=1}^{N-1} \sigma_{1,ee}(2k)\sigma_{3,ee}(2(N-k)) = \frac{1}{15}\left\{21\sigma_5\left(\frac{N}{2}\right) - 5\,(3N-2)\,\sigma_3\left(\frac{N}{2}\right) - \sigma_1\left(\frac{N}{2}\right)\right\}.$$

$$\begin{aligned}(10)\qquad &\sum_{k=1}^{N-1} \sigma_{1,ee}(2k)\sigma_{5,ee}(2(N-k)) \\ &= \frac{8}{63}\left\{20\sigma_7\left(\frac{N}{2}\right) - 21\,(N-1)\,\sigma_5\left(\frac{N}{2}\right) + \sigma_1\left(\frac{N}{2}\right)\right\}.\end{aligned}$$

$$\begin{aligned}(11)\qquad &\sum_{k=1}^{N-1} \sigma_{1,ee}(2k)\sigma_{7,ee}(2(N-k)) \\ &= \frac{8}{15}\left\{11\sigma_9\left(\frac{N}{2}\right) - (3N-4)\,\sigma_7\left(\frac{N}{2}\right) - \sigma_1\left(\frac{N}{2}\right)\right\}.\end{aligned}$$

$$\begin{aligned}(12)\qquad &\sum_{k=1}^{N-1} \sigma_{1,ee}(2k)\sigma_{11,ee}(2(N-k)) \\ &= \frac{256}{4095}\left\{691\sigma_{13}\left(\frac{N}{2}\right) - 1365\,(N-2)\,\sigma_{11}\left(\frac{N}{2}\right) - 691\sigma_1\left(\frac{N}{2}\right)\right\}.\end{aligned}$$

$$(13)\qquad \sum_{k=1}^{N-1} \sigma_{3,ee}(2k)\sigma_{3,ee}(2(N-k)) = \frac{8}{15}\left\{\sigma_7\left(\frac{N}{2}\right) - \sigma_3\left(\frac{N}{2}\right)\right\}.$$

$$\begin{aligned}(14)\qquad &\sum_{k=1}^{N-1} \sigma_{3,ee}(2k)\sigma_{5,ee}(2(N-k)) \\ &= \frac{16}{315}\left\{11\sigma_9\left(\frac{N}{2}\right) - 21\sigma_5\left(\frac{N}{2}\right) + 10\sigma_3\left(\frac{N}{2}\right)\right\}.\end{aligned}$$

$$\begin{aligned}(15)\qquad &\sum_{k=1}^{N-1} \sigma_{5,ee}(2k)\sigma_{7,ee}(2(N-k)) \\ &= \frac{128}{315}\left\{\sigma_{13}\left(\frac{N}{2}\right) + 20\sigma_7\left(\frac{N}{2}\right) - 21\sigma_5\left(\frac{N}{2}\right)\right\}.\end{aligned}$$

$$\begin{aligned}(16)\qquad &\sum_{k=1}^{N-1} \sigma_{3,ee}(2k)\sigma_{9,ee}(2(N-k)) \\ &= \frac{256}{165}\left\{\sigma_{13}\left(\frac{N}{2}\right) - 11\sigma_9\left(\frac{N}{2}\right) + 10\sigma_3\left(\frac{N}{2}\right)\right\}.\end{aligned}$$

(17) *For* $m=1$ *and* $n\geq 2$,

$$\sum_{k=1}^{N-1}\sigma_{1,ee}(2k)\sigma_{1,ee}(2^n(N-k))$$
$$=\frac{1}{12}\left[2\left\{(2^n-2)\,\sigma_3(N)-(2^n-12)\,\sigma_3\left(\frac{N}{2}\right)\right\}\right.$$
$$\left.-\,(2^n-2)\,(3N-1)\,\sigma_1(N)-(2^n-6+12N)\,\sigma_1\left(\frac{N}{2}\right)\right].$$

(18) *For* $m=1$ *and* $n\geq 2$,

$$\sum_{k=1}^{N-1}\sigma_{1,ee}(2k)\sigma_{3,ee}\,(2^n(N-k))$$
$$=\frac{1}{840}\left\{5\,(8^n-8)\,\sigma_5(N)-(5\cdot 8^n-1216)\,\sigma_5\left(\frac{N}{2}\right)\right.$$
$$-5\,(8^n-8)\,(3N-2)\,\sigma_3(N)+5\,(8^n-64)\,(3N-2)\,\sigma_3\left(\frac{N}{2}\right)$$
$$\left.-56\sigma_1\left(\frac{N}{2}\right)\right\}.$$

(19) *For* $m=1$ *and* $n\geq 2$,

$$\sum_{k=1}^{N-1}\sigma_{3,ee}(2k)\sigma_{1,ee}(2^n(N-k))$$
$$=\frac{1}{30}\left\{(2^n-2)\,\sigma_5(N)-(2^n-44)\,\sigma_5\left(\frac{N}{2}\right)-5\,(3\cdot 2^n-4)\,\sigma_3\left(\frac{N}{2}\right)\right.$$
$$\left.-\,(2^n-2)\,\sigma_1(N)+(2^n-4)\,\sigma_1\left(\frac{N}{2}\right)\right\}.$$

(20) *For* $m\geq 2$ *and* $n=0$,

$$\sum_{k=1}^{N-1}\sigma_{1,ee}\,(2^m k)\,\sigma_{1,ee}(N-k)$$
$$=\frac{1}{24}\left\{(2^m-2)\,\sigma_3(N)-(2^m-10)\,\sigma_3\left(\frac{N}{2}\right)+32\sigma_3\left(\frac{N}{4}\right)\right.$$
$$-\,(2^m-2)\,(3N-2)\,\sigma_1(N)$$
$$\left.+\,(2^m-4)\,(3N-2)\,\sigma_1\left(\frac{N}{2}\right)-2\,(3\cdot 2^m N-2)\,\sigma_1\left(\frac{N}{4}\right)\right\}.$$

(21) *For $m \geq 2$ and $n = 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,ee}(2^m k)\sigma_{1,ee}(2(N-k))$$
$$= \frac{1}{12}\left[2\left\{\left(2^m-2\right)\sigma_3(N) - \left(2^m-12\right)\sigma_3\left(\frac{N}{2}\right)\right\}\right.$$
$$\left.-\left(2^m-2\right)(3N-1)\,\sigma_1(N) - \left(2^m-6+12N\right)\sigma_1\left(\frac{N}{2}\right)\right].$$

(22) *For $m \geq 2$ and $n = 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,ee}(2^m k)\sigma_{3,ee}(2(N-k))$$
$$= \frac{1}{30}\left\{\left(2^m-2\right)\sigma_5(N) - \left(2^m-44\right)\sigma_5\left(\frac{N}{2}\right) - 5\left(3\cdot 2^m N-4\right)\sigma_3\left(\frac{N}{2}\right)\right.$$
$$\left.-\left(2^m-2\right)\sigma_1(N) + \left(2^m-4\right)\sigma_1\left(\frac{N}{2}\right)\right\}.$$

(23) *For $m \geq 2$ and $n = 1$,*

$$\sum_{k=1}^{N-1} \sigma_{3,ee}\left(2^m k\right)\sigma_{1,ee}(2(N-k))$$
$$= \frac{1}{840}\left\{5\left(8^m-8\right)\sigma_5(N) - \left(5\cdot 8^m-1216\right)\sigma_5\left(\frac{N}{2}\right)\right.$$
$$-5\left(8^m-8\right)(3N-2)\,\sigma_3(N) + 5\left(8^m-64\right)(3N-2)\,\sigma_3\left(\frac{N}{2}\right)$$
$$\left.-56\sigma_1\left(\frac{N}{2}\right)\right\}.$$

(24)
$$\sum_{k=1}^{N-1} \sigma_{3,ee}\left(3^m k\right)\sigma_{1,ee}\left(3^n(N-k)\right)$$
$$= \frac{1}{24}\left[2\left(4\cdot 3^{n+m+2} - 3^{n+2} - 3^{m+2} + 2\right)\sigma_3\left(\frac{N}{4}\right)\right.$$
$$-18\left(4\cdot 3^{n+m} - 3^n - 3^m - 2\right)\sigma_3\left(\frac{N}{12}\right)$$
$$+\left\{2\left(3^{n+1}+3^{m+1}-2\right) - 3\left(2\cdot 3^{n+m+1} - 3^n - 3^m\right)N\right\}\sigma_1\left(\frac{N}{4}\right)$$
$$\left.+3\left\{-2\left(3^n+3^m-2\right) + 3\left(2\cdot 3^{n+m} - 3^n - 3^m\right)N\right\}\sigma_1\left(\frac{N}{12}\right)\right].$$

**Example 4.3.** *Let us consider the Theorem 4.1 where $p=5$, $s=1$, $s'=2$, $m=2$ and $n=3$. Then we obtain*

$$\sum_{k=1}^{N-1}\sigma_{1,ee}(5^2k)\sigma_{2,ee}(5^3(N-k))$$
$$=\frac{1}{12}\left\{\left(5^{2+1}-1\right)\left(5^{2(3+1)}-1\right)\sum_{k<N/4}\sigma_1(k)\sigma_2\left(\frac{N}{4}-k\right)\right.$$
$$+\left(5^{2+1}-1\right)\left(25-5^{2(3+1)}\right)\sum_{k<N/20}\sigma_1\left(\frac{N}{4}-5k\right)\sigma_2(k)$$
$$+\left(5-5^{2+1}\right)\left(5^{2(3+1)}-1\right)\sum_{k<N/20}\sigma_1(k)\sigma_2\left(\frac{N}{4}-5k\right)$$
$$\left.+\left(5-5^{2+1}\right)\left(25-5^{2(3+1)}\right)\sum_{k<N/20}\sigma_1(k)\sigma_1\left(\frac{N}{20}-k\right)\right\}.$$

*In Table 2, we represent some values of* $\sum_{k=1}^{N-1}\sigma_{1,ee}(5^2k)\sigma_{2,ee}(5^3(N-k))$.

| $N$ | 4 | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|
| $\sum_{k=1}^{N-1}\sigma_{1,ee}(5^2k)\sigma_{2,ee}(5^3(N-k))$ | 0 | 4036448 | 32291584 | 117056992 | 314842944 |

*TABLE 2. Some values of* $\sum_{k=1}^{N-1}\sigma_{1,ee}(5^2k)\sigma_{2,ee}(5^3(N-k))$.

**Theorem 4.4.** *Let*

$$I_{p^m,p^n}(N):=\sum_{k=1}^{N-1}\sigma_{s,ee}\left(p^mk\right)\sigma_{s',oe}\left(p^n(N-k)\right).$$

*Then we obtain the Table 3.*

| $m$ | $n$ | $I_{2^m,2^n}(N)$ |
|---|---|---|
| 0 | 0 | $2^s\{\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-2k)-2^{s'}\sum_{k<N/4}\sigma_s(\frac{N}{4}-k)\sigma_{s'}(k)\}$ |
| 0 | $n\geq 1$ | $2^s\{\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(N-4k)-2^{s'}\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-2k)\}$ |
| 1 | 0 | $2^s\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)-2^{s'}\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)\}$ |
| 1 | $n\geq 1$ | $2^s\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)-2^{s'}\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)\}$ |
| $m\geq 2$ | $n=0$ | $\frac{2^s}{2^s-1}\{(2^{s(m-1)}-1)\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)$ $+(2^s-2^{s(m-1)})\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)$ $-2^{s'}(2^{s(m-1)}-1)\sum_{k<N/4}\sigma_s(N-4k)\sigma_{s'}(k)$ $+2^{s'}(2^{s(m-1)}-2^s)\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)\}$ |
| $m\geq 2$ | $n\geq 1$ | $\frac{2^s}{2^s-1}\{(2^{s(m-1)}-1)\sum_{k=1}^{N-1}\sigma_s(k)\sigma_{s'}(N-k)$ $-2^{s'}(2^{s(m-1)}-1)\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)$ $+(2^s-2^{s(m-1)})\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)$ $-2^{s'}(2^s-2^{s(m-1)})\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)\}$ |

*TABLE 3. Formulas for $I_{2^m,2^n}(N)$*

*And for $n, m \in \mathbb{N}\cup\{0\}$ with odd prime $p$, we have*

$$\begin{aligned}
\sum_{k=1}^{N-1}\sigma_{s,ee}(p^m k)\sigma_{s',oe}(p^n(N-k)) &= \frac{2^s}{(p^s-1)(p^{s'}-1)} \\
&\times\Bigg[\left(p^{s(m+1)}-1\right)\left(p^{s'(n+1)}-1\right) \\
&\quad\times\left\{\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right)-2^{s'}\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right)\right\} \\
&\quad+\left(p^{s(m+1)}-1\right)\left(p^{s'}-p^{s'(n+1)}\right) \\
&\quad\times\left\{\sum_{4k+2pm=N}\sigma_s(k)\sigma_{s'}(m)-2^{s'}\sum_{k<N/4p}\sigma_s\left(\frac{N}{4}-pk\right)\sigma_{s'}(k)\right\} \\
&\quad+\left(p^s-p^{s(m+1)}\right)\left(p^{s'(n+1)}-1\right) \\
&\quad\times\left\{\sum_{k<N/4p}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2pk\right)-2^{s'}\sum_{k<N/4p}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-pk\right)\right\}
\end{aligned}$$

$$+\left(p^{s}-p^{s(m+1)}\right)\left(p^{s'}-p^{s'(n+1)}\right)$$
$$\times\left\{\sum_{k<N/4p}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2p}-2k\right)-2^{s'}\sum_{k<N/4p}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4p}-k\right)\right\}\Bigg].$$

*Proof.* It is similar to Theorem 4.1. □

**Explicit evaluations on some specific situations :**

**Corollary 4.5.** *We have*

(1) *For $m=0$ and $n\geq 1$,*

$$\sum_{k=1}^{N-1}\sigma_{1,ee}(k)\sigma_{1,oe}(2^{n}(N-k))$$
$$=\frac{1}{24}\left\{\sigma_3(N)-5\sigma_3\left(\frac{N}{2}\right)-16\sigma_3\left(\frac{N}{4}\right)-(3N-2)\,\sigma_1(N)\right.$$
$$\left.+2\,(3N-2)\,\sigma_1\left(\frac{N}{2}\right)-2\sigma_1\left(\frac{N}{4}\right)\right\}.$$

(2)
$$\sum_{k=1}^{N-1}\sigma_{1,ee}(2k)\sigma_{1,oe}(N-k)$$
$$=\frac{1}{12}\left\{6\sigma_3\left(\frac{N}{2}\right)-16\sigma_3\left(\frac{N}{4}\right)-3N\sigma_1\left(\frac{N}{2}\right)+2\,(3N-1)\,\sigma_1\left(\frac{N}{4}\right)\right\}.$$

(3)
$$\sum_{k=1}^{N-1}\sigma_{1,ee}(2k)\sigma_{3,oe}(N-k)$$
$$=\frac{1}{120}\left\{13\sigma_5\left(\frac{N}{2}\right)-160\sigma_5\left(\frac{N}{4}\right)-5\,(3N-2)\,\sigma_3\left(\frac{N}{2}\right)\right.$$
$$\left.+40\,(3N-2)\,\sigma_3\left(\frac{N}{4}\right)+7\sigma_1\left(\frac{N}{2}\right)\right\}.$$

(4)
$$\sum_{k=1}^{N-1}\sigma_{3,ee}(2k)\sigma_{1,oe}(N-k)$$
$$=\frac{1}{30}\left\{11\sigma_5\left(\frac{N}{2}\right)-32\sigma_5\left(\frac{N}{4}\right)-10\sigma_3\left(\frac{N}{2}\right)-\sigma_1\left(\frac{N}{2}\right)\right.$$
$$\left.+2\sigma_1\left(\frac{N}{4}\right)\right\}.$$

(5) *For $m=1$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,ee}(2k)\sigma_{1,oe}(2^n(N-k))$$
$$= \frac{1}{12}\left[2\left\{\sigma_3(N) - 6\sigma_3\left(\frac{N}{2}\right)\right\} - (3N-1)\,\sigma_1(N) + 3\,(2N-1)\,\sigma_1\left(\frac{N}{2}\right)\right].$$

(6) *For $m=1$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,ee}(2k)\sigma_{3,oe}(2^n(N-k))$$
$$= \frac{1}{120}\left\{5\sigma_5(N) - 152\sigma_5\left(\frac{N}{2}\right) - 5\,(3N-2)\,\sigma_3(N)\right.$$
$$\left. +40\,(3N-2)\,\sigma_3\left(\frac{N}{2}\right) + 7\sigma_1\left(\frac{N}{2}\right)\right\}.$$

(7) *For $m=1$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{3,ee}(2k)\sigma_{1,oe}(2^n(N-k))$$
$$= \frac{1}{30}\left\{\sigma_5(N) - 22\sigma_5\left(\frac{N}{2}\right) - 10\sigma_3\left(\frac{N}{2}\right) - \sigma_1(N) + 2\sigma_1\left(\frac{N}{2}\right)\right\}.$$

(8) *For $m \geq 2$ and $n=0$,*

$$\sum_{k=1}^{N-1} \sigma_{1,ee}(2^m k)\sigma_{1,oe}(N-k)$$
$$= \frac{1}{24}\left\{(2^m-2)\,\sigma_3(N) - (2^m-14)\,\sigma_3\left(\frac{N}{2}\right) - 32\sigma_3\left(\frac{N}{4}\right)\right.$$
$$- (2^m-2)\,\sigma_1(N) + (2^m - 2 - 3\cdot 2^m N)\,\sigma_1\left(\frac{N}{2}\right)$$
$$\left. +2\,(3\cdot 2^m N - 2)\,\sigma_1\left(\frac{N}{4}\right)\right\}.$$

(9) *For $m \geq 2$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,ee}(2^m k)\sigma_{1,oe}(2^n(N-k))$$
$$= \frac{1}{24}\left[4\left\{(2^m-1)\,\sigma_3(N) - (2^m+4)\,\sigma_3\left(\frac{N}{2}\right)\right\}\right.$$
$$\left. - (2^m - 4 + 3\cdot 2^m N)\,\sigma_1(N) + \left(2^m - 8 + 3\cdot 2^{m+1}N\right)\sigma_1\left(\frac{N}{2}\right)\right].$$

(10) *For $m \geq 2$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,ee}(2^m k)\sigma_{3,oe}(2^n(N-k))$$
$$= \frac{1}{240}\left\{2\left(2^{m+2}-3\right)\sigma_5(N) - 8\,(2^m+36)\,\sigma_5\left(\frac{N}{2}\right) - 5\,(3\cdot 2^m N - 4)\,\sigma_3(N)\right.$$
$$\left. +40\,(3\cdot 2^m N - 4)\,\sigma_3\left(\frac{N}{2}\right) + 7\,(2^m-2)\,\sigma_1(N) - 7\,(2^m-4)\,\sigma_1\left(\frac{N}{2}\right)\right\}.$$

(11) *For $m \geq 2$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{3,ee}(2^m k)\sigma_{1,oe}(2^n(N-k))$$
$$= \frac{1}{840}\left\{(5\cdot 8^m - 12)\,\sigma_5(N) - (5\cdot 8^m + 576)\,\sigma_5\left(\frac{N}{2}\right) - 5\,(8^m-8)\,\sigma_3(N)\right.$$
$$\left. +5\,(8^m-64)\,\sigma_3\left(\frac{N}{2}\right) - 28\sigma_1(N) + 56\sigma_1\left(\frac{N}{2}\right)\right\}.$$

**Theorem 4.6.** *Let*

$$J_{p^m,p^n}(N) := \sum_{k=1}^{N-1} \sigma_{s,ee}\left(p^m k\right)\sigma_{s',eo}\left(p^n(N-k)\right).$$

*Then we obtain the Table 4.*

| $m$ | $n$ | $J_{2^m,2^n}(N)$ |
|---|---|---|
| 0 | 0 | $2^{s+s'}\{\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-2k)-\sum_{k<N/4}\sigma_s(\frac{N}{4}-k)\sigma_{s'}(k)\}$ |
| 0 | $n\geq 1$ | $2^{s+s'n}\{\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(N-4k)-\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-2k)\}$ |
| 1 | 0 | $2^{s+s'}\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)-\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)\}$ |
| 1 | $n\geq 1$ | $2^{s+s'n}\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)-\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)\}$ |
| $m\geq 2$ | $n=0$ | $\frac{2^{s+s'}}{2^s-1}\{(2^{s(m-1)}-1)\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)$ $+(2^s-2^{s(m-1)})\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)$ $-(2^{s(m-1)}-1)\sum_{k<N/4}\sigma_s(N-4k)\sigma_{s'}(k)$ $+(2^{s(m-1)}-2^s)\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)\}$ |
| $m\geq 2$ | $n\geq 1$ | $\frac{2^{s+s'n}}{2^s-1}\{(2^{s(m-1)}-1)\sum_{k=1}^{N-1}\sigma_s(k)\sigma_{s'}(N-k)$ $-(2^{s(m-1)}-1)\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)$ $+(2^s-2^{s(m-1)})\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)$ $-(2^s-2^{s(m-1)})\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)\}$ |

*TABLE 4. Formulas for $J_{2^m,2^n}(N)$*

*And for $n, m \in \mathbb{N}\cup\{0\}$ with odd prime $p$, we have*

$$\sum_{k=1}^{N-1}\sigma_{s,ee}(p^m k)\sigma_{s',eo}(p^n(N-k)) = \frac{2^{s+s'}}{(p^s-1)(p^{s'}-1)}$$
$$\times\Bigg[\left(p^{s(m+1)}-1\right)\left(p^{s'(n+1)}-1\right)$$
$$\times\left\{\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right)-\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right)\right\}$$
$$+\left(p^{s(m+1)}-1\right)\left(p^{s'}-p^{s'(n+1)}\right)$$
$$\times\left\{\sum_{4k+2pm=N}\sigma_s(k)\sigma_{s'}(m)-\sum_{k<N/4p}\sigma_s\left(\frac{N}{4}-pk\right)\sigma_{s'}(k)\right\}$$
$$+\left(p^s-p^{s(m+1)}\right)\left(p^{s'(n+1)}-1\right)$$
$$\times\left\{\sum_{k<N/4p}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2pk\right)-\sum_{k<N/4p}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-pk\right)\right\}$$

$$+\left(p^{s}-p^{s(m+1)}\right)\left(p^{s'}-p^{s'(n+1)}\right)$$
$$\times\left\{\sum_{k<N/4p}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2p}-2k\right)-\sum_{k<N/4p}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4p}-k\right)\right\}\Bigg].$$

*Proof.* The proof is similar as of Theorem 4.1. □

**Corollary 4.7.** *We have*

(1) *For* $m=0$ *and* $n\geq 1$,

$$\sum_{k=1}^{N-1}\sigma_{1,ee}(k)\sigma_{1,eo}(2^n(N-k))$$
$$=\frac{2^n}{24}\left[\sigma_3^*(N)-(3N-2)\,\sigma_1^*(N)-6N\sigma_1\left(\frac{N}{4}\right)\right].$$

(2)
$$\sum_{k=1}^{N-1}\sigma_{1,ee}(2k)\sigma_{1,eo}(N-k)$$
$$=\frac{1}{12}\left[16\sigma_3^*\left(\frac{N}{2}\right)-(9N-2)\,\sigma_1^*\left(\frac{N}{2}\right)-3N\sigma_1\left(\frac{N}{4}\right)\right].$$

(3)
$$\sum_{k=1}^{N-1}\sigma_{1,ee}(2k)\sigma_{3,eo}(N-k)=\frac{1}{3}\left[4\sigma_5^*\left(\frac{N}{2}\right)-(3N-2)\,\sigma_3^*\left(\frac{N}{2}\right)\right].$$

(4)
$$\sum_{k=1}^{N-1}\sigma_{3,ee}(2k)\sigma_{1,eo}(N-k)$$
$$=\frac{1}{30}\left[32\sigma_5^*\left(\frac{N}{2}\right)-15N\sigma_3\left(\frac{N}{2}\right)-2\sigma_1^*\left(\frac{N}{2}\right)\right].$$

(5) *For* $m=1$ *and* $n\geq 1$,

$$\sum_{k=1}^{N-1}\sigma_{1,ee}(2k)\sigma_{1,eo}(2^n(N-k))=\frac{2^n}{12}\left[2\sigma_3^*(N)-3N\sigma_1(N)+\sigma_1^*(N)\right].$$

(6) *For* $m=1$ *and* $n\geq 1$,

$$\sum_{k=1}^{N-1}\sigma_{1,ee}(2k)\sigma_{3,eo}(2^n(N-k))=\frac{8^n}{24}\left[\sigma_5^*(N)-(3N-2)\,\sigma_3^*(N)\right].$$

(7) *For $m = 1$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{3,ee}(2k)\sigma_{1,eo}(2^n(N-k)) = \frac{2^n}{30}\left\{\sigma_5^*(N) - 15N\sigma_3\left(\frac{N}{2}\right) - \sigma_1^*(N)\right\}.$$

(8) *For $m \geq 2$ and $n = 0$,*

$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{1,ee}(2^m k)\sigma_{1,eo}(N-k) \\
&= \frac{1}{24}\left[3\left(2^m - 2\right)\sigma_3(N) - \left(3 \cdot 2^m - 38\right)\sigma_3\left(\frac{N}{2}\right) - 32\sigma_3\left(\frac{N}{4}\right)\right. \\
&\qquad - 3N\left(2^m - 2\right)\sigma_1(N) + \left\{4 - 3\left(2^m + 4\right)N\right\}\sigma_1\left(\frac{N}{2}\right) \\
&\qquad \left. +2\left(3 \cdot 2^m N - 2\right)\sigma_1\left(\frac{N}{4}\right)\right].
\end{aligned}$$

(9) *For $m \geq 2$ and $n \geq 1$,*

$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{1,ee}(2^m k)\sigma_{1,eo}(2^n(N-k)) \\
&= \frac{2^n}{12}\left[\left(3 \cdot 2^m - 4\right)\sigma_3^*(N) - \left\{3N\left(2^m - 1\right) - 1\right\}\sigma_1(N)\right. \\
&\qquad \left. + \left\{3\left(2^m - 2\right)N - 1\right\}\sigma_1\left(\frac{N}{2}\right)\right].
\end{aligned}$$

(10) *For $m \geq 2$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,ee}(2^m k)\sigma_{3,eo}(2^n(N-k)) = \frac{2^{3n}}{48}\left(3 \cdot 2^m - 4\right)\left\{\sigma_5^*(N) - \sigma_3^*(N)\right\}.$$

(11) *For $m \geq 2$ and $n \geq 1$,*

$$\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{3,ee}(2^m k)\sigma_{1,eo}(2^n(N-k)) \\
&= \frac{2^n}{1680}\left[\left(15 \cdot 8^m - 64\right)\sigma_5^*(N) - 15N\left(8^m - 8\right)\sigma_3(N)\right. \\
&\qquad \left. +15N\left(8^m - 64\right)\sigma_3\left(\frac{N}{2}\right) - 56\sigma_1^*(N)\right].
\end{aligned}$$

**Theorem 4.8.** *Let*

$$K_{p^m,p^n}(N) := \sum_{k=1}^{N-1} \sigma_{s,ee}\left(p^m k\right) \sigma_{s',oo}\left(p^n(N-k)\right).$$

*Then we obtain the Table 5.*

| $m$ | $n$ | $K_{2^m,2^n}(N)$ |
|---|---|---|
| $0$ | $0$ | $2^s\{\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(N-4k) - (2^{s'}+1)\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-2k)$ $+2^{s'}\sum_{k<N/4}\sigma_s(\frac{N}{4}-k)\sigma_{s'}(k)\}$ |
| $0$ | $n\geq 1$ | $0$ |
| $1$ | $0$ | $2^s\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k) - (2^{s'}+1)\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)$ $+2^{s'}\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)\}$ |
| $1$ | $n\geq 1$ | $0$ |
| $m\geq 2$ | $n=0$ | $\frac{2^s}{2^s-1}\{(2^{s(m-1)}-1)\sum_{k=1}^{N-1}\sigma_s(k)\sigma_{s'}(N-k)$ $+(2^s-2^{s(m-1)})\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)$ $-(2^{s'}+1)(2^{s(m-1)}-1)\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)$ $-(2^{s'}+1)(2^s-2^{s(m-1)})\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)$ $+2^{s'}(2^{s(m-1)}-1)\sum_{k<N/4}\sigma_s(N-4k)\sigma_{s'}(k)$ $+2^{s'}(2^s-2^{s(m-1)})\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)\}$ |
| $m\geq 2$ | $n\geq 1$ | $0$ |

TABLE 5. Formulas for $K_{2^m,2^n}(N)$

*And for $n,m\in\mathbb{N}\cup\{0\}$ with odd prime $p$, we have*

$$\sum_{k=1}^{N-1}\sigma_{s,ee}(p^m k)\sigma_{s',oo}(p^n(N-k)) = \frac{2^s}{(p^s-1)(p^{s'}-1)}$$

$$\times\left[\left(p^{s(m+1)}-1\right)\left(p^{s'(n+1)}-1\right)\left\{\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(N-4k)\right.\right.$$

$$\left.-\left(2^{s'}+1\right)\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right)+2^{s'}\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right)\right\}$$

$$+\left(p^{s(m+1)}-1\right)\left(p^{s'}-p^{s'(n+1)}\right)\left\{\sum_{4k+pm=N}\sigma_s(k)\sigma_{s'}(m)\right.$$

$$\left.-\left(2^{s'}+1\right)\sum_{4k+2pm=N}\sigma_s(k)\sigma_{s'}(m)+2^{s'}\sum_{k<N/4p}\sigma_s\left(\frac{N}{4}-pk\right)\sigma_{s'}(k)\right\}$$

$$+\left(p^s - p^{s(m+1)}\right)\left(p^{s'(n+1)} - 1\right)\left\{\sum_{k<N/4p}\sigma_s(k)\sigma_{s'}(N-4pk)\right.$$
$$\left.-\left(2^{s'}+1\right)\sum_{k<N/4p}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2pk\right)+2^{s'}\sum_{k<N/4p}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-pk\right)\right\}$$
$$+\left(p^s - p^{s(m+1)}\right)\left(p^{s'} - p^{s'(n+1)}\right)\left\{\sum_{k<N/4p}\sigma_s(k)\sigma_{s'}\left(\frac{N}{p}-4k\right)\right.$$
$$\left.\left.-\left(2^{s'}+1\right)\sum_{k<N/4p}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2p}-2k\right)+2^{s'}\sum_{k<N/4p}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4p}-k\right)\right\}\right].$$

*Proof.* The proof is similar as of Theorem 4.1. □

**Corollary 4.9.** *We have*

(1)
$$\sum_{k=1}^{N-1}\sigma_{1,ee}(2k)\sigma_{1,oo}(N-k) = \frac{1}{12}\left[2\left\{\sigma_3^*(N)-8\sigma_3^*\left(\frac{N}{2}\right)\right\}-(3N-1)\left\{\sigma_1^*(N)-2\sigma_1^*\left(\frac{N}{2}\right)\right\}\right].$$

(2)
$$\sum_{k=1}^{N-1}\sigma_{1,ee}(2k)\sigma_{3,oo}(N-k) = \frac{1}{24}\left[\sigma_5^*(N)-32\sigma_5^*\left(\frac{N}{2}\right)-(3N-2)\left\{\sigma_3^*(N)-8\sigma_3^*\left(\frac{N}{2}\right)\right\}\right].$$

(3)
$$\sum_{k=1}^{N-1}\sigma_{3,ee}(2k)\sigma_{1,oo}(N-k) = \frac{1}{30}\left\{\sigma_5^*(N)-32\sigma_5^*\left(\frac{N}{2}\right)-\sigma_1^*(N)+2\sigma_1^*\left(\frac{N}{2}\right)\right\}.$$

(4) *For $m \geq 2$ and $n = 0$,*

$$\sum_{k=1}^{N-1}\sigma_{1,ee}(2^m k)\sigma_{1,oo}(N-k)$$
$$=\frac{1}{24}\left[\left(3\cdot 2^m-2\right)\sigma_3(N)-3\left(2^m+10\right)\sigma_3\left(\frac{N}{2}\right)+32\sigma_3\left(\frac{N}{4}\right)\right.$$

$$- (3 \cdot 2^m N - 2) \left\{ \sigma_1^*(N) - 2\sigma_1^*\left(\frac{N}{2}\right)\right\}\Bigg].$$

**Theorem 4.10.** *Let*

$$L_{p^m,p^n}(N) := \sum_{k=1}^{N-1} \sigma_{s,oe}\left(p^m k\right) \sigma_{s',oe}\left(p^n(N-k)\right).$$

*Then we obtain the Table 6.*

| $m$ | $n$ | $L_{2^m,2^n}(N)$ |
|---|---|---|
| 0 | 0 | $\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k) - 2^{s'}\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)$ $-2^s\{\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-2k) - 2^{s'}\sum_{k<N/4}\sigma_s(\frac{N}{4}-k)\sigma_{s'}(k)\}$ |
| 0 | $n \geq 1$ | $\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k) - 2^{s'}\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)$ $-2^s\{\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(N-4k) - 2^{s'}\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-2k)\}$ |
| 1 | 0 | $\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k) - 2^{s'}\sum_{k<N/4}\sigma_s(N-4k)\sigma_{s'}(k)$ $-2^s\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k) - 2^{s'}\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)\}$ |
| 1 | $n \geq 1$ | $\sum_{k=1}^{N-1}\sigma_s(k)\sigma_{s'}(N-k) - 2^{s'}\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)$ $-2^s\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k) - 2^{s'}\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)\}$ |
| $m \geq 2$ | 0 | $\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k) - 2^s\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)$ $-2^{s'}\{\sum_{k<N/4}\sigma_s(N-4k)\sigma_{s'}(k) - 2^s\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)\}$ |
| $m \geq 2$ | $n \geq 1$ | $\sum_{k=1}^{N-1}\sigma_s(k)\sigma_{s'}(N-k) - 2^s\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)$ $-2^{s'}\{\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k) - 2^s\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)\}$ |

*TABLE 6. Formulas for $L_{2^m,2^n}(N)$*

*And for $n, m \in \mathbb{N}\cup\{0\}$ with odd prime $p$, we have*

$$\sum_{k=1}^{N-1} \sigma_{s,oe}(p^m k)\sigma_{s',oe}(p^n(N-k)) = \frac{1}{(p^s-1)(p^{s'}-1)}$$

$$\times \Bigg[\left(p^{s(m+1)}-1\right)\left(p^{s'(n+1)}-1\right)\Bigg\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-k\right)$$

$$- 2^{s'}\sum_{k<N/4}\sigma_s\left(\frac{N}{2}-2k\right)\sigma_{s'}(k) - 2^s\Bigg(\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right)$$

$$-2^{s'}\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right)\Bigg)\Bigg\} + \left(p^{s(m+1)}-1\right)\left(p^{s'}-p^{s'(n+1)}\right)$$

$$\times \left\{ \sum_{k<N/2p} \sigma_s\left(\frac{N}{2} - pk\right)\sigma_{s'}(k) - 2^{s'} \sum_{k<N/4p} \sigma_s\left(\frac{N}{2} - 2pk\right)\sigma_{s'}(k) \right.$$

$$\left. -2^s \left( \sum_{4k+2pm=N} \sigma_s(k)\sigma_{s'}(m) - 2^{s'} \sum_{k<N/4p} \sigma_s\left(\frac{N}{4} - pk\right)\sigma_{s'}(k) \right) \right\}$$

$$+ \left(p^s - p^{s(m+1)}\right)\left(p^{s'(n+1)} - 1\right) \left\{ \sum_{k<N/2p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2} - pk\right) \right.$$

$$- 2^{s'} \sum_{2pk+4m=N} \sigma_s(k)\sigma_{s'}(m) - 2^s \left( \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2} - 2pk\right) \right.$$

$$\left.\left. -2^{s'} \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4} - pk\right) \right) \right\} + \left(p^s - p^{s(m+1)}\right)\left(p^{s'} - p^{s'(n+1)}\right)$$

$$\times \left\{ \sum_{k<N/2p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2p} - k\right) - 2^{s'} \sum_{k<N/4p} \sigma_s\left(\frac{N}{2p} - 2k\right)\sigma_{s'}(k) \right.$$

$$\left.\left. -2^s \left( \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2p} - 2k\right) - 2^{s'} \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4p} - k\right) \right) \right\} \right].$$

*Proof.* The proof is similar as of Theorem 4.1. □

**Corollary 4.11.** *We have*

(1) *For $m = 0$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,oe}(k)\sigma_{1,oe}(2^n(N-k)) = \sum_{k=1}^{N-1} \sigma_{1,oe}(2k)\sigma_{1,oe}(N-k)$$
$$= \frac{1}{24}\left\{ \sigma_3(N) - 7\sigma_3\left(\frac{N}{2}\right) + 16\sigma_3\left(\frac{N}{4}\right) - \sigma_1^*(N) + 2\sigma_1\left(\frac{N}{4}\right) \right\}.$$

(2) *For $m = 1$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,oe}(2k)\sigma_{1,oe}(2^n(N-k))$$
$$= \frac{1}{12}\left\{ \sigma_3(N) + 4\sigma_3\left(\frac{N}{2}\right) - \sigma_1(N) + 2\sigma_1\left(\frac{N}{2}\right) \right\}.$$

(3) *For $m = 1$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,oe}(2k)\sigma_{3,oe}(2^n(N-k)) = \sum_{k=1}^{N-1} \sigma_{3,oe}(2k)\sigma_{1,oe}(2^n(N-k))$$
$$= \frac{1}{240}\left[3\left\{\sigma_5(N) + 48\sigma_5\left(\frac{N}{2}\right)\right\} - 10\left\{\sigma_3(N) - 8\sigma_3\left(\frac{N}{2}\right)\right\}\right.$$
$$\left. +7\left\{\sigma_1(N) - 2\sigma_1\left(\frac{N}{2}\right)\right\}\right].$$

(4) *For $m \geq 2$ and $n = 0$,*

$$\sum_{k=1}^{N-1} \sigma_{1,oe}(2^m k)\sigma_{1,oe}(N-k)$$
$$= \frac{1}{24}\left\{\sigma_3(N) - 7\sigma_3\left(\frac{N}{2}\right) + 16\sigma_3\left(\frac{N}{4}\right) - \sigma_1^*(N) + 2\sigma_1\left(\frac{N}{4}\right)\right\}.$$

(5) *For $m \geq 2$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,oe}(2^m k)\sigma_{1,oe}(2^n(N-k))$$
$$= \frac{1}{12}\left\{\sigma_3(N) + 4\sigma_3\left(\frac{N}{2}\right) - \sigma_1(N) + 2\sigma_1\left(\frac{N}{2}\right)\right\}.$$

(6) *For $m \geq 2$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,oe}(2^m k)\sigma_{3,oe}(2^n(N-k)) = \sum_{k=1}^{N-1} \sigma_{3,oe}(2^m k)\sigma_{1,oe}(2^n(N-k))$$
$$= \frac{1}{240}\left[3\left\{\sigma_5(N) + 48\sigma_5\left(\frac{N}{2}\right)\right\} - 10\left\{\sigma_3(N) - 8\sigma_3\left(\frac{N}{2}\right)\right\}\right.$$
$$\left. +7\left\{\sigma_1(N) - 2\sigma_1\left(\frac{N}{2}\right)\right\}\right].$$

**Theorem 4.12.** *Let*

$$M_{p^m,p^n}(N) := \sum_{k=1}^{N-1} \sigma_{s,oe}\left(p^m k\right)\sigma_{s',eo}\left(p^n(N-k)\right).$$

*Then we obtain the Table 7.*

| $m$ | $n$ | $M_{2^m,2^n}(N)$ |
|---|---|---|
| 0 | 0 | $2^{s'}[\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)-\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)$ $-2^s\{\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-2k)-\sum_{k<N/4}\sigma_s(\frac{N}{4}-k)\sigma_{s'}(k)\}]$ |
| 0 | $n\geq 1$ | $2^{s'n}[\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)-\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)$ $-2^s\{\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(N-4k)-\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-2k)\}]$ |
| 1 | 0 | $2^{s'}[\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)-\sum_{k<N/4}\sigma_s(N-4k)\sigma_{s'}(k)$ $-2^s\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)-\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)\}]$ |
| 1 | $n\geq 1$ | $2^{s'n}[\sum_{k=1}^{N-1}\sigma_s(k)\sigma_{s'}(N-k)-\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)$ $-2^s\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)-\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)\}]$ |
| $m\geq 2$ | 0 | $2^{s'}[\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)-\sum_{k<N/4}\sigma_s(N-4k)\sigma_{s'}(k)$ $-2^s\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)-\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)\}]$ |
| $m\geq 2$ | $n\geq 1$ | $2^{s'n}[\sum_{k=1}^{N-1}\sigma_s(k)\sigma_{s'}(N-k)-\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)$ $-2^s\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)-\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)\}]$ |

*TABLE 7. Formulas for $M_{2^m,2^n}(N)$*

*And for $n, m \in \mathbb{N}\cup\{0\}$ with odd prime $p$, we have*

$$\sum_{k=1}^{N-1}\sigma_{s,oe}(p^m k)\sigma_{s',eo}(p^n(N-k)) = \frac{2^{s'}}{(p^s-1)(p^{s'}-1)}$$

$$\times\Bigg[\left(p^{s(m+1)}-1\right)\left(p^{s'(n+1)}-1\right)\Bigg\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-k\right)$$

$$-\sum_{k<N/4}\sigma_s\left(\frac{N}{2}-2k\right)\sigma_{s'}(k)-2^s\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right)$$

$$+2^s\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right)\Bigg\}+\left(p^{s(m+1)}-1\right)\left(p^{s'}-p^{s'(n+1)}\right)$$

$$\times\Bigg\{\sum_{k<N/2p}\sigma_s\left(\frac{N}{2}-pk\right)\sigma_{s'}(k)-\sum_{k<N/4p}\sigma_s\left(\frac{N}{2}-2pk\right)\sigma_{s'}(k)$$

$$-2^s\sum_{4k+2pm=N}\sigma_s(k)\sigma_{s'}(m)+2^s\sum_{k<N/4p}\sigma_s\left(\frac{N}{4}-pk\right)\sigma_{s'}(k)\Bigg\}$$

$$+\left(p^s - p^{s(m+1)}\right)\left(p^{s'(n+1)} - 1\right)\left\{\sum_{k<N/2p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2} - pk\right)\right.$$

$$- \sum_{2pk+4m=N} \sigma_s(k)\sigma_{s'}(m) - 2^s \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2} - 2pk\right)$$

$$\left.+2^s \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4} - pk\right)\right\} + \left(p^s - p^{s(m+1)}\right)\left(p^{s'} - p^{s'(n+1)}\right)$$

$$\times\left\{\sum_{k<N/2p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2p} - k\right) - \sum_{k<N/4p} \sigma_s\left(\frac{N}{2p} - 2k\right)\sigma_{s'}(k)\right.$$

$$\left.\left.-2^s \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2p} - 2k\right) + 2^s \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4p} - k\right)\right\}\right].$$

*Proof.* The proof is similar as of Theorem 4.1. □

**Corollary 4.13.** *We have*
(1) *For* $m = 0$ *and* $n \geq 1$,

$$\sum_{k=1}^{N-1} \sigma_{1,oe}(k)\sigma_{1,eo}(2^n(N-k))$$
$$= \frac{2^{n-3}}{3}\left\{\sigma_3^*(N) - \sigma_1^*(N) - 3N\sigma_1^*\left(\frac{N}{2}\right) + 3N\sigma_1\left(\frac{N}{4}\right)\right\}.$$

$$\sum_{k=1}^{N-1} \sigma_{1,oe}(2k)\sigma_{1,eo}(N-k)$$

(2)
$$= \frac{1}{24}\left\{3\sigma_3^*(N) - 16\sigma_3^*\left(\frac{N}{2}\right) - 3N\sigma_1(N) + 2\left(3N-1\right)\sigma_1\left(\frac{N}{2}\right)\right.$$
$$\left.+2\sigma_1\left(\frac{N}{4}\right)\right\}.$$

(3) *For* $m = 1$ *and* $n \geq 1$,

$$\sum_{k=1}^{N-1} \sigma_{1,oe}(2k)\sigma_{1,eo}(2^n(N-k))$$
$$= \frac{2^{n-3}}{3}\left[4\sigma_3^*(N) - \left(3N+1\right)\sigma_1(N) + \left(6N+1\right)\sigma_1\left(\frac{N}{2}\right)\right].$$

(4) *For $m=1$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,oe}(2k)\sigma_{3,eo}(2^n(N-k)) = \frac{8^{n-1}}{3}\left\{\sigma_5^*(N) - \sigma_3^*(N)\right\}.$$

(5) *For $m=1$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{3,oe}(2k)\sigma_{1,eo}(2^n(N-k))$$
$$= \frac{2^{n-4}}{15}\left[8\sigma_5^*(N) - 15N\left\{\sigma_3(N) - 8\sigma_3\left(\frac{N}{2}\right)\right\} + 7\sigma_1^*(N)\right].$$

(6) *For $m \geq 2$ and $n=0$,*

$$\sum_{k=1}^{N-1} \sigma_{1,oe}(2^m k)\sigma_{1,eo}(N-k)$$
$$= \frac{1}{24}\left\{3\sigma_3^*(N) - 16\sigma_3^*\left(\frac{N}{2}\right) - 3N\sigma_1(N) + 2(3N-1)\sigma_1\left(\frac{N}{2}\right)\right.$$
$$\left. + 2\sigma_1\left(\frac{N}{4}\right)\right\}.$$

(7) *For $m \geq 2$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,oe}(2^m k)\sigma_{1,eo}(2^n(N-k))$$
$$= \frac{2^{n-3}}{3}\left[4\sigma_3^*(N) - (3N+1)\sigma_1(N) + (6N+1)\sigma_1\left(\frac{N}{2}\right)\right].$$

(8) *For $m \geq 2$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,oe}(2^m k)\sigma_{3,eo}(2^n(N-k)) = \frac{8^{n-1}}{3}\left\{\sigma_5^*(N) - \sigma_3^*(N)\right\}.$$

(9) *For $m \geq 2$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{3,oe}(2^m k)\sigma_{1,eo}(2^n(N-k))$$
$$= \frac{2^{n-4}}{15}\left[8\sigma_5^*(N) - 15N\left\{\sigma_3(N) - 8\sigma_3\left(\frac{N}{2}\right)\right\} + 7\sigma_1^*(N)\right].$$

**Theorem 4.14.** *Let*

$$N_{p^m,p^n}(N) := \sum_{k=1}^{N-1} \sigma_{s,oe}\left(p^m k\right) \sigma_{s',oo}\left(p^n(N-k)\right).$$

*Then we obtain the Table 8.*

| $m$ | $n$ | $N_{2^m,2^n}(N)$ |
|---|---|---|
| 0 | 0 | $\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)-(2^{s'}+1)\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)$ $+2^{s'}\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)-2^s\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(N-4k)$ $+2^s(2^{s'}+1)\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-2k)$ $-2^{s+s'}\sum_{k<N/4}\sigma_s(\frac{N}{4}-k)\sigma_{s'}(k)$ |
| 0 | $n\geq 1$ | 0 |
| 1 | 0 | $\sum_{k=1}^{N-1}\sigma_s(k)\sigma_{s'}(N-k)-(2^{s'}+1)\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)$ $+2^{s'}\sum_{k<N/4}\sigma_s(N-4k)\sigma_{s'}(k)-2^s\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)$ $+2^s(2^{s'}+1)\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)$ $-2^{s+s'}\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)$ |
| 1 | $n\geq 1$ | 0 |
| $m\geq 2$ | 0 | $\sum_{k=1}^{N-1}\sigma_s(k)\sigma_{s'}(N-k)-2^s\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)$ $-(2^{s'}+1)\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)$ $+2^s(2^{s'}+1)\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)$ $+2^{s'}\sum_{k<N/4}\sigma_s(N-4k)\sigma_{s'}(k)-2^{s+s'}\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)$ |
| $m\geq 2$ | $n\geq 1$ | 0 |

*TABLE 8. Formulas for $N_{2^m,2^n}(N)$*

*And for $n, m \in \mathbb{N}\cup\{0\}$ with odd prime $p$, we have*

$$\sum_{k=1}^{N-1}\sigma_{s,oe}(p^m k)\sigma_{s',oo}(p^n(N-k)) = \frac{1}{(p^s-1)(p^{s'}-1)}$$
$$\times\left[\left(p^{s(m+1)}-1\right)\left(p^{s'(n+1)}-1\right)\left\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)\right.\right.$$
$$-\left(2^{s'}+1\right)\sum_{k<N/2}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-k\right)+2^{s'}\sum_{k<N/4}\sigma_s\left(\frac{N}{2}-2k\right)\sigma_{s'}(k)$$
$$-2^s\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(N-4k)+2^s\left(2^{s'}+1\right)\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right)$$

$$
\begin{aligned}
&-2^{s+s'} \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right)\Bigg\} + \left(p^{s(m+1)}-1\right)\left(p^{s'}-p^{s'(n+1)}\right) \\
&\times \Bigg\{ \sum_{2k+pm=N} \sigma_s(k)\sigma_{s'}(m) - \left(2^{s'}+1\right) \sum_{k<N/2p} \sigma_s\left(\frac{N}{2}-pk\right)\sigma_{s'}(k) \\
&+ 2^{s'} \sum_{k<N/4p} \sigma_s\left(\frac{N}{2}-2pk\right)\sigma_{s'}(k) - 2^s \sum_{4k+pm=N} \sigma_s(k)\sigma_{s'}(m) \\
&+ 2^s\left(2^{s'}+1\right) \sum_{4k+2pm=N} \sigma_s(k)\sigma_{s'}(m) - 2^{s+s'} \sum_{k<N/4p} \sigma_s\left(\frac{N}{4}-pk\right)\sigma_{s'}(k)\Bigg\} \\
&+ \left(p^s-p^{s(m+1)}\right)\left(p^{s'(n+1)}-1\right)\Bigg\{ \sum_{k<N/2p} \sigma_s(k)\sigma_{s'}(N-2pk) \\
&- \left(2^{s'}+1\right) \sum_{k<N/2p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-pk\right) + 2^{s'} \sum_{2pk+4m=N} \sigma_s(k)\sigma_{s'}(m) \\
&- 2^s \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}(N-4pk) + 2^s\left(2^{s'}+1\right) \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2pk\right) \\
&- 2^{s+s'} \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-pk\right)\Bigg\} + \left(p^s-p^{s(m+1)}\right)\left(p^{s'}-p^{s'(n+1)}\right) \\
&\times \Bigg\{ \sum_{k<N/2p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{p}-2k\right) - \left(2^{s'}+1\right) \sum_{k<N/2p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2p}-k\right) \\
&+ 2^{s'} \sum_{k<N/4p} \sigma_s\left(\frac{N}{2p}-2k\right)\sigma_{s'}(k) - 2^s \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{p}-4k\right) \\
&+2^s\left(2^{s'}+1\right) \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2p}-2k\right) - 2^{s+s'} \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4p}-k\right)\Bigg\}\Bigg].
\end{aligned}
$$

*Proof.* It is similar as of Theorem 4.1. □

**Corollary 4.15.** *We have*

$$
\begin{aligned}
(1) \qquad &\sum_{k=1}^{N-1} \sigma_{1,oe}(2k)\sigma_{1,oo}(N-k) \\
&= \frac{1}{24}\left\{\sigma_3^*(N) + 16\sigma_3^*\left(\frac{N}{2}\right) - \sigma_1^*(N) + 2\sigma_1^*\left(\frac{N}{2}\right)\right\}.
\end{aligned}
$$

(2) *For $m \geq 2$ and $n = 0$,*

$$\sum_{k=1}^{N-1} \sigma_{1,oe}(2^m k)\sigma_{1,oo}(N-k)$$
$$= \frac{1}{24}\left\{\sigma_3^*(N) + 16\sigma_3^*\left(\frac{N}{2}\right) - \sigma_1^*(N) + 2\sigma_1^*\left(\frac{N}{2}\right)\right\}.$$

**Theorem 4.16.** *Let*

$$Q_{p^m,p^n}(N) := \sum_{k=1}^{N-1} \sigma_{s,eo}\left(p^m k\right)\sigma_{s',eo}\left(p^n(N-k)\right).$$

*Then we obtain the Table 9.*

| $m$ | $n$ | $Q_{2^m,2^n}(N)$ |
|---|---|---|
| 0 | 0 | $2^{s+s'}\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k) - \sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)$ $- \sum_{k<N/4}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-2k) + \sum_{k<N/4}\sigma_s(\frac{N}{4}-k)\sigma_{s'}(k)\}$ |
| 0 | $n \geq 1$ | $2^{s+s'n}\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k) - \sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)$ $- \sum_{k<N/4}\sigma_s(k)\sigma_{s'}(N-4k) + \sum_{k<N/4}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-2k)\}$ |
| 1 | 0 | $2^{s+s'}\{\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k) - \sum_{k<N/4}\sigma_s(N-4k)\sigma_{s'}(k)$ $- \sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k) + \sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)\}$ |
| 1 | $n \geq 1$ | $2^{s+s'n}\{\sum_{k=1}^{N-1}\sigma_s(k)\sigma_{s'}(N-k) - \sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)$ $- \sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k) + \sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)\}$ |
| $m \geq 2$ | 0 | $2^{s'+sm}\{\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k) - \sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)$ $- \sum_{k<N/4}\sigma_s(N-4k)\sigma_{s'}(k) + \sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)\}$ |
| $m \geq 2$ | $n \geq 1$ | $2^{s'n+sm}\{\sum_{k=1}^{N-1}\sigma_s(k)\sigma_{s'}(N-k) - \sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)$ $- \sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k) + \sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)\}$ |

*TABLE 9. Formulas for $Q_{2^m,2^n}(N)$*

*And for $n, m \in \mathbb{N} \cup \{0\}$ with odd prime $p$, we have*

$$\sum_{k=1}^{N-1} \sigma_{s,eo}(p^m k)\sigma_{s',eo}(p^n(N-k)) = \frac{2^{s+s'}}{(p^s-1)(p^{s'}-1)}$$
$$\times \left[\left(p^{s(m+1)}-1\right)\left(p^{s'(n+1)}-1\right)\left\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-k\right)\right.\right.$$

$$
\begin{aligned}
&- \sum_{k<N/4} \sigma_s\left(\frac{N}{2}-2k\right)\sigma_{s'}(k) - \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right) \\
&+ \sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right)\Bigg\} + \left(p^{s(m+1)}-1\right)\left(p^{s'}-p^{s'(n+1)}\right) \\
&\times \Bigg\{ \sum_{k<N/2p} \sigma_s\left(\frac{N}{2}-pk\right)\sigma_{s'}(k) - \sum_{k<N/4p} \sigma_s\left(\frac{N}{2}-2pk\right)\sigma_{s'}(k) \\
&- \sum_{4k+2pm=N} \sigma_s(k)\sigma_{s'}(m) + \sum_{k<N/4p} \sigma_s\left(\frac{N}{4}-pk\right)\sigma_{s'}(k)\Bigg\} \\
&+ \left(p^{s}-p^{s(m+1)}\right)\left(p^{s'(n+1)}-1\right)\Bigg\{ \sum_{k<N/2p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-pk\right) \\
&- \sum_{2pk+4m=N} \sigma_s(k)\sigma_{s'}(m) - \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2pk\right) \\
&+ \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-pk\right)\Bigg\} + \left(p^{s}-p^{s(m+1)}\right)\left(p^{s'}-p^{s'(n+1)}\right) \\
&\times \Bigg\{ \sum_{k<N/2p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2p}-k\right) - \sum_{k<N/4p} \sigma_s\left(\frac{N}{2p}-2k\right)\sigma_{s'}(k) \\
&- \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2p}-2k\right) + \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4p}-k\right)\Bigg\}\Bigg].
\end{aligned}
$$

*Proof.* The proof is similar as of Theorem 4.1. □

**Corollary 4.17.** *We have*
(1) *For* $m = 0$ *and* $n \geq 1$,

$$
\begin{aligned}
&\sum_{k=1}^{N-1} \sigma_{1,eo}(k)\sigma_{1,eo}(2^n(N-k)) \\
&= 2^{n-3}\left[\sigma_3(N) - \sigma_3\left(\frac{N}{2}\right) - N\left\{\sigma_1(N)+\sigma_1\left(\frac{N}{2}\right)\right\} + 2N\sigma_1\left(\frac{N}{4}\right)\right].
\end{aligned}
$$

(2)
$$\sum_{k=1}^{N-1} \sigma_{1,eo}(2k)\sigma_{1,eo}(N-k)$$
$$= \frac{1}{4}\left[\sigma_3(N) - \sigma_3\left(\frac{N}{2}\right) - N\left\{\sigma_1(N) + \sigma_1\left(\frac{N}{2}\right)\right\} + 2N\sigma_1\left(\frac{N}{4}\right)\right].$$

(3) *For $m = 1$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,eo}(2k)\sigma_{1,eo}(2^n(N-k))$$
$$= 2^{n-1}\left[\sigma_3(N) - \sigma_3\left(\frac{N}{2}\right) - N\left\{\sigma_1(N) - \sigma_1\left(\frac{N}{2}\right)\right\}\right].$$

(4) *For $m = 1$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,eo}(2k)\sigma_{3,eo}(2^n(N-k))$$
$$= 8^{n-1}\left[\sigma_5(N) - \sigma_5\left(\frac{N}{2}\right) - N\left\{\sigma_3(N) - \sigma_3\left(\frac{N}{2}\right)\right\}\right].$$

(5) *For $m = 1$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{3,eo}(2k)\sigma_{1,eo}(2^n(N-k))$$
$$= 2^{n-1}\left[\sigma_5(N) - \sigma_5\left(\frac{N}{2}\right) - N\left\{\sigma_3(N) - \sigma_3\left(\frac{N}{2}\right)\right\}\right].$$

(6) *For $m \geq 2$ and $n = 0$,*

$$\sum_{k=1}^{N-1} \sigma_{1,eo}\left(2^m k\right)\sigma_{1,eo}(N-k)$$
$$= 2^{m-3}\left[\sigma_3(N) - \sigma_3\left(\frac{N}{2}\right) - N\left\{\sigma_1(N) + \sigma_1\left(\frac{N}{2}\right)\right\} + 2N\sigma_1(\frac{N}{4})\right].$$

(7) *For $m \geq 2$ and $n \geq 1$,*

$$\sum_{k=1}^{N-1} \sigma_{1,eo}\left(2^m k\right)\sigma_{1,eo}\left(2^n(N-k)\right)$$
$$= 2^{n+m-2}\left[\sigma_3(N) - \sigma_3\left(\frac{N}{2}\right) - N\left\{\sigma_1(N) - \sigma_1\left(\frac{N}{2}\right)\right\}\right].$$

(8) *For* $m \geq 2$ *and* $n \geq 1$,

$$\sum_{k=1}^{N-1} \sigma_{1,eo}\left(2^m k\right) \sigma_{3,eo}\left(2^n(N-k)\right) = 2^{3n+m-4}\left[\sigma_5(N) - \sigma_5\left(\frac{N}{2}\right) - N\left\{\sigma_3(N) - \sigma_3\left(\frac{N}{2}\right)\right\}\right].$$

(9) *For* $m \geq 2$ *and* $n \geq 1$,

$$\sum_{k=1}^{N-1} \sigma_{3,oe}(2^m k)\sigma_{1,eo}(2^n(N-k)) = 2^{n+3m-4}\left[\sigma_5(N) - \sigma_5\left(\frac{N}{2}\right) - N\left\{\sigma_3(N) - \sigma_3\left(\frac{N}{2}\right)\right\}\right].$$

**Theorem 4.18.** *Let*

$$R_{p^m,p^n}(N) := \sum_{k=1}^{N-1} \sigma_{s,eo}\left(p^m k\right) \sigma_{s',oo}\left(p^n(N-k)\right).$$

*Then we obtain the Table 10.*

| $m$ | $n$ | $R_{2^m,2^n}(N)$ |
|---|---|---|
| 0 | 0 | $2^s\{\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k) - (2^{s'}+1)\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)$ $+2^{s'}\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k) - \sum_{k<N/4}\sigma_s(k)\sigma_{s'}(N-4k)$ $+(2^{s'}+1)\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-2k)$ $-2^{s'}\sum_{k<N/4}\sigma_s(\frac{N}{4}-k)\sigma_{s'}(k)\}$ |
| 0 | $n \geq 1$ | 0 |
| 1 | 0 | $2^s\{\sum_{k=1}^{N-1}\sigma_s(k)\sigma_{s'}(N-k) - (2^{s'}+1)\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)$ $+2^{s'}\sum_{k<N/4}\sigma_s(N-4k)\sigma_{s'}(k) - \sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)$ $+(2^{s'}+1)\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)$ $-2^{s'}\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)\}$ |
| 1 | $n \geq 1$ | 0 |
| $m \geq 2$ | 0 | $2^{sm}\{\sum_{k=1}^{N-1}\sigma_s(k)\sigma_{s'}(N-k) - \sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)$ $-(2^{s'}+1)\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)$ $+(2^{s'}+1)\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)$ $+2^{s'}\sum_{k<N/4}\sigma_s(N-4k)\sigma_{s'}(k) - 2^{s'}\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)\}$ |
| $m \geq 2$ | $n \geq 1$ | 0 |

*TABLE 10. Formulas for* $R_{2^m,2^n}(N)$

*And for* $n, m \in \mathbb{N} \cup \{0\}$ *with odd prime* $p$*, we have*

$$\sum_{k=1}^{N-1} \sigma_{s,eo}(p^m k)\sigma_{s',oo}(p^n(N-k)) = \frac{2^s}{(p^s-1)(p^{s'}-1)}$$

$$\times \Bigg[ \left(p^{s(m+1)}-1\right)\left(p^{s'(n+1)}-1\right) \Bigg\{ \sum_{k<N/2} \sigma_s(k)\sigma_{s'}(N-2k)$$

$$-(2^{s'}+1)\sum_{k<N/2} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-k\right) + 2^{s'} \sum_{k<N/4} \sigma_s\left(\frac{N}{2}-2k\right)\sigma_{s'}(k)$$

$$-\sum_{k<N/4} \sigma_s(k)\sigma_{s'}(N-4k) + \left(2^{s'}+1\right)\sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right)$$

$$-2^{s'}\sum_{k<N/4} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right)\Bigg\} + \left(p^{s(m+1)}-1\right)\left(p^{s'}-p^{s'(n+1)}\right)$$

$$\times \Bigg\{ \sum_{2k+pm=N} \sigma_s(k)\sigma_{s'}(m) - \left(2^{s'}+1\right)\sum_{k<N/2p} \sigma_s\left(\frac{N}{2}-pk\right)\sigma_{s'}(k)$$

$$+2^{s'}\sum_{k<N/4p} \sigma_s\left(\frac{N}{2}-2pk\right)\sigma_{s'}(k) - \sum_{4k+pm=N} \sigma_s(k)\sigma_{s'}(m)$$

$$+\left(2^{s'}+1\right)\sum_{4k+2pm=N} \sigma_s(k)\sigma_{s'}(m) - 2^{s'}\sum_{k<N/4p} \sigma_s\left(\frac{N}{4}-pk\right)\sigma_{s'}(k)\Bigg\}$$

$$+\left(p^s-p^{s(m+1)}\right)\left(p^{s'(n+1)}-1\right)\Bigg\{ \sum_{k<N/2p} \sigma_s(k)\sigma_{s'}(N-2pk)$$

$$-\left(2^{s'}+1\right)\sum_{k<N/2p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-pk\right) + 2^{s'}\sum_{2pk+4m=N} \sigma_s(k)\sigma_{s'}(m)$$

$$-\sum_{k<N/4p} \sigma_s(k)\sigma_{s'}(N-4pk) + \left(2^{s'}+1\right)\sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2pk\right)$$

$$-2^{s'}\sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-pk\right)\Bigg\} + \left(p^s-p^{s(m+1)}\right)\left(p^{s'}-p^{s'(n+1)}\right)$$

$$\times \Bigg\{ \sum_{k<N/2p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{p}-2k\right) - \left(2^{s'}+1\right)\sum_{k<N/2p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2p}-k\right)$$

$$+2^{s'}\sum_{k<N/4p} \sigma_s\left(\frac{N}{2p}-2k\right)\sigma_{s'}(k) - \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{p}-4k\right)$$

$$+\left(2^{s'}+1\right)\sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2p}-2k\right) - 2^{s'}\sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4p}-k\right)\Bigg\}\Bigg].$$

*Proof.* It is similar as of Theorem 4.1. □

**Corollary 4.19.** *We have*

(1)
$$\sum_{k=1}^{N-1}\sigma_{1,eo}(2k)\sigma_{1,oo}(N-k) = \frac{1}{4}\left[\sigma_3(N)-\sigma_3\left(\frac{N}{2}\right)-N\left\{\sigma_1(N)-3\sigma_1\left(\frac{N}{2}\right)+2\sigma_1\left(\frac{N}{4}\right)\right\}\right].$$

(2) *For $m \geq 2$ and $n = 0$,*

$$\sum_{k=1}^{N-1}\sigma_{1,eo}(2^m k)\sigma_{1,oo}(N-k) = 2^{m-3}\left[\sigma_3(N)-\sigma_3\left(\frac{N}{2}\right)-N\left\{\sigma_1(N)-3\sigma_1\left(\frac{N}{2}\right)+2\sigma_1\left(\frac{N}{4}\right)\right\}\right].$$

**Theorem 4.20.** *Let*

$$S_{p^m,p^n}(N) := \sum_{k=1}^{N-1}\sigma_{s,oo}\left(p^m k\right)\sigma_{s',oo}\left(p^n(N-k)\right).$$

*Then we obtain the Table 11.*

| $m$ | $n$ | $S_{2^m,2^n}(N)$ |
|---|---|---|
| 0 | 0 | $\sum_{k=1}^{N-1}\sigma_s(k)\sigma_{s'}(N-k)-(2^{s'}+1)\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k)$ $+2^{s'}\sum_{k<N/4}\sigma_s(N-4k)\sigma_{s'}(k)-(2^s+1)\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k)$ $+(2^s+1)(2^{s'}+1)\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-k)$ $-2^{s'}(2^s+1)\sum_{k<N/4}\sigma_s(\frac{N}{2}-2k)\sigma_{s'}(k)$ $+2^s\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(N-4k)-2^s(2^{s'}+1)\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(\frac{N}{2}-2k)$ $+2^{s+s'}\sum_{k<N/4}\sigma_s(\frac{N}{4}-k)\sigma_{s'}(k)$ |
| 0 | $n \geq 1$ | 0 |
| $m \geq 1$ | $n \geq 0$ | 0 |

TABLE 11. *Formulas for $S_{2^m,2^n}(N)$*

*And for $n, m \in \mathbb{N}\cup\{0\}$ with odd prime $p$, we have*

$$\sum_{k=1}^{N-1}\sigma_{s,oo}(p^m k)\sigma_{s',oo}(p^n(N-k)) = \frac{1}{(p^s-1)(p^{s'}-1)}$$
$$\times\left[(p^{s(m+1)}-1)(p^{s'(n+1)}-1)\left\{\sum_{k=1}^{N-1}\sigma_s(k)\sigma_{s'}(N-k)\right.\right.$$

$$- (2^{s'}+1)\sum_{k<N/2}\sigma_s(N-2k)\sigma_{s'}(k) + 2^{s'}\sum_{k<N/4}\sigma_s(N-4k)\sigma_{s'}(k)$$

$$- (2^{s}+1)\sum_{k<N/2}\sigma_s(k)\sigma_{s'}(N-2k) + (2^s+1)(2^{s'}+1)\sum_{k<N/2}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-k\right)$$

$$- 2^{s'}(2^{s}+1)\sum_{k<N/4}\sigma_s\left(\frac{N}{2}-2k\right)\sigma_{s'}(k) + 2^{s}\sum_{k<N/4}\sigma_s(k)\sigma_{s'}(N-4k)$$

$$-2^{s}(2^{s'}+1)\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2k\right) + 2^{s+s'}\sum_{k<N/4}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-k\right)\Bigg\}$$

$$+ (p^{s(m+1)}-1)(p^{s'}-p^{s'(n+1)})\Bigg\{\sum_{k<N/p}\sigma_s(N-pk)\sigma_{s'}(k)$$

$$- (2^{s'}+1)\sum_{k<N/2p}\sigma_s(N-2pk)\sigma_{s'}(k) + 2^{s'}\sum_{k<N/4p}\sigma_s(N-4pk)\sigma_{s'}(k)$$

$$- (2^{s}+1)\sum_{2k+pm=N}\sigma_s(k)\sigma_{s'}(m) + (2^s+1)(2^{s'}+1)\sum_{k<N/2p}\sigma_s\left(\frac{N}{2}-pk\right)\sigma_{s'}(k)$$

$$- 2^{s'}(2^{s}+1)\sum_{k<N/4p}\sigma_s\left(\frac{N}{2}-2pk\right)\sigma_{s'}(k) + 2^{s}\sum_{4k+pm=N}\sigma_s(k)\sigma_{s'}(m)$$

$$-2^{s}(2^{s'}+1)\sum_{4k+2pm=N}\sigma_s(k)\sigma_{s'}(m) + 2^{s+s'}\sum_{k<N/4p}\sigma_s\left(\frac{N}{4}-pk\right)\sigma_{s'}(k)\Bigg\}$$

$$+ (p^{s}-p^{s(m+1)})(p^{s'(n+1)}-1)\Bigg\{\sum_{k<N/p}\sigma_s(k)\sigma_{s'}(N-pk)$$

$$- (2^{s'}+1)\sum_{pk+2m=N}\sigma_s(k)\sigma_{s'}(m) + 2^{s'}\sum_{pk+4m=N}\sigma_s(k)\sigma_{s'}(m)$$

$$- (2^{s}+1)\sum_{k<N/2p}\sigma_s(k)\sigma_{s'}(N-2pk) + (2^s+1)(2^{s'}+1)\sum_{k<N/2p}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-pk\right)$$

$$- 2^{s'}(2^{s}+1)\sum_{2pk+4m=N}\sigma_s(k)\sigma_{s'}(m) + 2^{s}\sum_{k<N/4p}\sigma_s(k)\sigma_{s'}(N-4pk)$$

$$- 2^{s}(2^{s'}+1)\sum_{k<N/4p}\sigma_s(k)\sigma_{s'}\left(\frac{N}{2}-2pk\right) + 2^{s+s'}\sum_{k<N/4p}\sigma_s(k)\sigma_{s'}\left(\frac{N}{4}-pk\right)\Bigg\}$$

$$+ (p^{s}-p^{s(m+1)})(p^{s'}-p^{s'(n+1)})\Bigg\{\sum_{k<N/p}\sigma_s(k)\sigma_{s'}\left(\frac{N}{p}-k\right)$$

$$- (2^{s'}+1)\sum_{k<N/2p}\sigma_s\left(\frac{N}{p}-2k\right)\sigma_{s'}(k) + 2^{s'}\sum_{k<N/4p}\sigma_s\left(\frac{N}{p}-4k\right)\sigma_{s'}(k)$$

$$- (2^s+1) \sum_{k<N/2p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{p}-2k\right) + (2^s+1)(2^{s'}+1) \sum_{k<N/2p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2p}-k\right)$$
$$- 2^{s'}(2^s+1) \sum_{k<N/4p} \sigma_s\left(\frac{N}{2p}-2k\right)\sigma_{s'}(k) + 2^s \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{p}-4k\right)$$
$$\left.\left. -2^s(2^{s'}+1) \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{2p}-2k\right) + 2^{s+s'} \sum_{k<N/4p} \sigma_s(k)\sigma_{s'}\left(\frac{N}{4p}-k\right)\right\}\right].$$

*Proof.* It is similar as of Theorem 4.1. □

**Corollary 4.21.** *We have*

$$\sum_{k=1}^{N-1} \sigma_{1,oo}(k)\sigma_{3,oo}(N-k) = \sum_{k=1}^{N-1} \sigma_{3,oo}(k)\sigma_{1,oo}(N-k)$$
$$= \sigma_5\left(\frac{N}{2}\right) - \sigma_5\left(\frac{N}{4}\right)$$

*and*

$$\sum_{k=1}^{N-1} \sigma_{1,oo}(k)\sigma_{5,oo}(N-k) = \sum_{k=1}^{N-1} \sigma_{5,oo}(k)\sigma_{1,oo}(N-k)$$
$$= \frac{1}{17}\left[32\left\{\sigma_7\left(\frac{N}{2}\right) - \sigma_7\left(\frac{N}{4}\right)\right\} - 15b\left(\frac{N}{2}\right)\right].$$

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