Honam Mathematical J. **35** (2013), No. 2, pp. 179–200
http://dx.doi.org/10.5831/HMJ.2013.35.2.179

A TIME-INDEPENDENT CONDITIONAL FOURIER-FEYNMAN TRANSFORM AND CONVOLUTION PRODUCT ON AN ANALOGUE OF WIENER SPACE

Dong Hyun Cho

Abstract. Let $C[0,t]$ denote the function space of all real-valued continuous paths on $[0,t]$. Define $X_n : C[0,t] \to \mathbb{R}^{n+1}$ by $X_n(x) = (x(t_0), x(t_1), \cdots, x(t_n))$, where $0 = t_0 < t_1 < \cdots < t_n < t$ is a partition of $[0,t]$. In the present paper, using a simple formula for the conditional expectation given the conditioning function X_n, we evaluate the $L_p(1 \le p \le \infty)$-analytic conditional Fourier-Feynman transform and the conditional convolution product of the cylinder functions which have the form

$$f((v_1,x),\cdots,(v_r,x)) \text{ for } x \in C[0,t],$$

where $\{v_1,\cdots,v_r\}$ is an orthonormal subset of $L_2[0,t]$ and $f \in L_p(\mathbb{R}^r)$. We then investigate several relationships between the conditional Fourier-Feynman transform and the conditional convolution product of the cylinder functions.

1. Introduction and preliminaries

Let $C_0[0,t]$ denote the Wiener space, that is, the space of real-valued continuous functions x on the closed interval $[0,t]$ with $x(0) = 0$. On the space $C_0[0,t]$, the concept of an analytic Fourier-Feynman transform was introduced by Brue [1]. Huffman, Park and Skoug [11] defined a convolution product on $C_0[0,t]$ and then, established various relationships between the analytic Fourier-Feynman transform and the convolution product. Furthermore, Chang and Skoug [4] introduced the

Received March 18, 2013. Accepted April 29, 2013.
2010 Mathematics Subject Classification. 28C20, 44A35, 60G15, 60H05.
Key words and phrases. analogue of Wiener measure, analytic conditional Feynman integral, analytic conditional Fourier-Feynman transform, analytic conditional Wiener integral, conditional convolution product.
This work was supported by Kyonggi University Research Grant 2011.

concepts of conditional Fourier-Feynman transform and conditional convolution product on the Wiener space $C_0[0,t]$. In that paper, they also examined the effects that drift has on the conditional Fourier-Feynman transform, the conditional convolution product, and various relationships that occur between them. Further works were studied by Chang, Cho, Kim, Song and Yoo [3, 8]. In fact, Cho and his coauthors [3] introduced the L_1-analytic conditional Fourier-Feynman transform and the conditional convolution product over Wiener paths in abstract Wiener space and then, established their relationships between them of certain cylinder type functions. Cho [8] extended the relationships between the conditional convolution product and the $L_p(1 \le p \le 2)$-analytic conditional Fourier-Feynman transform of the same kind of cylinder functions. Moreover, on $C[0,t]$, the space of the real-valued continuous paths on $[0,t]$, Kim [14] extended the relationships between the conditional convolution product and the $L_p(1 \le p \le \infty)$-analytic conditional Fourier-Feynman transform of the functions in a Banach algebra which corresponds to the Cameron-Storvick's Banach algebra $\mathcal{S}$ [2]. Cho [5, 6] established several relationships between the L_1-analytic conditional Fourier-Feynman transform and the conditional convolution product of the cylinder functions on $C[0,t]$. In particular, he [5] derived an evaluation formula for the $L_p(1 \le p \le \infty)$-analytic conditional Fourier-Feynman transform and the conditional convolution product of the same cylinder functions with the conditioning function $X_{n+1} : C[0,t] \to \mathbb{R}^{n+2}$ given by $X_{n+1}(x) = (x(t_0), x(t_1), \cdots, x(t_n), x(t_{n+1}))$ where $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ is a partition of $[0,t]$, and then, proved their relationships. Note that X_{n+1} depends on the present time t, that is, the expectation is taken over the paths which pass through a particular point at the time t.

In this paper, we further develop the relationships in [3, 5, 6, 8, 14] on the more generalized space $(C[0,t], w_\varphi)$, the analogue of the Wiener space associated with the probability measure φ on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ [12, 16, 17]. For the conditioning function $X_n : C[0,t] \to \mathbb{R}^{n+1}$ given by $X_n(x) = (x(t_0), x(t_1), \cdots, x(t_n))$ which is independent of the present time t, we proceed to study the relationships between the conditional convolution product and the analytic conditional Fourier-Feynman transform of the cylinder functions defined on $C[0,t]$. In fact, using a simple formula for the conditional w_φ-integrals given X_n, we evaluate the $L_p(1 \le p \le \infty)$-analytic conditional Fourier-Feynman transform and the conditional convolution product for the functions of the form

$$f((v_1,x),\cdots,(v_r,x)) \text{ for } w_\varphi\text{-a.e. } x\in C[0,t],$$

where $\{v_1,\cdots,v_r\}$ is an orthonormal set in $L_2[0,t]$ and $f\in L_p(\mathbb{R}^r)$. We then investigate several relationships between the conditional Fourier-Feynman transforms and the conditional convolution products of the cylinder functions. Finally, we show that the L_p-analytic conditional Fourier-Feynman transform $T_q^{(p)}[[(F*G)_q|X_n](\cdot,\vec{\xi}_n)|X_n]$ of the conditional convolution product $[(F*G)_q|X_n]$ for the cylinder functions F and G, can be expressed by the formula

$$\begin{aligned}&T_q^{(p)}[[(F*G)_q|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)\\ =\ &\Big[T_q^{(p)}[F|X_n]\Big(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\vec{\zeta}_n+\vec{\xi}_n)\Big)\Big]\Big[T_q^{(p)}[G|X_n]\Big(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\vec{\zeta}_n-\vec{\xi}_n)\Big)\Big]\end{aligned}$$

for a nonzero real q, w_φ-a.e. $y\in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n,\vec{\zeta}_n\in\mathbb{R}^{n+1}$. Thus the analytic conditional Fourier-Feynman transform of the conditional convolution product for the cylinder functions, can be interpreted as the product of the analytic conditional Fourier-Feynman transform of each function.

Throughout this paper, let $\mathbb{C}$ and $\mathbb{C}_+$ denote the set of the complex numbers and the set of the complex numbers with positive real parts, respectively.

Now, we introduce the concrete form of the probability measure w_φ on the Borel class $\mathcal{B}(C[0,t])$ of $C[0,t]$. For a positive real t, let $C=C[0,t]$ be the space of all real-valued continuous functions on the closed interval $[0,t]$ with the supremum norm. For $\vec{t}=(t_0,t_1,\cdots,t_n)$ with $0=t_0<t_1<\cdots<t_n\le t$, let $J_{\vec{t}}: C[0,t]\to\mathbb{R}^{n+1}$ be the function given by $J_{\vec{t}}(x)=(x(t_0),x(t_1),\cdots,x(t_n))$. For B_j $(j=0,1,\cdots,n)$ in $\mathcal{B}(\mathbb{R})$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0,t]$ is called an interval and let $\mathcal{I}$ be the set of all such intervals. For a probability measure φ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, let

$$\begin{aligned}m_\varphi\Big[J_{\vec{t}}^{-1}\Big(\prod_{j=0}^n B_j\Big)\Big] &= \Big[\prod_{j=1}^n\frac{1}{2\pi(t_j-t_{j-1})}\Big]^{\frac{1}{2}}\int_{B_0}\int_{\prod_{j=1}^n B_j}\\ &\qquad\exp\Big\{-\frac{1}{2}\sum_{j=1}^n\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\Big\}d\vec{u}\,d\varphi(u_0).\end{aligned}$$

Then $\mathcal{B}(C[0,t])$ coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique probability measure w_φ on $(C[0,t],\mathcal{B}(C[0,t]))$ such that $w_\varphi(I)=m_\varphi(I)$ for all I in $\mathcal{I}$. This measure w_φ is called an analogue of the Wiener measure associated with the probability measure φ [12, 16, 17, 19].

Let $\{e_k : k = 1, 2, \cdots\}$ be a complete orthonormal subset of $L_2[0,t]$ such that each e_k is of bounded variation. For v in $L_2[0,t]$ and x in $C[0,t]$, let $(v,x) = \lim_{n\to\infty} \sum_{k=1}^{n} \langle v, e_k\rangle \int_0^t e_k(s)dx(s)$ if the limit exists, where $\langle \cdot, \cdot \rangle$ denotes the inner product over $L_2[0,t]$. (v,x) is called the Paley-Wiener-Zygmund integral of v according to x. Note that we also denote the dot product on the r-dimensional Euclidean space $\mathbb{R}^r$ by $\langle \cdot, \cdot \rangle_{\mathbb{R}^r}$.

Applying Theorem 3.5 in [12], we can easily prove the following theorem.

Theorem 1.1. *Let $\{h_1, h_2, \cdots, h_r\}$ be an orthonormal subset of $L_2[0,t]$. For $i = 1, 2, \cdots, r$, let $Z_i(x) = (h_i, x)$ on $C[0,t]$. Then $Z_1, Z_2, \cdots, Z_r$ are independent and each Z_i has the standard normal distribution. Moreover, if $f : \mathbb{R}^r \to \mathbb{R}$ is Borel measurable, then*

$$\begin{aligned} &\int_C f(Z_1(x), Z_2(x), \cdots, Z_r(x))dw_\varphi(x) \\ \stackrel{*}{=}\ &\left(\frac{1}{2\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} f(u_1, u_2, \cdots, u_r) \exp\left\{-\frac{1}{2}\sum_{j=1}^{r} u_j^2\right\} d\vec{u}, \end{aligned}$$

where $\stackrel{*}{=}$ means that if either side exists then both sides exist and they are equal.

Let $F : C[0,t] \to \mathbb{C}$ be integrable and X be a random vector on $C[0,t]$ assuming that the value space of X is a normed space equipped with the Borel σ-algebra. Then, we have the conditional expectation $E[F|X]$ of F given X from a well known probability theory [15]. Furthermore, there exists a P_X-integrable complex-valued function ψ on the value space of X such that $E[F|X](x) = (\psi \circ X)(x)$ for w_φ-a.e. $x \in C[0,t]$, where P_X is the probability distribution of X. The function ψ is called the conditional w_φ-integral of F given X and it is also denoted by $E[F|X]$.

Throughout this paper, let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ be a partition of $[0,t]$ unless otherwise specified. For any x in $C[0,t]$, define the polygonal function $[x]$ of x by

$$\begin{aligned}[x](s) = &\sum_{j=1}^{n+1} \chi_{(t_{j-1},t_j]}(s)\left(\frac{t_j - s}{t_j - t_{j-1}} x(t_{j-1}) + \frac{s - t_{j-1}}{t_j - t_{j-1}} x(t_j)\right) \\ &+ \chi_{\{t_0\}}(s)x(t_0)\end{aligned} \tag{1.1}$$

for $s \in [0,t]$, where $\chi_{(t_{j-1},t_j]}$ and $\chi_{\{t_0\}}$ denote the indicator functions. Similarly, for $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_{n+1}) \in \mathbb{R}^{n+2}$, define the polygonal function $[\vec{\xi}_{n+1}]$ of $\vec{\xi}_{n+1}$ by (1.1) where $x(t_j)$ is replaced by ξ_j for $j = 0, 1, \cdots, n+1$.

In the following theorem, we introduce a simple formula for the conditional w_φ-integrals on $C[0,t]$ [7].

Theorem 1.2. *Let $X_n : C[0,t] \to \mathbb{R}^{n+1}$ be given by*

$$X_n(x) = (x(t_0), x(t_1), \cdots, x(t_n)). \tag{1.2}$$

Moreover let F be integrable on $C[0,t]$ and P_{X_n} be the probability distribution of X_n on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$. Then, for P_{X_n}-a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$,

$$\begin{aligned} E[F|X_n](\vec{\xi}_n) = & \left[\frac{1}{2\pi(t-t_n)}\right]^{\frac{1}{2}} \int_{\mathbb{R}} E[F(x - [x] + [\vec{\xi}_{n+1}])] \\ & \times \exp\left\{-\frac{(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\right\} d\xi_{n+1} \end{aligned} \tag{1.3}$$

where $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1})$ for $\xi_{n+1} \in \mathbb{R}$.

For a function $F : C[0,t] \to \mathbb{C}$ and $\lambda > 0$, let $F^\lambda(x) = F(\lambda^{-\frac{1}{2}}x)$ and $X_n^\lambda(x) = X_n(\lambda^{-\frac{1}{2}}x)$, where X_n is given by (1.2). Suppose that $E[F^\lambda]$ exists for each $\lambda > 0$. Under the notations as used in Theorem 1.2, we can obtain by (1.3)

$$\begin{aligned} E[F^\lambda|X_n^\lambda](\vec{\xi}_n) = & \left[\frac{\lambda}{2\pi(t-t_n)}\right]^{\frac{1}{2}} \int_{\mathbb{R}} E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}_{n+1}])] \\ & \times \exp\left\{-\frac{\lambda}{2}\frac{(\xi_{n+1}-\xi_n)^2}{t-t_n}\right\} d\xi_{n+1} \end{aligned} \tag{1.4}$$

for $P_{X_n^\lambda}$-a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$, where $P_{X_n^\lambda}$ is the probability distribution of X_n^λ on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$. For $y \in C[0,t]$, let $K_F^\lambda(y, \vec{\xi}_n)$ be given by (1.4) where $[\vec{\xi}_{n+1}]$ is replaced by $y + [\vec{\xi}_{n+1}]$. If $K_F^\lambda(0, \vec{\xi}_n)$ has the analytic extension $J_\lambda^*(F)(\vec{\xi}_n)$ on $\mathbb{C}_+$ as a function of λ, then it is called the conditional analytic Wiener w_φ-integral of F given X_n with parameter λ and denoted by $E^{anw_\lambda}[F|X_n](\vec{\xi}_n) = J_\lambda^*(F)(\vec{\xi}_n)$ for $\vec{\xi}_n \in \mathbb{R}^{n+1}$. Moreover, if for a nonzero real q, $E^{anw_\lambda}[F|X_n](\vec{\xi}_n)$ has the limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman w_φ-integral of F given X_n with parameter q and denoted by $E^{anf_q}[F|X_n](\vec{\xi}_n) = \lim_{\lambda \to -iq} E^{anw_\lambda}[F|X_n](\vec{\xi}_n)$.

2. A time-independent conditional Fourier-Feynman transform

For a given extended real number p with $1 < p \le \infty$, suppose that p and p' are related by $\frac{1}{p} + \frac{1}{p'} = 1$(possibly $p' = 1$ if $p = \infty$). Let F_n and F be measurable functions such that $\lim_{n\to\infty} \int_C |F_n(x) - F(x)|^{p'} dw_\varphi(x) = 0$. Then we write $\underset{n\to\infty}{\text{l.i.m.}}(w^{p'})(F_n) = F$ and call F the limit in the mean of order p'. A similar definition is understood when n is replaced by a continuously varying parameter.

We now define the conditional analytic Fourier-Feynman transform of the functions on $C[0,t]$.

Definition 2.1. *Let F be defined on $C[0,t]$ and X_n be given by (1.2). For $\lambda \in \mathbb{C}_+$ and w_φ-a.e. $y \in C[0,t]$, let $T_\lambda[F|X_n](y, \vec{\xi}_n) = E^{anw_\lambda}[F(y + \cdot)|X_n](\vec{\xi}_n)$ for P_{X_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$ if it exists. For a nonzero real q and w_φ-a.e. $y \in C[0,t]$, define the L_1-analytic conditional Fourier-Feynman transform $T_q^{(1)}[F|X_n]$ of F given X_n by the formula $T_q^{(1)}[F|X_n](y, \vec{\xi}_n) = E^{anf_q}[F(y + \cdot)|X_n](\vec{\xi}_n)$ for P_{X_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$ if it exists. For $1 < p \le \infty$, define the L_p-analytic conditional Fourier-Feynman transform $T_q^{(p)}[F|X_n]$ of F given X_n by the formula $T_q^{(p)}[F|X_n](\cdot, \vec{\xi}_n) = \underset{\lambda\to -iq}{\text{l.i.m.}}(w^{p'})(T_\lambda[F|X_n](\cdot, \vec{\xi}_n))$ for P_{X_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, where λ approaches to $-iq$ through $\mathbb{C}_+$.*

For each $j = 1, \cdots, n+1$, let $\alpha_j = \frac{1}{\sqrt{t_j - t_{j-1}}}\chi_{(t_{j-1}, t_j]}$ on $[0,t]$. Let V be the subspace of $L_2[0,t]$ generated by $\{\alpha_1, \cdots, \alpha_{n+1}\}$ and $V^\perp$ denote the orthogonal complement of V. Let $\mathcal{P}$ and $\mathcal{P}^\perp$ be the orthogonal projections from $L_2[0,t]$ to V and $V^\perp$, respectively.

Throughout this paper, let $\{v_1, v_2, \cdots, v_r\}$ be an orthonormal subset of $L_2[0,t]$ such that $\{\mathcal{P}^\perp v_1, \cdots, \mathcal{P}^\perp v_r\}$ is an independent set unless otherwise specified. Let $\{e_1, \cdots, e_r\}$ be the orthonormal set obtained from $\{\mathcal{P}^\perp v_1, \cdots, \mathcal{P}^\perp v_r\}$ by the Gram-Schmidt orthonormalization process. Now, for $l = 1, \cdots, r$, let $\mathcal{P}^\perp v_l = \sum_{j=1}^r \alpha_{lj} e_j$ be the linear combinations of the e_js and let $A = [\alpha_{jl}]_{r\times r}$ be the transpose of the coefficient matrix of the combinations. We can also regard A as the linear transformation $T_A : \mathbb{R}^r \to \mathbb{R}^r$ given by

$$T_A \vec{z} = \vec{z} A, \tag{2.1}$$

where $\vec{z}$ is an arbitrary row-vector in $\mathbb{R}^r$. Note that A is invertible so that T_A is an isomorphism. Let

$$(\mathcal{P}\vec{v})(t) = ((\mathcal{P}v_1)(t), \cdots, (\mathcal{P}v_r)(t)) \tag{2.2}$$

and for $\vec{\xi_n} = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$ let

$$(\vec{\xi_n}, (\mathcal{P}\vec{v})(\vec{t})) = \left(\sum_{j=1}^{n} (\xi_j - \xi_{j-1})(\mathcal{P}v_1)(t_j), \cdots, \sum_{j=1}^{n} (\xi_j - \xi_{j-1})(\mathcal{P}v_r)(t_j) \right). \tag{2.3}$$

Furthermore, let

$$\Gamma(t, A) = \frac{1}{1 + (t - t_n)\|(\mathcal{P}\vec{v})(t)A^{-1}\|_{\mathbb{R}^r}^2} \tag{2.4}$$

and for $\lambda \in \mathbb{C}_+$, $\vec{z} \in \mathbb{R}^r$ let

$$\Phi(\lambda, \vec{z}) = \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \exp\left\{ -\frac{\lambda}{2} [\|\vec{z}\|_{\mathbb{R}^r}^2 - (t - t_n)\Gamma(t, A)\langle \vec{z}, (\mathcal{P}\vec{v})(t)A^{-1}\rangle_{\mathbb{R}^r}^2] \right\}. \tag{2.5}$$

Let $(\vec{v}, x) = ((v_1, x), \cdots, (v_r, x))$ for $x \in C[0, t]$. For $1 \le p \le \infty$, let $\mathcal{A}_r^{(p)}$ be the space of the cylinder functions F_r of the form

$$F_r(x) = f_r(\vec{v}, x) \tag{2.6}$$

for w_φ-a.e. $x \in C[0, t]$, where $f_r \in L_p(\mathbb{R}^r)$. Note that, without loss of generality, we can take f_r to be Borel measurable.

With the above notations we have the following lemma [6].

Lemma 2.2. *Let $\lambda \in \mathbb{C}_+$ and k be an integrable function on $\mathbb{R}^r$. Furthermore, for $\vec{\xi_n} = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$, let $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1})$ and let*

$$\begin{aligned} H(\lambda, k, \vec{\xi_n}) =& \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \left[\frac{\lambda}{2\pi(t - t_n)}\right]^{\frac{1}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}^r} k((\vec{v}, [\vec{\xi}_{n+1}]) + T_A\vec{z}) \\ & \times \exp\left\{ -\frac{\lambda}{2}\|\vec{z}\|_{\mathbb{R}^r}^2 - \frac{\lambda(\xi_{n+1} - \xi_n)^2}{2(t - t_n)} \right\} d\vec{z}\, d\xi_{n+1} \end{aligned} \tag{2.7}$$

where T_A is given by (2.1). Then we have

$$H(\lambda, k, \vec{\xi_n}) = (\Gamma(t, A))^{\frac{1}{2}} \int_{\mathbb{R}^r} k((\vec{\xi_n}, (\mathcal{P}\vec{v})(\vec{t})) + T_A\vec{z})\Phi(\lambda, \vec{z})d\vec{z} \tag{2.8}$$

where $(\mathcal{P}\vec{v})(t)$, $(\vec{\xi_n}, (\mathcal{P}\vec{v})(\vec{t}))$, $\Gamma(t, A)$ and $\Phi(\lambda, \vec{z})$ are given by (2.2), (2.3), (2.4) and (2.5), respectively.

Theorem 2.3. *Let X_n and $F_r \in \mathcal{A}_r^{(p)}(1 \le p \le \infty)$ be given by (1.2) and (2.6), respectively. Then for $\lambda \in \mathbb{C}_+$, w_φ-a.e. $y \in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $T_\lambda[F_r|X_n](y,\vec{\xi}_n)$ exists and it is given by*

$$T_\lambda[F_r|X_n](y,\vec{\xi}_n) = H(\lambda, k_{f_r}(y), \vec{\xi}_n) \tag{2.9}$$

where $k_{f_r}(y)(\vec{u}) = f_r((\vec{v},y)+\vec{u})$ for $\vec{u} \in \mathbb{R}^r$ and H is given by (2.8). Furthermore, as a function of y, $T_\lambda[F_r|X_n](\cdot,\vec{\xi}_n) \in \mathcal{A}_r^{(p)}$.

Proof. For $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$, let $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1})$. For $\lambda > 0$, w_φ-a.e. $y \in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, we have by Theorem 2.2 in [5]

$$\begin{aligned} K_{F_r}^\lambda(y,\vec{\xi}_n) &= \left[\frac{\lambda}{2\pi(t-t_n)}\right]^{\frac{1}{2}} \int_{\mathbb{R}} E[F_r(\lambda^{-\frac{1}{2}}(x-[x]) + y + [\vec{\xi}_{n+1}])] \\ &\quad \times \exp\left\{-\frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\right\} d\xi_{n+1} \\ &= \left[\frac{\lambda}{2\pi(t-t_n)}\right]^{\frac{1}{2}} \int_{\mathbb{R}} \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} f_r((\vec{v},y) + (\vec{v},[\vec{\xi}_{n+1}]) + T_A\vec{z}) \\ &\quad \times \exp\left\{-\frac{\lambda}{2}\|\vec{z}\|_{\mathbb{R}^r}^2 - \frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\right\} d\vec{z}\, d\xi_{n+1} \\ &= H(\lambda, k_{f_r}(y), \vec{\xi}_n) \end{aligned}$$

where H is given by (2.7) replacing k by $k_{f_r}(y)$. By (2.8) of Lemma 2.2 we have

$$K_{F_r}^\lambda(y,\vec{\xi}_n) = (\Gamma(t,A))^{\frac{1}{2}} \int_{\mathbb{R}^r} f_r((\vec{v},y) + (\vec{\xi}_n, (\mathcal{P}\vec{v})(\vec{t})) + T_A\vec{z})\Phi(\lambda,\vec{z})d\vec{z}$$

where $\Gamma(t,A)$ and $\Phi(\lambda,\vec{z})$ are given by (2.4) and (2.5), respectively. Since

$$\begin{aligned} &\|\vec{z}\|_{\mathbb{R}^r}^2 - (t-t_n)\Gamma(t,A)\langle\vec{z},(\mathcal{P}\vec{v})(t)A^{-1}\rangle_{\mathbb{R}^r}^2 \\ &=\Gamma(t,A)[\|\vec{z}\|_{\mathbb{R}^r}^2 + (t-t_n)[\|\vec{z}\|_{\mathbb{R}^r}^2\|(\mathcal{P}\vec{v})(t)A^{-1}\|_{\mathbb{R}^r}^2 - \langle\vec{z},(\mathcal{P}\vec{v})(t)A^{-1}\rangle_{\mathbb{R}^r}^2]] \\ &\ge\Gamma(t,A)\|\vec{z}\|_{\mathbb{R}^r}^2 \end{aligned} \tag{2.10}$$

by the Cauchy-Schwarz's inequality, we have

$$|\Phi(\lambda,\vec{z})| \le \left(\frac{|\lambda|}{2\pi}\right)^{\frac{r}{2}} \exp\left\{-\frac{\Gamma(t,A)\mathrm{Re}\lambda}{2}\|\vec{z}\|_{\mathbb{R}^r}^2\right\} \le \left(\frac{|\lambda|}{2\pi}\right)^{\frac{r}{2}} \tag{2.11}$$

for any $\lambda \in \mathbb{C}_+$ and $\vec{z} \in \mathbb{R}^r$. Now, by the Morera's theorem with aids of Hölder's inequality and the dominated convergence theorem, we have

(2.9) for $\lambda \in \mathbb{C}_+$. To prove $T_\lambda[F_r|X_n](\cdot,\vec{\xi}_n) \in \mathcal{A}_r^{(p)}$, let $\lambda \in \mathbb{C}_+$ and for $\vec{u} \in \mathbb{R}^r$ let

$$\gamma(\vec{u}) = (\Gamma(t,A))^{\frac{1}{2}} \int_{\mathbb{R}^r} f_r(\vec{u} + (\vec{\xi}_n, (\mathcal{P}\vec{v})(\vec{t})) + T_A\vec{z})\Phi(\lambda,\vec{z})d\vec{z}.$$

Then we have

$$\begin{aligned}\gamma(\vec{u}) &= (\Gamma(t,A))^{\frac{1}{2}} \int_{\mathbb{R}^r} f_r(T_A(((\vec{\xi}_n, (\mathcal{P}\vec{v})(\vec{t})) + \vec{u})A^{-1} - \vec{z}))\Phi(\lambda,\vec{z})d\vec{z} \\ &= (\Gamma(t,A))^{\frac{1}{2}}(f_r(T_A\cdot) * \Phi(\lambda,\cdot))(((\vec{\xi}_n, (\mathcal{P}\vec{v})(\vec{t})) + \vec{u})A^{-1}).\end{aligned}$$

By the change of variable theorem

$$\int_{\mathbb{R}^r} |f_r(T_A\vec{u})|^p d\vec{u} = |\det(A^{-1})| \int_{\mathbb{R}^r} |f_r(\vec{u})|^p d\vec{u} < \infty \tag{2.12}$$

if $1 \le p < \infty$ so that $f_r(T_A\cdot)$ is in $L_p(\mathbb{R}^r)$. Since $\Phi(\lambda,\cdot) \in L_1(\mathbb{R}^r)$, we have $f_r(T_A\cdot) * \Phi(\lambda,\cdot) \in L_p(\mathbb{R}^r)$ for $1 \le p \le \infty$ by the Young's inequality in [10, p.232]. Now $\gamma = (\Gamma(t,A))^{\frac{1}{2}}(f_r(T_A\cdot) * \Phi(\lambda,\cdot))(((\vec{\xi}_n, (\mathcal{P}\vec{v})(\vec{t})) + \cdot)A^{-1}) \in L_p(\mathbb{R}^r)$ by the change of variable theorem which completes the proof. □

From Theorem 3.2 of [6], we have the following theorem.

Theorem 2.4. *Let X_n and $F_r \in \mathcal{A}_r^{(1)}$ be given by (1.2) and (2.6), respectively. Then for a nonzero real q, w_φ-a.e. $y \in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $T_q^{(1)}[F_r|X_n](y,\vec{\xi}_n)$ exists and it is given by (2.9) replacing λ by $-iq$. Furthermore, as a function of y, $T_q^{(1)}[F_r|X_n](\cdot,\vec{\xi}_n) \in \mathcal{A}_r^{(\infty)}$.*

If $\{v_1, v_2, \cdots, v_r\}$ is an orthonormal subset of $V^\perp$, then $\mathcal{P}^\perp v_l = v_l$ and $\mathcal{P}v_l = 0$ for $l = 1, \cdots, r$ so that $(\mathcal{P}\vec{v})(t) = 0$. Furthermore, A is the identity matrix, $(\vec{\xi}_n, (\mathcal{P}\vec{v})(\vec{t})) = \vec{0} \in \mathbb{R}^r$ and $\Gamma(t,A) = 1$. Hence we have the following theorem by Theorems 1.1, 2.3 and 2.4, and Lemmas 1.1 and 1.2 of [13].

Theorem 2.5. *Let $\{e_1, e_2, \cdots, e_r\}$ be an orthonormal subset of $V^\perp$. Let X_n be given by (1.2) and $F_r \in \mathcal{A}_r^{(p)}(1 \le p \le 2)$ be given by (2.6) replacing $\{v_1, \cdots, v_r\}$ by $\{e_1, \cdots, e_r\}$. Then for a nonzero real q, w_φ-a.e. $y \in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $T_q^{(p)}[F_r|X_n](y,\vec{\xi}_n)$ exists and it is given by*

$$T_q^{(p)}[F_r|X_n](y,\vec{\xi}_n) = (f_r * \Psi(-iq,\cdot))(\vec{e},y)$$

where $(\vec{e},y)=((e_1,y),\cdots,(e_r,y))$ *and* $\Psi(\lambda,\vec{z})=(\frac{\lambda}{2\pi})^{\frac{r}{2}}\exp\{-\frac{\lambda}{2}\|\vec{z}\|^2_{\mathbb{R}^r}\}$ *for* $\lambda\in\mathbb{C}_+$ *or* $\lambda=-iq$. *Furthermore, as a function of* y, $T_q^{(p)}[F_r|X_n](\cdot,\vec{\xi}_n)\in\mathcal{A}_r^{(p')}$, *where* $\frac{1}{p}+\frac{1}{p'}=1$ *if* $1<p\le 2$ *and* $p'=\infty$ *if* $p=1$.

Remark 2.6. *An example of the orthonormal subset* $\{e_1,\cdots,e_r\}$ *of* $V^{\perp}$ *is given by* [9, Remark 2.3].

Theorem 2.7. *Let* X_n *and* $F_r\in\mathcal{A}_r^{(p)}(1\le p\le\infty)$ *be given by (1.2) and (2.6), respectively. For* w_φ-*a.e.* $y\in C[0,t]$ *and* P_{X_n}-*a.e.* $\vec{\xi}_n,\vec{\zeta}_n\in\mathbb{R}^{n+1}$, *let* $F_{r1}(y,\vec{\xi}_n,\vec{\zeta}_n)=f_r((\vec{v},y)+(\vec{\xi}_n+\vec{\zeta}_n,(\mathcal{P}\vec{v})(\vec{t})))$ *where* $(\vec{\xi}_n+\vec{\zeta}_n,(\mathcal{P}\vec{v})(\vec{t}))$ *is given by (2.3) replacing* $\vec{\xi}_n$ *by* $\vec{\xi}_n+\vec{\zeta}_n$. *Then for a nonzero real* q, *we have*

$$\int_C\bigg|T_{\overline{\lambda}}[T_\lambda[F_r|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)\\ -(\Gamma(t,A))^{\frac{1}{2}}F_{r1}(y,\vec{\xi}_n,\vec{\zeta}_n)\int_{\mathbb{R}^r}\Phi(1,\vec{z})d\vec{z}\bigg|^p dw_\varphi(y)\to 0$$

for $1\le p<\infty$ *and for* $1\le p\le\infty$

$$T_{\overline{\lambda}}[T_\lambda[F_r|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)\longrightarrow(\Gamma(t,A))^{\frac{1}{2}}F_{r1}(y,\vec{\xi}_n,\vec{\zeta}_n)\int_{\mathbb{R}^r}\Phi(1,\vec{z})d\vec{z}$$

as λ *approaches to* $-iq$ *through* $\mathbb{C}_+$, *where* $\Gamma(t,A)$ *and* $\Phi(1,\vec{z})$ *are given by (2.4) and (2.5), respectively.*

Proof. Note that $T_{\overline{\lambda}}[T_\lambda[F_r|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)$ is well-defined by Theorem 2.3. For $\lambda\in\mathbb{C}_+$, w_φ-a.e. $y\in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n,\vec{\zeta}_n\in\mathbb{R}^{n+1}$, we have by Theorem 3.3 in [6]

$$\begin{aligned}&T_{\overline{\lambda}}[T_\lambda[F_r|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)\\ =\ &(\Gamma(t,A))^{\frac{1}{2}}\int_{\mathbb{R}^r}f_r(T_A(((\vec{\xi}_n+\vec{\zeta}_n,(\mathcal{P}\vec{v})(\vec{t}))+(\vec{v},y))A^{-1}-\vec{z}))\\ &\times\Phi\bigg(\frac{|\lambda|^2}{2\mathrm{Re}\lambda},\vec{z}\bigg)d\vec{z}\end{aligned}$$

where $\Gamma(t,A)$ and $\Phi(\frac{|\lambda|^2}{2\mathrm{Re}\lambda},\vec{z})$ are given by (2.4) and (2.5), respectively. Let $\kappa=\int_{\mathbb{R}^r}\Phi(1,\vec{z})d\vec{z}$, $\Phi_1(\vec{z})=\kappa^{-1}\Phi(1,\vec{z})$ for $\vec{z}\in\mathbb{R}^r$ and let $\epsilon=$

$(\frac{2\mathrm{Re}\lambda}{|\lambda|^2})^{\frac{1}{2}} > 0$. Then

$$\begin{aligned}
&\kappa^{-1}(\Gamma(t,A))^{-\frac{1}{2}} T_{\overline{\lambda}}[T_\lambda[F_r|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n) \\
= \ & \epsilon^{-r}\kappa^{-1}\int_{\mathbb{R}^r} f_r(T_A(((\vec{\xi}_n+\vec{\zeta}_n,(\mathcal{P}\vec{v})(\vec{t}))+(\vec{v},y))A^{-1}-\vec{z}))\Phi\bigg(1,\frac{\vec{z}}{\epsilon}\bigg)d\vec{z} \\
= \ & \epsilon^{-r}\bigg(f_r(T_A\cdot)*\Phi_1\bigg(\frac{\cdot}{\epsilon}\bigg)\bigg)(((\vec{\xi}_n+\vec{\zeta}_n,(\mathcal{P}\vec{v})(\vec{t}))+(\vec{v},y))A^{-1}).
\end{aligned}$$

Clearly, we have $\Phi(1,\cdot)\in L_1(\mathbb{R}^r)$ by (2.11) and $\int_{\mathbb{R}^r}\Phi_1(\vec{z})d\vec{z}=1$. Furthermore, we have $f_r(T_A\cdot)\in L_p(\mathbb{R}^r)\,(1\le p\le\infty)$ by (2.12). Now we have by Theorem 1.1, Theorem 1.18 of [18] and the change of variable theorem

$$\begin{aligned}
&\int_C\bigg|T_{\overline{\lambda}}[T_\lambda[F_r|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n) \\
&-(\Gamma(t,A))^{\frac{1}{2}}F_{r1}(y,\vec{\xi}_n,\vec{\zeta}_n)\int_{\mathbb{R}^r}\Phi(1,\vec{z})d\vec{z}\bigg|^p dw_\varphi(y) \\
= \ & \kappa^p(\Gamma(t,A))^{\frac{p}{2}}\int_C|\kappa^{-1}(\Gamma(t,A))^{-\frac{1}{2}}T_{\overline{\lambda}}[T_\lambda[F_r|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n) \\
&-F_{r1}(y,\vec{\xi}_n,\vec{\zeta}_n)|^p dw_\varphi(y) \\
= \ & \kappa^p(\Gamma(t,A))^{\frac{p}{2}}\int_C\bigg|\epsilon^{-r}\bigg(f_r(T_A\cdot)*\Phi_1\bigg(\frac{\cdot}{\epsilon}\bigg)\bigg)(((\vec{\xi}_n+\vec{\zeta}_n,(\mathcal{P}\vec{v})(\vec{t})) \\
&+(\vec{v},y))A^{-1})-f_r(T_A((\vec{\xi}_n+\vec{\zeta}_n,(\mathcal{P}\vec{v})(\vec{t}))+(\vec{v},y))A^{-1})\bigg|^p dw_\varphi(y) \\
\le \ & \kappa^p|\det(A)|(\Gamma(t,A))^{\frac{p}{2}}\bigg(\frac{1}{2\pi}\bigg)^{\frac{r}{2}}\int_{\mathbb{R}^r}\bigg|\epsilon^{-r}\bigg(f_r(T_A\cdot)*\Phi_1\bigg(\frac{\cdot}{\epsilon}\bigg)\bigg)(\vec{u}) \\
&-f_r(T_A\vec{u}))\bigg|^p d\vec{u}\longrightarrow 0
\end{aligned}$$

as λ approaches $-iq$ through $\mathbb{C}_+$ if $1\le p<\infty$. Let $1\le p\le\infty$. By (2.11), we have

$$\begin{aligned}
0 \ &\le \ \psi(\vec{u})\equiv \operatorname{ess.sup}\{|\Phi_1(\vec{z})|:\|\vec{z}\|_{\mathbb{R}^r}\ge\|\vec{u}\|_{\mathbb{R}^r}\} \\
&\le \ \kappa^{-1}\bigg(\frac{1}{2\pi}\bigg)^{\frac{r}{2}}\exp\bigg\{-\frac{\Gamma(t,A)}{2}\|\vec{u}\|^2_{\mathbb{R}^r}\bigg\}
\end{aligned}$$

so that $\psi(\vec{u})$ is an L_1-function of $\vec{u}$. Consequently, we have by Theorem 1.25 of [18]

$$\begin{aligned}
&\lim_{\lambda\to -iq} T_{\overline{\lambda}}[T_\lambda[F_r|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)\\
=\ &\kappa(\Gamma(t,A))^{\frac{1}{2}}\lim_{\epsilon\to 0}\epsilon^{-r}\left(f_r(T_A\cdot)*\Phi_1\left(\frac{\cdot}{\epsilon}\right)\right)(((\vec{v},y)\\
&+(\vec{\xi}_n+\vec{\zeta}_n,(\mathcal{P}\vec{v})(\vec{t})))A^{-1})\\
=\ &(\Gamma(t,A))^{\frac{1}{2}}F_{r1}(y,\vec{\xi}_n,\vec{\zeta}_n)\int_{\mathbb{R}^r}\Phi(1,\vec{z})d\vec{z}
\end{aligned}$$

which completes the proof. □

3. A time-independent conditional convolution product

In this section we evaluate the time-independent conditional convolution product of the cylinder functions with the conditioning function X_n given by (1.2).

Definition 3.1. *Let X_n be given by (1.2), and F and G be defined on $C[0,t]$. Define the conditional convolution product $[(F*G)_\lambda|X_n]$ of F and G given X_n by the formula, for w_φ-a.e. $y\in C[0,t]$,*

$$\begin{aligned}
&[(F*G)_\lambda|X_n](y,\vec{\xi}_n)\\
=\ &\begin{cases} E^{anw_\lambda}\left[F\left(\dfrac{y+\cdot}{\sqrt{2}}\right)G\left(\dfrac{y-\cdot}{\sqrt{2}}\right)\middle|X_n\right](\vec{\xi}_n), & \lambda\in\mathbb{C}_+;\\ E^{anf_q}\left[F\left(\dfrac{y+\cdot}{\sqrt{2}}\right)G\left(\dfrac{y-\cdot}{\sqrt{2}}\right)\middle|X_n\right](\vec{\xi}_n), & \lambda=-iq;\ q\in\mathbb{R}-\{0\}\end{cases}
\end{aligned}$$

if they exist for P_{X_n}-a.e. $\vec{\xi}_n\in\mathbb{R}^{n+1}$. If $\lambda=-iq$, we replace $[(F*G)_\lambda|X_n]$ by $[(F*G)_q|X_n]$.

Theorem 3.2. *Let $F_r\in\mathcal{A}_r^{(p_1)}$, $G_r\in\mathcal{A}_r^{(p_2)}$ and f_r, g_r be related by (2.6), respectively, where $1\le p_1,p_2\le\infty$. Furthermore, let $\frac{1}{p_1}+\frac{1}{p_1'}=1$, $\frac{1}{p_2}+\frac{1}{p_2'}=1$ and X_n be given by (1.2). Then for $\lambda\in\mathbb{C}_+$, w_φ-a.e. $y\in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n\in\mathbb{R}^{n+1}$, $[(F_r*G_r)_\lambda|X_n](y,\vec{\xi}_n)$ exists and it is given by*

$$[(F_r*G_r)_\lambda|X_n](y,\vec{\xi}_n)=H(\lambda,k_{f_r,g_r}(y),\vec{\xi}_n)$$

where $k_{f_r,g_r}(y)(\vec{u})=f_r(\frac{1}{\sqrt{2}}[(\vec{v},y)+\vec{u}])g_r(\frac{1}{\sqrt{2}}[(\vec{v},y)-\vec{u}])$ for $\vec{u}\in\mathbb{R}^r$ and H is given by (2.8). Furthermore, as functions of y, $[(F_r*G_r)_\lambda|X_n](\cdot,\vec{\xi}_n)\in\mathcal{A}_r^{(1)}$ if either $p_2\le p_1'$ or $p_1\le p_2'$, $[(F_r*G_r)_\lambda|X_n](\cdot,\vec{\xi}_n)\in\mathcal{A}_r^{(p_2)}$ if $p_2\ge p_1'$ and $[(F_r*G_r)_\lambda|X_n](\cdot,\vec{\xi}_n)\in\mathcal{A}_r^{(p_1)}$ if $p_1\ge p_2'$.

Proof. Using the same method as used in the proof of Theorem 3.4 of [6], for $\lambda > 0$, w_φ-a.e. $y \in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi_n} \in \mathbb{R}^{n+1}$,

$$\begin{aligned} &[(F_r * G_r)_\lambda | X_n](y, \vec{\xi_n}) \\ = \;& (\Gamma(t,A))^{\frac{1}{2}} \int_{\mathbb{R}^r} f_r\bigg(\frac{1}{\sqrt{2}}[(\vec{v}, y) + (\vec{\xi_n}, (\mathcal{P}\vec{v})(\vec{t})) + T_A \vec{z}]\bigg) \\ & \times g_r\bigg(\frac{1}{\sqrt{2}}[(\vec{v}, y) - (\vec{\xi_n}, (\mathcal{P}\vec{v})(\vec{t})) - T_A \vec{z}]\bigg)\Phi(\lambda, \vec{z}) d\vec{z} \end{aligned}$$

where $(\vec{\xi_n}, (\mathcal{P}\vec{v})(\vec{t}))$, $\Gamma(t,A)$ and $\Phi(\lambda, \vec{z})$ are given by (2.3), (2.4) and (2.5), respectively. Now, let $\lambda \in \mathbb{C}_+$ and for $\vec{u} \in \mathbb{R}^r$, let

$$\begin{aligned} (3.1)\quad \gamma_1(\vec{u}) \;=\; & (\Gamma(t,A))^{\frac{1}{2}} \int_{\mathbb{R}^r} f_r\bigg(\frac{1}{\sqrt{2}}[\vec{u} + (\vec{\xi_n}, (\mathcal{P}\vec{v})(\vec{t})) + T_A \vec{z}]\bigg) \\ & \times g_r\bigg(\frac{1}{\sqrt{2}}[\vec{u} - (\vec{\xi_n}, (\mathcal{P}\vec{v})(\vec{t})) - T_A \vec{z}]\bigg)\Phi(\lambda, \vec{z}) d\vec{z} \end{aligned}$$

formally and suppose that $p_2 \le p_1'$. Since $0 < \Gamma(t,A) \le 1$, we have by the change of variable theorem

$$\int_{\mathbb{R}^r} |\gamma_1(\vec{u})| d\vec{u} \le |\det(A^{-1})| \int_{\mathbb{R}^r} |f_{r1}(\vec{p})| (|g_{r1}| * |\Phi_1|)(\vec{p}) d\vec{p}$$

where $f_{r1}(\vec{p}) = f_r(\vec{p} + \frac{1}{\sqrt{2}}(\vec{\xi_n}, (\mathcal{P}\vec{v})(\vec{t})))$, $g_{r1}(\vec{p}) = g_r(\vec{p} - \frac{1}{\sqrt{2}}(\vec{\xi_n}, (\mathcal{P}\vec{v})(\vec{t})))$ and $\Phi_1(\vec{p}) = \Phi(\lambda, \frac{1}{\sqrt{2}}\vec{p}A^{-1})$. Now let $\frac{1}{p_2} + \frac{1}{q} = \frac{1}{p_1'} + 1$ with $1 \le q \le \infty$. By the change of variable theorem, we have for $1 \le q < \infty$

$$\int_{\mathbb{R}^r} |\Phi_1(\vec{p})|^q d\vec{p} \le |\det(A)| \bigg(\frac{|\lambda|}{2\pi}\bigg)^{\frac{qr}{2}} \int_{\mathbb{R}^r} \exp\bigg\{-\frac{q\Gamma(t,A)\mathrm{Re}\,\lambda}{4}\|\vec{z}\|_{\mathbb{R}^r}^2\bigg\} d\vec{z} < \infty$$

by (2.10) and (2.11) so that $\Phi_1 \in L_q(\mathbb{R}^r)$ for $1 \le q \le \infty$. Now by the general form of Young's inequality [10, Theorem 8.9] and Hölder's inequality,

$$\int_{\mathbb{R}^r} |\gamma_1(\vec{u})| d\vec{u} \le |\det(A^{-1})| \|f_{r1}\|_{p_1} \|g_{r1}\|_{p_2} \|\Phi_1\|_q < \infty$$

which shows that $\gamma_1 \in L_1(\mathbb{R}^r)$ and hence $[(F_r * G_r)_\lambda | X_n](\cdot, \vec{\xi_n}) \in \mathcal{A}_r^{(1)}$. Similarly, $[(F_r * G_r)_\lambda | X_n](\cdot, \vec{\xi_n}) \in \mathcal{A}_r^{(1)}$ if $p_1 \le p_2'$. Suppose that $p_1' \le p_2$. Then, by Hölder's inequality, Young's inequality and the change of variable theorem, we can prove

$$\int_{\mathbb{R}^r} |\gamma_1(\vec{u})|^{p_2} d\vec{u} \le |\det(A)| [|\det(A^{-1})| 2^{\frac{r}{2}}]^{\frac{p_2}{p_1}+1} \|f_r\|_{p_1}^{p_2} \|\Phi(\lambda, \cdot)\|_{p_1'}^{p_2} \|g_r\|_{p_2}^{p_2} < \infty$$

if $1 < p_1' \le p_2 < \infty$ and

$$\int_{\mathbb{R}^r} |\gamma_1(\vec{u})|^{p_2} d\vec{u} \le 2^{\frac{r}{2}} \|f_r\|_\infty^{p_2} \|\Phi(\lambda,\cdot)\|_1^{p_2} \|g_r\|_{p_2}^{p_2} < \infty$$

if $1 = p_1' \le p_2 < \infty$. Furthermore, we have for $\vec{u} \in \mathbb{R}^r$

$$|\gamma_1(\vec{u})| \le \|g_r\|_\infty [|\det(A^{-1})| 2^{\frac{r}{2}}]^{\frac{1}{p_1}} \|f_r\|_{p_1} \|\Phi(\lambda,\cdot)\|_{p_1'}$$

if $1 < p_1' \le p_2 = \infty$ and

$$|\gamma_1(\vec{u})| \le \|g_r\|_\infty \|f_r\|_\infty \|\Phi(\lambda,\cdot)\|_1$$

if $1 = p_1'$ and $p_2 = \infty$. Now we have $\gamma_1 \in L_{p_2}(\mathbb{R}^r)$ so that $[(F_r * G_r)_\lambda | X_n](\cdot, \vec{\xi}_n) \in \mathcal{A}_r^{(p_2)}$. Similarly, we can prove $[(F_r * G_r)_\lambda | X_n](\cdot, \vec{\xi}_n) \in \mathcal{A}_r^{(p_1)}$ if $p_1 \ge p_2'$. Note that the existence of $[(F_r * G_r)_\lambda | X_n]$ follows from the dominated convergence theorem and Morera's theorem. The theorem now follows. □

Theorem 3.3. *Let X_n be given by (1.2) and q be a nonzero real number. Then for $\lambda \in \mathbb{C}_+$ or $\lambda = -iq$, and $\vec{\xi}_n \in \mathbb{R}^{n+1}$, we have the followings:*

(1) if $F_r \in A_r^{(1)}$ and $G_r \in A_r^{(1)}$, then $[(F_r * G_r)_\lambda | X_n](\cdot, \vec{\xi}_n) \in A_r^{(1)}$,

(2) if $F_r \in A_r^{(2)}$ and $G_r \in A_r^{(2)}$, then $[(F_r * G_r)_\lambda | X_n](\cdot, \vec{\xi}_n) \in A_r^{(\infty)}$,

(3) if $F_r \in A_r^{(1)}$ and $G_r \in A_r^{(2)}$, then $[(F_r * G_r)_\lambda | X_n](\cdot, \vec{\xi}_n) \in A_r^{(2)}$,

(4) if $F_r \in A_r^{(1)}$ and $G_r \in A_r^{(1)} \cap A_r^{(2)}$, then $[(F_r * G_r)_\lambda | X_n](\cdot, \vec{\xi}_n) \in A_r^{(1)} \cap A_r^{(2)}$, and

(5) if $F_r \in A_r^{(1)}$ and $G_r \in A_r^{(\infty)}$, then $[(F_r * G_r)_\lambda | X_n](\cdot, \vec{\xi}_n) \in A_r^{(\infty)}$.

Proof. Let F_r, G_r and f_r, g_r be related by (2.6), respectively.

(1) The result follows from Theorem 3.4 of [6].

(2) For $\lambda \in \mathbb{C}_+$ or $\lambda = -iq$ let γ_1 be given by (3.1). Then it is not difficult to show that for $\vec{u} \in \mathbb{R}^r$

$$|\gamma_1(\vec{u})| \le 2^{\frac{r}{2}} |\det(A^{-1})| \|\Phi(\lambda,\cdot)\|_\infty \|f_r\|_2 \|g_r\|_2 < \infty$$

by Hölder's inequality and the change of variable theorem. By the dominated convergence theorem, $[(F_r * G_r)_q | X_n]$ exists and the result follows.

(3) For $\lambda \in \mathbb{C}_+$ or $\lambda = -iq$ let γ_1 be given by (3.1). Then we have by the change of variable theorem and Hölder's inequality

$$\int_{\mathbb{R}^r} |\gamma_1(\vec{u})|^2 d\vec{u} \le 2^{\frac{r}{2}} |\det(A^{-1})|^2 \|\Phi(\lambda,\cdot)\|_\infty^2 \|f_r\|_1^2 \|g_r\|_2^2 < \infty$$

so that the result follows.

(4) The result follows from (1) and (3).

(5) It follows immediately from $F_r \in \mathcal{A}_r^{(1)}$ and the dominated convergence theorem. □

Now applying the same method as used in the proof of Theorem 4.2 of [6], we have the following theorem from Theorems 2.3 and 3.2.

Theorem 3.4. *Let X_n be given by (1.2) and $F_r, G_r \in \cup_{1\le p\le\infty} A_r^{(p)}$ be given by (2.6). Then for $\lambda \in \mathbb{C}_+$, w_φ-a.e. $y \in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n, \vec{\zeta}_n \in \mathbb{R}^{n+1}$, we have*

$$\begin{aligned}&T_\lambda[[(F_r * G_r)_\lambda|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)\\ =\ &\left[T_\lambda[F_r|X_n]\left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_n+\vec{\xi}_n)\right)\right]\left[T_\lambda[G_r|X_n]\left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_n-\vec{\xi}_n)\right)\right].\end{aligned}$$

We have the following relationships between the conditional Fourier-Feynman transform and the conditional convolution product from Theorems 2.5, 3.3, 3.4 and Theorem 4.2 of [6].

Theorem 3.5. *Let X_n be given by (1.2) and q be a nonzero real. Then we have the followings:*

(1) if $F_r, G_r \in A_r^{(1)}$ are given by (2.6), then we have for w_φ-a.e. $y \in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n, \vec{\zeta}_n \in \mathbb{R}^{n+1}$,

$$\begin{aligned}&T_q^{(1)}[[(F_r * G_r)_q|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)\\ =\ &\left[T_q^{(1)}[F_r|X_n]\left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_n+\vec{\xi}_n)\right)\right]\left[T_q^{(1)}[G_r|X_n]\left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_n-\vec{\xi}_n)\right)\right],\end{aligned}$$

(2) if $F_r \in A_r^{(1)}$ and $G_r \in A_r^{(2)}$ are given by (2.6) where $\{v_1, \cdots, v_r\} \subset V^\perp$, then we have for w_φ-a.e. $y \in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n, \vec{\zeta}_n \in \mathbb{R}^{n+1}$,

$$\begin{aligned}&T_q^{(2)}[[(F_r * G_r)_q|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)\\ =\ &\left[T_q^{(1)}[F_r|X_n]\left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_n+\vec{\xi}_n)\right)\right]\left[T_q^{(2)}[G_r|X_n]\left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_n-\vec{\xi}_n)\right)\right].\end{aligned}$$

4. Evaluation formulas for bounded cylinder functions

Let $\hat{\mathrm{M}}(\mathbb{R}^r)$ be the set of all functions ϕ on $\mathbb{R}^r$ defined by

$$\phi(\vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle\vec{u},\vec{z}\rangle_{\mathbb{R}^r}\}d\rho(\vec{z}), \tag{4.1}$$

where ρ is a complex Borel measure of bounded variation over $\mathbb{R}^r$. For w_φ-a.e. $x \in C[0,t]$, let Φ_2 be given by

$$\Phi_2(x) = \phi(\vec{v}, x) \tag{4.2}$$

where ϕ is given by (4.1).

Now we have the following theorem.

Theorem 4.1. *Let* $1 \le p \le \infty$, A^T *be the transpose of* A *and* $T_{A^T}\vec{u} = \vec{u}A^T$ *for* $\vec{u} \in \mathbb{R}^r$. *Let* X_n *and* Φ_2 *be given by (1.2) and (4.2), respectively. Then for* $\lambda \in \mathbb{C}_+$, w_φ*-a.e.* $y \in C[0,t]$ *and* P_{X_n}*-a.e.* $\vec{\xi_n} \in \mathbb{R}^{n+1}$, $T_\lambda[\Phi_2|X_n](y,\vec{\xi_n})$ *exists and it is given by*

$$
(4.3)\quad \begin{aligned} T_\lambda[\Phi_2|X_n](y,\vec{\xi_n}) = \int_{\mathbb{R}^r} \exp\bigg\{ & i\langle(\vec{v},y),\vec{u}\rangle_{\mathbb{R}^r} + i\langle(\vec{\xi_n},(\mathcal{P}\vec{v})(\vec{t})),\vec{u}\rangle_{\mathbb{R}^r} \\ & - \frac{1}{2\lambda}[\|T_{A^T}\vec{u}\|^2_{\mathbb{R}^r} + (t-t_n)\langle(\mathcal{P}\vec{v})(t),\vec{u}\rangle^2_{\mathbb{R}^r}]\bigg\} d\rho(\vec{u}) \end{aligned}
$$

where $(\mathcal{P}\vec{v})(t)$ *and* $(\vec{\xi_n},(\mathcal{P}\vec{v})(\vec{t}))$ *are given by (2.2) and (2.3), respectively. For nonzero real* q, w_φ*-a.e.* $y \in C[0,t]$ *and* P_{X_n}*-a.e.* $\vec{\xi_n} \in \mathbb{R}^{n+1}$, $T_q^{(p)}[F|X_n](y,\vec{\xi_n})$ *also exists and it is given by (4.3) replacing* λ *by* $-iq$. *Furthermore, as a function of* y, $T_q^{(p)}[\Phi_2|X_n](\cdot,\vec{\xi_n}) \in \mathcal{A}_r^{(\infty)}$.

Proof. For $\vec{\xi_n} = (\xi_0,\xi_1,\cdots,\xi_n) \in \mathbb{R}^{n+1}$, let $\vec{\xi}_{n+1} = (\xi_0,\xi_1,\cdots,\xi_n,\xi_{n+1})$. For $\lambda > 0$, w_φ-a.e. $y \in C[0,t]$ and $\vec{\xi_n} \in \mathbb{R}^{n+1}$, we have by Theorem 4.1 of [5]

$$
\begin{aligned}
& K^\lambda_{\Phi_2}(y,\vec{\xi_n}) \\
= & \left[\frac{\lambda}{2\pi(t-t_n)}\right]^{\frac12} \int_{\mathbb{R}}\int_{\mathbb{R}^r} \exp\bigg\{ i\langle(\vec{v},y)+(\vec{v},[\vec{\xi}_{n+1}]),\vec{u}\rangle_{\mathbb{R}^r} - \frac{1}{2\lambda}\|T_A\vec{u}\|^2_{\mathbb{R}^r} \\
& -\frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\bigg\} d\rho(\vec{u})d\xi_{n+1} \\
= & \left[\frac{\lambda}{2\pi(t-t_n)}\right]^{\frac12} \int_{\mathbb{R}}\int_{\mathbb{R}^r} \exp\bigg\{ i\langle(\vec{v},y)+(\vec{\xi_n},(\mathcal{P}\vec{v})(\vec{t})),\vec{u}\rangle_{\mathbb{R}^r} + i(\xi_{n+1}-\xi_n) \\
& \times\langle(\mathcal{P}\vec{v})(t),\vec{u}\rangle_{\mathbb{R}^r} - \frac{1}{2\lambda}\|T_A\vec{u}\|^2_{\mathbb{R}^r} - \frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\bigg\} d\rho(\vec{u})d\xi_{n+1} \\
= & \int_{\mathbb{R}^r} \exp\bigg\{ i\langle(\vec{v},y)+(\vec{\xi_n},(\mathcal{P}\vec{v})(\vec{t})),\vec{u}\rangle_{\mathbb{R}^r} - \frac{1}{2\lambda}[\|T_A\vec{u}\|^2_{\mathbb{R}^r} + (t-t_n) \\
& \times\langle(\mathcal{P}\vec{v})(t),\vec{u}\rangle^2_{\mathbb{R}^r}]\bigg\} d\rho(\vec{u})
\end{aligned}
$$

where the last equality follows from the well known integration formula

$$
(4.4)\quad \int_{\mathbb{R}} \exp\{-au^2+ibu\}du = \left(\frac{\pi}{a}\right)^{\frac12} \exp\left\{-\frac{b^2}{4a}\right\}
$$

for $a \in \mathbb{C}_+$ and any real b. By the analytic continuation, we have (4.3) for $\lambda \in \mathbb{C}_+$. For $p = 1$, the final result follows from the dominated convergence theorem. Now let $1 < p \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Further, let $T_q^{(p)}[\Phi_2|X_n](y, \vec{\xi}_n)$ be formally given by (4.3) replacing λ by $-iq$. Then we have

$$|T_\lambda[\Phi_2|X_n](y, \vec{\xi}_n) - T_q^{(p)}[\Phi_2|X_n](y, \vec{\xi}_n)|^{p'} \le (2\|\rho\|)^{p'}$$

so that by the dominated convergence theorem

$$\int_C |T_\lambda[\Phi_2|X_n](y, \vec{\xi}_n) - T_q^{(p)}[\Phi_2|X_n](y, \vec{\xi}_n)|^{p'} dw_\varphi(y)$$

converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$, which completes the proof. □

Theorem 4.2. *Let $1 \le p \le \infty$. Let X_n and Φ_2 be given by (1.2) and (4.2), respectively. For w_φ-a.e. $y \in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n, \vec{\zeta}_n \in \mathbb{R}^{n+1}$, let $\Phi_3(y, \vec{\xi}_n, \vec{\zeta}_n) = \phi((\vec{v}, y) + (\vec{\xi}_n + \vec{\zeta}_n, (\mathcal{P}\vec{v})(\vec{t})))$ where $(\vec{\xi}_n + \vec{\zeta}_n, (\mathcal{P}\vec{v})(\vec{t}))$ is given by (2.3) replacing $\vec{\xi}_n$ by $\vec{\xi}_n + \vec{\zeta}_n$. Then for a nonzero real q, we have*

$$\|T_{\overline{\lambda}}[T_\lambda[\Phi_2|X_n](\cdot, \vec{\xi}_n)|X_n](\cdot, \vec{\zeta}_n) - \Phi_3(\cdot, \vec{\xi}_n, \vec{\zeta}_n)\|_p \longrightarrow 0 \tag{4.5}$$

as λ approaches to $-iq$ through $\mathbb{C}_+$.

Proof. By Theorem 4.1, $T_{\overline{\lambda}}[T_\lambda[\Phi_2|X_n](\cdot, \vec{\xi}_n)|X_n](y, \vec{\zeta}_n)$ is well-defined so that we have for $\lambda \in \mathbb{C}_+$

$$\begin{aligned}
& T_{\overline{\lambda}}[T_\lambda[\Phi_2|X_n](\cdot, \vec{\xi}_n)|X_n](y, \vec{\zeta}_n) \\
= & \int_{\mathbb{R}^r} \exp\bigg\{ i\langle(\vec{v}, y) + (\vec{\zeta}_n, (\mathcal{P}\vec{v})(\vec{t})), \vec{u}\rangle_{\mathbb{R}^r} - \frac{1}{2\overline{\lambda}}[\|T_{A^T}\vec{u}\|^2_{\mathbb{R}^r} + (t - t_n) \\
& \times \langle(\mathcal{P}\vec{v})(t), \vec{u}\rangle^2_{\mathbb{R}^r}] + i\langle(\vec{\xi}_n, (\mathcal{P}\vec{v})(\vec{t})), \vec{u}\rangle_{\mathbb{R}^r} - \frac{1}{2\lambda}[\|T_{A^T}\vec{u}\|^2_{\mathbb{R}^r} + (t - t_n) \\
& \times \langle(\mathcal{P}\vec{v})(t), \vec{u}\rangle^2_{\mathbb{R}^r}] \bigg\} d\rho(\vec{u}) \\
= & \int_{\mathbb{R}^r} \exp\bigg\{ i\langle(\vec{v}, y) + (\vec{\xi}_n + \vec{\zeta}_n, (\mathcal{P}\vec{v})(\vec{t})), \vec{u}\rangle_{\mathbb{R}^r} - \frac{\operatorname{Re}\lambda}{|\lambda|^2}[\|T_{A^T}\vec{u}\|^2_{\mathbb{R}^r} \\
& + (t - t_n)\langle(\mathcal{P}\vec{v})(t), \vec{u}\rangle^2_{\mathbb{R}^r}] \bigg\} d\rho(\vec{u}).
\end{aligned}$$

Then we have

$$
\begin{aligned}
&|T_{\overline{\lambda}}[T_\lambda[\Phi_2|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)-\Phi_3(y,\vec{\xi}_n,\vec{\zeta}_n)|\\
=\ &\bigg|\int_{\mathbb{R}^r}\bigg[\exp\bigg\{i\langle(\vec{v},y)+(\vec{\xi}_n+\vec{\zeta}_n,(\mathcal{P}\vec{v})(\vec{t})),\vec{u}\rangle_{\mathbb{R}^r}-\frac{\operatorname{Re}\lambda}{|\lambda|^2}[\|T_{A^T}\vec{u}\|^2_{\mathbb{R}^r}+(t-\\
&t_n)\langle(\mathcal{P}\vec{v})(t),\vec{u}\rangle^2_{\mathbb{R}^r}]\bigg\}-\exp\{i\langle(\vec{v},y)+(\vec{\xi}_n+\vec{\zeta}_n,(\mathcal{P}\vec{v})(\vec{t})),\vec{u}\rangle_{\mathbb{R}^r}\}\bigg]d\rho(\vec{u})\bigg|\\
\le\ &\int_{\mathbb{R}^r}\bigg|\exp\bigg\{-\frac{\operatorname{Re}\lambda}{|\lambda|^2}[\|T_{A^T}\vec{u}\|^2_{\mathbb{R}^r}+(t-t_n)\langle(\mathcal{P}\vec{v})(t),\vec{u}\rangle^2_{\mathbb{R}^r}]\bigg\}-1\bigg|d|\rho|(\vec{u})
\end{aligned}
$$

so that the inequality is independent of y, and we have for $1\le p<\infty$

$$
\begin{aligned}
&\int_C|T_{\overline{\lambda}}[T_\lambda[\Phi_2|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)-\Phi_3(y,\vec{\xi}_n,\vec{\zeta}_n)|^p dw_\varphi(y)\\
\le\ &\bigg[\int_{\mathbb{R}^r}\bigg|\exp\bigg\{-\frac{\operatorname{Re}\lambda}{|\lambda|^2}[\|T_{A^T}\vec{u}\|^2_{\mathbb{R}^r}+(t-t_n)\langle(\mathcal{P}\vec{v})(t),\vec{u}\rangle^2_{\mathbb{R}^r}]\bigg\}-1\bigg|d|\rho|(\vec{u})\bigg]^p.
\end{aligned}
$$

Now we have (4.5) for $1\le p\le\infty$ as λ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem, which completes the proof. □

Theorem 4.3. *Let ϕ_4, ϕ_5 and ρ_4, ρ_5 be related by (4.1), respectively, and let $\Phi_4(x)=\phi_4(\vec{v},x)$ and $\Phi_5(x)=\phi_5(\vec{v},x)$ for w_φ-a.e. $x\in C[0,t]$. Furthermore, let X_n be given by (1.2). Then for $\lambda\in\mathbb{C}_+$, w_φ-a.e. $y\in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi}_n\in\mathbb{R}^{n+1}$, $[(\Phi_4*\Phi_5)_\lambda|X_n](y,\vec{\xi}_n)$ exists and it is given by*

$$
\begin{aligned}
&[(\Phi_4*\Phi_5)_\lambda|X_n](y,\vec{\xi}_n)\\
=\ &\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\exp\bigg\{\frac{i}{\sqrt{2}}[\langle(\vec{v},y),\vec{u}+\vec{w}\rangle_{\mathbb{R}^r}+\langle(\vec{\xi}_n,(\mathcal{P}\vec{v})(\vec{t})),\vec{u}-\vec{w}\rangle_{\mathbb{R}^r}]\\
&-\frac{1}{4\lambda}[\|T_{A^T}(\vec{u}-\vec{w})\|^2_{\mathbb{R}^r}+(t-t_n)\langle(\mathcal{P}\vec{v})(t),\vec{u}-\vec{w}\rangle^2_{\mathbb{R}^r}]\bigg\}d\rho_4(\vec{u})d\rho_5(\vec{w})
\end{aligned}
$$

where $(\mathcal{P}\vec{v})(t)$ and $(\vec{\xi}_n,(\mathcal{P}\vec{v})(\vec{t}))$ are given by (2.2) and (2.3), respectively, and T_{A^T} is as given in Theorem 4.1. For a nonzero real q, $[(\Phi_4*\Phi_5)_q|X_n](y,\vec{\xi}_n)$ is given by the right hand side of the above equality where λ is replaced by $-iq$. Furthermore, as a function of y, $[(\Phi_4*\Phi_5)_q|X_n](\cdot,\vec{\xi}_n)\in\mathcal{A}_r^{(\infty)}$.

Proof. For $\vec{\xi}_n=(\xi_0,\xi_1,\cdots,\xi_n)\in\mathbb{R}^{n+1}$, let $\vec{\xi}_{n+1}=(\xi_0,\xi_1,\cdots,\xi_n,\xi_{n+1})$. For $\lambda>0$, w_φ-a.e. $y\in C[0,t]$ and $\vec{\xi}_n\in\mathbb{R}^{n+1}$, we have by Theorem 4.3 of [5] and Fubini's theorem

$$
\begin{aligned}
&[(\Phi_4 * \Phi_5)_\lambda|X_n](y,\vec{\xi_n})\\
= &\Big[\frac{\lambda}{2\pi(t-t_n)}\Big]^{\frac{1}{2}}\int_{\mathbb{R}}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\exp\Big\{\frac{i}{\sqrt{2}}[\langle(\vec{v},y),\vec{u}+\vec{w}\rangle_{\mathbb{R}^r}+\langle(\vec{v},[\vec{\xi}_{n+1}]),\vec{u}\\
&-\vec{w}\rangle_{\mathbb{R}^r}]-\frac{1}{4\lambda}\|T_{A^T}(\vec{u}-\vec{w})\|^2_{\mathbb{R}^r}-\frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\Big\}d\rho_4(\vec{u})d\rho_5(\vec{w})d\xi_{n+1}\\
= &\Big[\frac{\lambda}{2\pi(t-t_n)}\Big]^{\frac{1}{2}}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{\mathbb{R}}\exp\Big\{\frac{i}{\sqrt{2}}[\langle(\vec{v},y),\vec{u}+\vec{w}\rangle_{\mathbb{R}^r}+\langle(\vec{\xi_n},(\mathcal{P}\vec{v})(\vec{t})),\\
&\vec{u}-\vec{w}\rangle_{\mathbb{R}^r}]+\frac{i}{\sqrt{2}}\langle(\mathcal{P}\vec{v})(t),\vec{u}-\vec{w}\rangle_{\mathbb{R}^r}(\xi_{n+1}-\xi_n)-\frac{1}{4\lambda}\|T_{A^T}(\vec{u}-\vec{w})\|^2_{\mathbb{R}^r}\\
&-\frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\Big\}d\xi_{n+1}d\rho_4(\vec{u})d\rho_5(\vec{w})\\
= &\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\exp\Big\{\frac{i}{\sqrt{2}}[\langle(\vec{v},y),\vec{u}+\vec{w}\rangle_{\mathbb{R}^r}+\langle(\vec{\xi_n},(\mathcal{P}\vec{v})(\vec{t})),\vec{u}-\vec{w}\rangle_{\mathbb{R}^r}]\\
&-\frac{1}{4\lambda}[\|T_{A^T}(\vec{u}-\vec{w})\|^2_{\mathbb{R}^r}+(t-t_n)\langle(\mathcal{P}\vec{v})(t),\vec{u}-\vec{w}\rangle^2_{\mathbb{R}^r}]\Big\}d\rho_4(\vec{u})d\rho_5(\vec{w})
\end{aligned}
$$

where the last equality follows from (4.4). By the dominated convergence theorem and Morera's theorem, we have the results. □

Now, we have the final theorem of our work.

Theorem 4.4. *Let X_n be given by (1.2), q be a nonzero real and $1 \le p \le \infty$. Furthermore, let Φ_4 and Φ_5 be as given in Theorem 4.3. Then we have for w_φ-a.e. $y \in C[0,t]$ and P_{X_n}-a.e. $\vec{\xi_n},\vec{\zeta_n} \in \mathbb{R}^{n+1}$*

$$
\begin{aligned}
&T_q^{(p)}[[(\Phi_4 * \Phi_5)_q|X_n](\cdot,\vec{\xi_n})|X_n](y,\vec{\zeta_n})\\
= &\Big[T_q^{(p)}[\Phi_4|X_n]\Big(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\vec{\zeta_n}+\vec{\xi_n})\Big)\Big]\Big[T_q^{(p)}[\Phi_5|X_n]\Big(\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}(\vec{\zeta_n}-\vec{\xi_n})\Big)\Big].
\end{aligned}
$$

Proof. For $\vec{\zeta_n} = (\zeta_0,\zeta_1,\cdots,\zeta_n) \in \mathbb{R}^{n+1}$, let $\vec{\zeta}_{n+1} = (\zeta_0,\zeta_1,\cdots,\zeta_n,\zeta_{n+1})$. For $\lambda > 0$, w_φ-a.e. $y \in C[0,t]$ and $\vec{\zeta_n} \in \mathbb{R}^{n+1}$, we have by Theorem 4.3

$$
\begin{aligned}
&T_\lambda[[(\Phi_4 * \Phi_5)_q|X_n](\cdot,\vec{\xi_n})|X_n](y,\vec{\zeta_n})\\
= &\Big[\frac{\lambda}{2\pi(t-t_n)}\Big]^{\frac{1}{2}}\Big(\frac{\lambda}{2\pi}\Big)^{\frac{r}{2}}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{\mathbb{R}}\int_{\mathbb{R}^r}\exp\Big\{\frac{i}{\sqrt{2}}[\langle(\vec{v},y)+(\vec{v},[\vec{\zeta}_{n+1}])\\
&+T_A\vec{z},\vec{u}+\vec{w}\rangle_{\mathbb{R}^r}+\langle(\vec{\xi_n},(\mathcal{P}\vec{v})(\vec{t})),\vec{u}-\vec{w}\rangle_{\mathbb{R}^r}]+\frac{1}{4qi}[\|T_{A^T}(\vec{u}-\vec{w})\|^2_{\mathbb{R}^r}
\end{aligned}
$$

$$+(t-t_n)\langle(\mathcal{P}\vec{v})(t),\vec{u}-\vec{w}\rangle_{\mathbb{R}^r}^2]-\frac{\lambda}{2}\|\vec{z}\|_{\mathbb{R}^r}^2-\frac{\lambda(\zeta_{n+1}-\zeta_n)^2}{2(t-t_n)}\Big\}d\vec{z}d\zeta_{n+1}$$
$$d\rho_4(\vec{u})d\rho_5(\vec{w})$$
$$= \int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\exp\Big\{\frac{i}{\sqrt{2}}[\langle(\vec{v},y)+(\vec{\zeta}_n+\vec{\xi}_n,(\mathcal{P}\vec{v})(\vec{t})),\vec{u}\rangle_{\mathbb{R}^r}+\langle(\vec{v},y)+(\vec{\zeta}_n-\vec{\xi}_n,$$
$$(\mathcal{P}\vec{v})(\vec{t})),\vec{w}\rangle_{\mathbb{R}^r}]+\frac{1}{4qi}[\|T_{A^T}(\vec{u}-\vec{w})\|_{\mathbb{R}^r}^2+(t-t_n)\langle(\mathcal{P}\vec{v})(t),\vec{u}-\vec{w}\rangle_{\mathbb{R}^r}^2]-$$
$$\frac{1}{4\lambda}[\|T_{A^T}(\vec{u}+\vec{w})\|_{\mathbb{R}^r}^2+(t-t_n)\langle(\mathcal{P}\vec{v})(t),\vec{u}+\vec{w}\rangle_{\mathbb{R}^r}^2]\Big\}d\rho_4(\vec{u})d\rho_5(\vec{w})$$

by using the same methods as used in the proof of Theorem 4.1 and Theorem 4.1 of [5]. Let $T_q^{(p)}[[(\Phi_4*\Phi_5)_q|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)$ be the right hand side of the last equality, where λ is replaced by $-iq$. The existence of $T_q^{(1)}[[(\Phi_4*\Phi_5)_q|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)$ follows from the dominated convergence theorem. Now let $1<p\le\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$. Then we have by the dominated convergence theorem

$$\int_C |T_\lambda[[(\Phi_4*\Phi_5)_q|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)$$
$$-T_q^{(p)}[[(\Phi_4*\Phi_5)_q|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)|^{p'}dw_\varphi(y)$$
$$\le \Big[\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\Big|\exp\Big\{-\frac{1}{4\lambda}[\|T_{A^T}(\vec{u}+\vec{w})\|_{\mathbb{R}^r}^2+(t-t_n)\langle(\mathcal{P}\vec{v})(t),\vec{u}+\vec{w}\rangle_{\mathbb{R}^r}^2]\Big\}$$
$$-\exp\Big\{\frac{1}{4qi}\|T_{A^T}(\vec{u}+\vec{w})\|_{\mathbb{R}^r}^2+(t-t_n)\langle(\mathcal{P}\vec{v})(t),\vec{u}+\vec{w}\rangle_{\mathbb{R}^r}^2]\Big\}\Big|d|\rho_4|(\vec{u})$$
$$d|\rho_5|(\vec{w})\Big]^{p'}\to 0$$

as λ approaches to $-iq$ through $\mathbb{C}_+$, which shows the existence of $T_q^{(p)}[[(\Phi_4*\Phi_5)_q|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)$. Now the equality in the theorem follows from Theorems 3.4, 4.1 and 4.3. □

Remark 4.5. *Without using Theorem 3.4, we can prove Theorem 4.4 with aids of Theorems 4.1 and 4.3.*

References

[1] Brue M. D., *A functional transform for Feynman integrals similar to the Fourier transform*, Thesis, Univ. of Minnesota, Minneapolis, 1972.

[2] Cameron R. H., Storvick D. A., *Some Banach algebras of analytic Feynman integrable functionals*, Lecture Notes in Mathematics 798, Springer, Berlin-New York, 1980.

[3] Chang K. S., Cho D. H., Kim B. S., Song T. S., Yoo I., *Conditional Fourier-Feynman transform and convolution product over Wiener paths in abstract Wiener space*, Integral Transforms Spec. Funct. **14(3)** (2003), 217-235.

[4] Chang S. J., Skoug D., *The effect of drift on conditional Fourier-Feynman transforms and conditional convolution products*, Int. J. Appl. Math. **2(4)** (2000), 505-527.

[5] Cho D. H., *A time-dependent conditional Fourier-Feynman transform and convolution product on an analogue of Wiener space*, Houston J. Math. 2012, submitted.

[6] Cho D. H., *Conditional integral transforms and conditional convolution products on a function space*, Integral Transforms Spec. Funct. **23(6)** (2012), 405-420.

[7] Cho D. H., *A simple formula for an analogue of conditional Wiener integrals and its applications II*, Czechoslovak Math. J. **59(2)** (2009), 431-452.

[8] Cho D. H., *Conditional Fourier-Feynman transform and convolution product over Wiener paths in abstract Wiener space: an L_p theory*, J. Korean Math. Soc. **41(2)** (2004), 265-294.

[9] Cho D. H., Kim B. J., Yoo I., *Analogues of conditional Wiener integrals and their change of scale transformations on a function space*, J. Math. Anal. Appl. **359** (2009), 421-438.

[10] Folland G. B., *Real analysis*, John Wiley & Sons, New York-Chichester-Brisbane, 1984.

[11] Huffman T., Park C., Skoug D., *Analytic Fourier-Feynman transforms and convolution*, Trans. Amer. Math. Soc. **347(2)** (1995), 661-673.

[12] Im M. K., Ryu K. S., *An analogue of Wiener measure and its applications*, J. Korean Math. Soc. **39(5)** (2002), 801-819.

[13] Johnson G. W., Skoug D. L., *The Cameron-Storvick function space integral: an $L(L_p, L_{p'})$-theory*, Nagoya Math. J. **60** (1976), 93-137.

[14] Kim M. J., *Conditional Fourier-Feynman transform and convolution product on a function space*, Int. J. Math. Anal. **3(10)** (2009), 457-471.

[15] Laha R. G., Rohatgi V. K., *Probability theory*, John Wiley & Sons, New York-Chichester-Brisbane, 1979.

[16] Ryu K. S., Im M. K., *A measure-valued analogue of Wiener measure and the measure-valued Feynman-Kac formula*, Trans. Amer. Math. Soc. **354(12)** (2002), 4921-4951.

[17] Ryu K. S., Im M. K., Choi K. S., *Survey of the theories for analogue of Wiener measure space*, Interdiscip. Inform. Sci. **15(3)** (2009), 319-337.

[18] Stein E. M., Weiss G., *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, 1971.

[19] Yeh J., *Stochastic processes and the Wiener integral*, Marcel Dekker, New York, 1973.

Dong Hyun Cho
Department of Mathematics, Kyonggi University,
Kyonggido Suwon 443-760, Korea.
E-mail: j94385@kyonggi.ac.kr