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MULTIPLICATION DEDEKIND MODULES

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Abstract. In this paper, we give some properties on multiplication Dedekind modules.

1. Introduction

Throughout this paper all rings will be commutative with identity and all modules will be unitary. Let M be an R-module, S the set of nonzero divisors of R and R_S the total quotient ring of R. For a nonzero ideal I of R, let $I^{-1} = \{x \in R_S | xI \subseteq R\}$. I is said to be an *invertible ideal* of R if $II^{-1} = R$. Put $T = \{t \in S | tm = 0$ for some $m \in M$ implies $m = 0\}$. Then T is a multiplicatively closed subset of S and if M is torsion free, then $T = S$ [8, Proposition 1.1]. An R module M is said to be *faithful* if Ann M $= [0 :_R M] = 0$. In particular, if M is a faithful multiplication module then M is torsion free by [3, Lemma 4.1] and so $T = S$. So in this case, $R_T = R_S$. Let N be a submodule of M. If $x = \frac{r}{t} \in R_T$ and $n \in N$, then we say that $xn \in M$ if there exists $m \in M$ such that $tm = rn$. Then this is a well defined operation [8, p.399]. For a submodule N of M, $N^{-1} = \{x \in R_T | xN \subseteq M\} = [M :_{R_T} N]$. We say that N is *invertible* in M if $NN^{-1} = M$ and M is called a *Dedekind* module providing that every nonzero submodule of M is invertible. An R-module M is said to be a D_1-*module* if non zero cyclic submodule of M is invertible. It is clear that every Dedekind module is a D_1-module.

An R-module M is called a *multiplication module* if every submodule N of M has the form IM for some ideal I of R. Note that $I \subseteq (N : M)$ and hence $N = IM \subseteq (N : M)M \subseteq N$, so that $N = (N : M)M$. A proper submodule N of an R-module M is called *prime* if whenever $rm \in N$ for some $r \in R$ and $m \in M$, then $m \in N$ or $rM \subseteq N$. An R-module

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M is called *cancellation* if for ideals I and J of R, $IM = JM$ implies $I = J$ [7]. Finitely generated faithful multiplication modules are cancellation [9, Corollary to Theorem 9]. In particular, faithful multiplication module over an integral domain is cancellation [5, Theorem 3.1].

In this paper we first extend Chisenese Remainder Theorem in ring theory to modules under some conditions and give some properties. Also, we define the product of submodules of a multiplication module and will give characterizations of faithful multiplication Dedekind modules.

2. Chinese Remainder Theorem in Module

In this section we extend Chinese Reminder Theorem in Ring to module. We begin with the following Lemma.

Lemma. *Let $I_1, \cdots, I_n$ be ideals of a ring R and M an R-module. Let $\phi : M \longrightarrow M/I_1M \bigoplus \cdots \bigoplus M/I_nM$ be given by $\phi(m) = (\phi_1(m), \cdots, \phi_n(m))$, where $\phi_i : M \longrightarrow M/I_iM$ is the natural R-module homomorphism. Then,*

(1) If $I_1, \cdots, I_n$ are comaximal then ϕ is surjective.

(2) If M is cancellation and ϕ is surjective then $I_1, \cdots, I_n$ are comaximal.

(3) ϕ is injective if and only if $\bigcap_{i=1}^n I_iM = 0$.

(4) If M is cancellation then ϕ is bijective if and only if $\bigcap_{i=1}^n I_iM = 0$ and $I_1, \cdots, I_n$ are comaximal.

Proof. (1) If $n = 1$, then it is clear. For $n = 2$, let $\phi : M \longrightarrow M/I_1M \bigoplus M/I_2M$. Let $(\overline{m_1}, \overline{m_2}) \in M/I_1M \bigoplus M/I_2M$. Since $I_1 + I_2 = R$, there exist $e_1 \in I_1, e_2 \in I_2$ such that $e_1 + e_2 = 1$. Put $m = e_1m_2 + e_2m_1$. Then $\phi_1(m) = \overline{e_1m_2 + e_2m_1} = \overline{e_2m_1} = \overline{(1 - e_1)m_1} = \overline{m_1}$. Similarly $\phi_2(m) = \overline{m}_2$. Hence there exist $m \in M$ such that $\phi(m) = (\phi_1(m), \phi_2(m)) = (\overline{m}_1, \overline{m}_2)$ and ϕ is surjective.

Assume that we have proved the result for $n = 1, \cdots, k-1$.

$R = R^{k-1} = (I_k + I_1)(I_k + I_2) \cdots (I_k + I_{k-1}) \subseteq I_k + I_1 \cdots I_{k-1} \subseteq I_k + \bigcap_{i=1}^{k-1} I_i$. So $R = I_k + \bigcap_{i=1}^{k-1} I_i$. So there exist $y \in \bigcap_{i=1}^{k-1} I_i$ and $z \in I_k$ such that $1 = y + z$.

For any element $(\overline{m_1}, \cdots, \overline{m_{k-1}}, \overline{m_k}) \in M/I_1M \bigoplus \cdots \bigoplus M/I_kM$, there exists $m \in M$ such that $\phi_i(m) = \overline{m_i}$ for $i = 1, \cdots, k-1$. Now let $n = zm + ym_k$. Then $\phi_k(n) = zm + ym_k + I_kM = ym_k + I_kM = (1 - z)m_k + I_kM = m_k + I_kM = \overline{m_k}$

For $i = 1, \cdots, k-1$, $\phi_i(n) = n + I_iM = zm + ym_k + I_iM = (1 - y)m + I_iM = m + I_iM = \phi_i(m) = \overline{m_i}$.

Hence $\phi(n) = (\phi_1(n), \cdots, \phi_{k-1}(n), \phi_k(n))= (\overline{m_1}, \cdots, \overline{m_k})$. So we complete our induction.

(2) Suppose that M is cancellation and ϕ is surjective. Then we can assume that a map $\phi_{ij} : M \longrightarrow M/I_iM \bigoplus M/I_jM$ defined by $\phi_{ij}(m) = (\phi_i(m), \phi_j(m))$ is surjective where $1 \leq i \neq j \leq n$. Let m be any element in M. Since ϕ_{ij} is surjective, there exist $m_i \in M, m_j \in M$ such that $\phi_{ij}(m_i) = (\phi_i(m_i), \phi_j(m_i)) = (m + I_iM, 0 + I_jM)$ and $\phi_{ij}(m_j) = (\phi_i(m_j), \phi_j(m_j) = (0 + I_iM, m + I_jM)$.

Here we get $m + I_iM = \phi_i(m_i) = m_i + I_iM$ and $0 + I_jM = \phi_j(m_i) = m_i + I_jM$. Also we have $0 + I_iM = \phi_i(m_j) = m_j + I_iM, m + I_jM = \phi_j(m_j) = m_j + I_jM$, Hence $m - m_i \in I_iM$, $m - m_j \in I_jM$, $m_i \in I_jM$ and $m_j \in I_iM$.

Therefore, $\phi_{ij}(m - (m_i + m_j)) = \phi_{ij}(m) - (\phi_{ij}(m_i) + \phi_{ij}(m_j)) = (m + I_iM, m + I_jM) - (m + I_iM, m + I_jM) = (\overline{0}, \overline{0})$.

Hence $m - (m_i + m_j) \in Ker\phi_{ij} = I_iM \bigcap I_jM \subseteq I_iM + I_jM$ since $m - m_i \in I_iM$, $m - m_j \in I_jM, m_i \in I_jM$ and $m_j \in I_iM$.

Moreover since $m_i + m_j \in I_jM + I_iM, m \in I_iM + I_jM$. So $M = I_iM + I_jM$. Now we get $R = I_i + I_j$ from the condition that M is cancellation.

(3) and (4) are trivial since $ker\phi = \bigcap_{i=1}^{n} I_iM$. □

Compare the following theorem with [4, Theorem 2.11].

Theorem 2.1. *Let R be an integral domain, M a faithful multiplication module over R and let $I_1, \cdots, I_n$ be ideals of R. Let $\phi : M \longrightarrow M/I_1M \bigoplus \cdots \bigoplus M/I_nM$ be given by $\phi(m) = (\phi_1(m), \cdots, \phi_n(m))$, where $\phi_i : M \longrightarrow M/I_iM$ is the natural R-module homomorphism. Then,*

(1) ϕ is surjective if and only if $I_1, \cdots, I_n$ are comaximal.

(2) ϕ is injective if and only if $I_1 \bigcap \cdots \bigcap I_n = 0$.

(3) ϕ is bijective if and only if $\bigcap_{i=1}^{n} I_i = 0$ and $I_1, \cdots, I_n$ are comaximal.

Proof. Since M is a faithful multiplication module over a domain, M is finitely generated [5, Theorem 3.1] and hence M is cancellation [9, Corollary to Theorem 9].

(1) It follows from Lemma (1) and (2).

(2) By [3, Theorem 1.6] we know that $I_1M \bigcap \cdots \bigcap I_nM = (I_1 \bigcap \cdots \bigcap I_n)M$.

Hence ϕ is injective if and only if $0 = I_1M \bigcap \cdots \bigcap I_nM$.

From fact that M is cancellation, we know that $0 = (I_1 \bigcap \cdots \bigcap I_n)M$ if and only if $0 = I_1 \bigcap \cdots \bigcap I_n$. □

Corollary 2.2. *Let R be an integral domain, M a faithful multiplication R-module and let $I_1, \cdots, I_n$ be comaximal ideals of R.*
Then $M/I_1M \bigcap \cdots \bigcap I_nM \cong M/I_1M \bigoplus \cdots \bigoplus M/I_nM$.

3. Multiplication Dedekind Modules

In this section we define the product of submodules in module and find some characterizations of faithful multiplication Dedekind modules.

Definition 3.1. *Let M be a module over a ring R and N be a submodule of M such that $N = IM$ for some ideal of R. Then we say that I is a presentation ideal of N. Let $N = IM$ and $K = JM$ for some ideals I and J of R. The product of N and K, NK,is defined by IJM.*

Note that it is possible that for a submodule N no such presentation exist. For example, if V is a vector space over any field with subspace $W(\neq 0, \neq V)$, then W has not any presentation. Clearly, every submodule of M has a presentation ideal if and only if M is a multiplication module. Also, the product of N and K is independent of presentation ideals of N and K and so, the product of submodules of multiplication module is well defined [2, Theorem 3.4]. Clearly, NK is a submodule of M and $NK \subseteq N \bigcap K$.

Now we generalize a well known property about Dedekind domain using the product of submodules in multiplication module. Compare the following Theorem with [6, Theorem 15, p.731].

Theorem 3.2. *Let M be a faithful multiplication module over a domain R. Then the following conditions are equivalent.*
(1) M is a Dedekind module.
(2) Every nonzero proper submodule of M can be expressed as a finite product of prime submodules.

Proof. (1) $\Rightarrow$ (2) Let $N(\neq 0)$ be a proper submodule of M. Since M is a multiplication module, there exists an ideal I of R such that $N = IM$ and $I(\neq 0)$ is proper. R is a Dedekind domain [8, Theorem 3.5], $I = \mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_s^{r_s}$ for some prime ideals $\mathfrak{p}_i$ of R [6, Theorem 15, p.731]. Hence $N = IM = \mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_s^{r_s} M = (\mathfrak{p}_1 M)^{r_1} \cdots (\mathfrak{p}_s M)^{r_s}$ and $\mathfrak{p}_i M$ is a prime submodules of M [3, Corollary 2.11].

(2) $\Rightarrow$ (1) Let $I \neq 0$ be any proper ideal of R. Then $IM(\neq 0)$ is a proper submodule of M since M is a cancellation module [5, Theorem 3.1]. $IM = P_1^{k_1} \cdots P_n^{k_n}$ where P_i is a prime submodule of M and $k_i \geq 1$. Here $\mathfrak{p}_i = (P_i : M)$ is a prime ideal of R [3, Corollary 2.3]. Since M is a

multiplication module, $P_i^{k_i} = [(P_i : M)M]^{k_i} = (P_i : M)^{k_i}M$. So $IM = (P_1 : M)^{k_1} \cdots (P_n : M)^{k_n}M$. From the fact that M is a cancellation module, $I = \mathfrak{p}_1^{k_1} \cdots \mathfrak{p}_n^{k_n}$ and hence R is a Dedekind Domain [6, Theorem 15, p.731] and M is a Dedekind Module [8, Theorem 3.4]. □

Proposition 3.3. *If M is a multiplication Dedekind module over a ring R then M is finitely generated.*

Proof. Since M is Dedekind, M is D_1-module and hence $R/annM$ is an integral domain [8, Corollary 2.2]. Hence R is an integral domain and M is finitely generated [5, Theorem 3.1]. □

An R-module M is *cyclic submodule module* , CSM, if every submodule of M is cyclic. So M is a cyclic.

Theorem 3.4. *Let M be a faithful multiplication Dedekind module over a ring R. For nonzero comaximal ideals $I_1 \cdots I_n$ of R, $M/I_1M \bigoplus \cdots \bigoplus M/I_nM$ is a CSM.*

Proof. Since R is an integral domain [8, Corollary 2.2], $I_1 \cdots I_n \neq 0$. $(I_1 \cdots I_n)M = (I_1 \bigcap \cdots \bigcap I_n)M = I_1M \bigcap \cdots \bigcap I_nM$ [3, Theorem 1.6] and $M/I_1M \bigcap \cdots \bigcap I_nM \cong M/I_1M \bigoplus \cdots \bigoplus M/I_nM$ by Corollary 2.2.

Since M is a cancellation module, $(I_1 \cdots I_n)M \neq 0$ and hence $M/I_1 \cdots I_nM$ is a CSM [1, Corollary 2.9].

Therefore $M/I_1M \bigcap \cdots \bigcap M/I_nM \cong M/I_1M \bigoplus \cdots \bigoplus M/I_nM$ is a CSM. □

A submodule P of an R-module M is called *indecomposable* if for all ideals I of R and submodules N of M, $P = IN$ implies that $P = N$ or $P = IM$.

Theorem 3.5. *Let M be a faithful multiplication Dedekind module over a ring R and P a proper submodule of M. Then P is prime if and only if P is indecomposable.*

Proof. Assume that P is not indecomposable submodule of M. Then there exist an ideal I of R and a submodule N of M such that $P = IN, P \neq N$ and $P \neq IM$. So there exist $n \in N - P$ and $\sum_{i=1}^{s} r_i m_i \in IM - P$. Here $r_i \in I, m_i \in M$. Hence there exist k $(1 \leq k \leq s)$ such that $r_k m_k \notin P$. Then $r_k n \in IN = P, r_k \notin (P : M)$ and $n \notin P$. This means that P is not prime. Conversely, suppose that P is indecomposable. By our assumption, R is an integral domain [8, Corollary 2.2]. If $P = 0$ then $P = 0M$ for a prime ideal 0 of R and hence P is prime. Let $P \neq 0$. By Theorem 3.2, $P = P_1^{r_1} \cdots P_s^{r_s}$ where P_i is a prime submodule of M. Put

$\mathfrak{p}_i = (P_i : M)$. Then $P_i = (P_i : M)M = \mathfrak{p}_i M$. Hence $P = \mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_n^{r_n} M$ where $r_i \geq 1$ and $\mathfrak{p}_i$ is a prime ideal of R [3, Corollary 2.11]. Since $P = (\mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_n^{r_n - 1})(\mathfrak{p}_n M)$ and P is indecomposable, $P = \mathfrak{p}_n M$ which is prime [3, Corollary 2.11] or $P = \mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_n^{r_n - 1} M$. If we continue this process then we conclude that P is prime. □

Theorem 3.6. *Let M be a faithful multiplication Dedekind module over a ring R. Then any non zero submodule N of M is finitely generated faithful multiplication.*

Proof. N is invertible and M is finitely generated by [5, Theorem 3.1] and [8, Corollary 2.2]. The result comes from [1, Proposition 1.1]. □

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