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CERTAIN CLASS OF QR-SUBMANIFOLDS OF MAXIMAL QR-DIMENSION IN QUATERNIONIC SPACE FORM

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Abstract. In this paper we determine certain class of n-dimensional QR-submanifolds of maximal QR-dimension isometrically immersed in a quaternionic space form, that is, a quaternionic Kähler manifold of constant Q-sectional curvature under the conditions (3.1) concerning with the second fundamental form and the induced almost contact 3-structure.

1. Introduction

Let M be a connected real n-dimensional submanifold of real codimension p of a quaternionic Kähler manifold $\overline{M}$ with quaternionic Kähler structure $\{F, G, H\}$. If there exists an r-dimensional subbundle ν of the normal bundle $TM^{\perp}$ such that

$$F\nu_x \subset \nu_x,\ G\nu_x \subset \nu_x,\ H\nu_x \subset \nu_x,$$
$$F\nu_x^{\perp} \subset T_xM,\ G\nu_x^{\perp} \subset T_xM,\ H\nu_x^{\perp} \subset T_xM$$

at each point x in M, then M is called a *QR-submanifold of r QR-dimension*, where $\nu^{\perp}$ denotes the complementary orthogonal distribution to ν in $TM^{\perp}$ (cf. [1], [4], [7], [9] and [10] etc.). Real hypersurfaces, which are typical examples of QR-submanifold with $r = 0$, have been investigated in many papers (cf. [11], [12] and [13] etc.) in connection with the shape operator and the induced almost contact 3-structure (for definition, see [8]).

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On the other hand, for a QR-submanifold M of maximal QR-dimension, (that is, $(p-1)$ QR-dimension) we can take a distinguished normal vector field ξ to M, where p is the codimension of M so that $\nu^\perp = \mathrm{Span}\{\xi\}$. Recently many authors (cf. [4], [7], [9] and [10]) studied QR-submanifolds M of maximal QR-dimension in a quaternionic space form under the following additional condition :

The distinguished normal vector field ξ is parallel with respect to the normal connection induced on the normal bundle of M.

In this paper we shall determine a QR-submanifold M of maximal QR-dimension isometrically immersed in a quaternionic space form which satisfy the assumptions

$$\begin{aligned} h(\phi X, Y) - h(X, \phi Y) &= 2g(\phi X, Y)\eta, \\ h(\psi X, Y) - h(X, \psi Y) &= 2g(\psi X, Y)\eta, \\ h(\theta X, Y) - h(X, \theta Y) &= 2g(\theta X, Y)\eta \end{aligned}$$

for a normal vector field η to M without the additional condition above, where it is denoted by h the second fundamental form, $\{\phi, \psi, \theta\}$ the induced almost contact 3-structure on M (see §2) and g the Riemannian metric tensor of M induced from that of $\overline{M}$.

All manifolds, submanifolds and geometric objects will be assumed to be connected, differentiable and of class C^∞, and all maps also be of class C^∞ if not stated otherwise.

2. Preliminaries

Let $\overline{M}$ be a real $(n+p)$-dimensional quaternionic Kählerian manifold. Then, by definition, there is a 3-dimensional vector bundle V consisting of tensor fields of type (1,1) over $\overline{M}$ satisfying the following conditions (a), (b) and (c) :

(a) In any coordinate neighborhood $\overline{\mathcal{U}}$, there is a local basis $\{F, G, H\}$ of V such that

$$\begin{aligned} &F^2 = -I,\ G^2 = -I,\ H^2 = -I, \\ &FG = -GF = H,\ GH = -HG = F,\ HF = -FH = G. \end{aligned} \tag{2.1}$$

(b) There is a Riemannian metric $\bar{g}$ which is Hermitian with respect to all of F, G and H.

(c) For the Riemannian connection $\overline{\nabla}$ with respect to $\bar{g}$

$$\begin{pmatrix} \overline{\nabla}F \\ \overline{\nabla}G \\ \overline{\nabla}H \end{pmatrix} = \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} F \\ G \\ H \end{pmatrix} \tag{2.2}$$

where p, q and r are local 1-forms defined in $\overline{\mathcal{U}}$. Such a local basis $\{F, G, H\}$ is called a *canonical local basis* of the bundle V in $\overline{\mathcal{U}}$ (cf. [5] and [6]).

For canonical local bases $\{F, G, H\}$ and $\{'F, 'G, 'H\}$ of V in coordinate neighborhoods $\overline{\mathcal{U}}$ and $'\overline{\mathcal{U}}$ respectively, it follows that in $\overline{\mathcal{U}} \cap {}'\overline{\mathcal{U}}$

$$\begin{pmatrix} 'F \\ 'G \\ 'H \end{pmatrix} = (s_{xy}) \begin{pmatrix} F \\ G \\ H \end{pmatrix} \qquad (x,\ y = 1,\ 2,\ 3)$$

where s_{xy} are local differentiable functions with $(s_{xy}) \in SO(3)$ as a consequence of (2.1). It is well known that every quaternionic Kählerian manifold is orientable (cf. [5] and [6]).

Now let M be an n-dimensional QR-submanifold of maximal QR-dimension, namely, $(p-1)$ QR-dimension isometrically immersed in $\overline{M}$. Then by definition there is a unit normal vector field ξ such that $\nu_x^{\perp} = \mathrm{Span}\{\xi\}$ at each point x in M. We set

$$U = -F\xi, \quad V = -G\xi, \quad W = -H\xi. \tag{2.3}$$

Denoting by $\mathcal{D}_x$ the maximal quaternionic invariant subspace

$$T_xM \cap FT_xM \cap GT_xM \cap HT_xM$$

of T_xM, we have $\mathcal{D}_x^{\perp} \supset \mathrm{Span}\{U, V, W\}$, where $\mathcal{D}_x^{\perp}$ is the complementary orthogonal subspace to $\mathcal{D}_x$ in T_xM. But, using (2.1) and (2.3), we can prove that $\mathcal{D}_x^{\perp} = \mathrm{Span}\{U, V, W\}$ (cf. [1] and [10]). Thus we have

$$T_xM = \mathcal{D}_x \oplus \mathrm{Span}\{U, V, W\}, \quad {}^{\forall}x \in M,$$

which together with (2.1) and (2.3) implies

$$FT_xM,\ GT_xM,\ HT_xM \subset T_xM \oplus \mathrm{Span}\{\xi\}.$$

Therefore, for any tangent vector field X and for a local orthonormal basis $\{\xi_\alpha\}_{\alpha=1,\ldots,p}$ ($\xi_1 := \xi$) of normal vectors to M, we have the following decompsition in tangential and normal components :

$$\begin{gathered} FX = \phi X + u(X)\xi, \quad GX = \psi X + v(X)\xi, \\ HX = \theta X + w(X)\xi, \end{gathered} \tag{2.4}$$

$$F\xi_\alpha = \sum_{\beta=2}^{p} P_{1\alpha\beta}\xi_\beta, \quad G\xi_\alpha = \sum_{\beta=2}^{p} P_{2\alpha\beta}\xi_\beta, \quad H\xi_\alpha = \sum_{\beta=2}^{p} P_{3\alpha\beta}\xi_\beta \tag{2.5}$$

where $\alpha = 2, \ldots, p$. Then it is easily seen that ϕ, ψ and θ are skew-symmetric endomorphisms acting on T_xM. Moreover, from (2.3), (2.4), (2.5) and the Hermitian property of $\{F, G, H\}$, it follows that

$$\begin{aligned} &g(U, X) = u(X), \quad g(V, X) = v(X) \quad g(W, X) = w(X), \\ &u(U) = 1, \quad v(V) = 1, \quad w(W) = 1, \\ &\phi U = 0, \quad \psi V = 0, \quad \theta W = 0. \end{aligned} \tag{2.6}$$

Next, applying F to the first equation of (2.4) and making use of (2.3), (2.4) and (2.6), we have

$$\phi^2 X = -X + u(X)U, \quad u(\phi X) = 0.$$

Similarly taking account of the second and the third equations of (2.4), we obtain consequently

$$\begin{gathered} \phi^2 X = -X + u(X)U, \quad \psi^2 X = -X + v(X)V, \\ \theta^2 X = -X + w(X)W, \end{gathered} \tag{2.7}$$

$$\begin{gathered} u(\phi X) = g(\phi X, U) = 0, \ v(\psi X) = g(\psi X, V) = 0, \\ w(\theta X) = g(\theta X, W) = 0. \end{gathered} \tag{2.8}$$

Applying G and H respectively to the first equation of (2.4) and using (2.1), (2.3) and (2.4), we get

$$\begin{aligned} &\theta X + w(X)\xi = -\psi(\phi X) - v(\phi X)\xi + u(X)V, \\ &\psi X + v(X)\xi = \theta(\phi X) + w(\phi X)\xi - u(X)W, \end{aligned}$$

respectively. Thus we can see that

$$\begin{aligned} &\psi(\phi X) = -\theta X + u(X)V, \quad v(\phi X) = -w(X), \\ &\theta(\phi X) = \psi X + u(X)W, \quad w(\phi X) = v(X). \end{aligned} \tag{2.9}$$

Therefore, according to similar method as the above, the second and the third equations of (2.4) also yield respectively

$$\begin{aligned} &\phi(\psi X) = \theta X + v(X)U, \quad u(\psi X) = w(X), \\ &\theta(\psi X) = -\phi X + v(X)W, \quad w(\psi X) = -u(X), \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\phi(\theta X) = -\psi X + w(X)U, \quad u(\theta X) = -v(X), \\ &\psi(\theta X) = \phi X + w(X)V, \quad v(\theta X) = u(X). \end{aligned} \tag{2.11}$$

Furthermore, from (2.8) joined with the skew-symmetry of ϕ, ψ and θ, it follows that

$$\begin{aligned} &\psi U = -W, \quad v(U) = 0, \quad \theta U = V, \quad w(U) = 0, \\ &\phi V = W, \quad u(V) = 0, \quad \theta V = -U, \quad w(V) = 0, \\ &\phi W = -V, \quad u(W) = 0, \quad \psi W = U, \quad v(W) = 0, \end{aligned} \tag{2.12}$$

where we have used (2.9), (2.10) and (2.11).

The equations (2.6)-(2.12) tell us that M admits the so-called almost contact 3-structure (for definition, see [8]) and consequently the dimension of M satisfies the equality $n = 4m + 3$ for some integer m.

On the other hand, the normal distribution ν is quaternionic invariant, and so we can take a local orthonormal basis $\{\xi, \xi_a, \xi_{a^*}, \xi_{a^{**}}, \xi_{a^{***}}\}_{a=1,\dots,q:=\frac{p-1}{4}}$ of normal vectors to M such that

$$\xi_{a^*} := F\xi_a, \quad \xi_{a^{**}} := G\xi_a, \quad \xi_{a^{***}} := H\xi_a. \tag{2.13}$$

Now let ∇ be the Levi-Civita connection on M and let $\nabla^\perp$ the normal connection of $TM^\perp$ induced from $\overline{\nabla}$. Then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.14}$$

$$\begin{aligned} \overline{\nabla}_X \xi = -AX + \nabla^\perp_X \xi = -AX + \sum_{a=1}^{q} \{ & s_a(X)\xi_a + s_{a^*}(X)\xi_{a^*} \\ & + s_{a^{**}}(X)\xi_{a^{**}} + s_{a^{***}}(X)\xi_{a^{***}} \}, \end{aligned} \tag{2.15$_1$}$$

$$\begin{aligned} \overline{\nabla}_X \xi_a = -A_a X - s_a(X)\xi + \sum_{b=1}^{q} \{ & s_{ab}(X)\xi_b \\ & + s_{ab^*}(X)\xi_{b^*} + s_{ab^{**}}(X)\xi_{b^{**}} + s_{ab^{***}}(X)\xi_{b^{***}} \}, \end{aligned} \tag{2.15$_2$}$$

$$\begin{aligned} \overline{\nabla}_X \xi_{a^*} = -A_{a^*} X - s_{a^*}(X)\xi + \sum_{b=1}^{q} \{ & s_{a^*b}(X)\xi_b \\ & + s_{a^*b^*}(X)\xi_{b^*} + s_{a^*b^{**}}(X)\xi_{b^{**}} + s_{a^*b^{***}}(X)\xi_{b^{***}} \}, \end{aligned} \tag{2.15$_3$}$$

$$\begin{aligned} \overline{\nabla}_X \xi_{a^{**}} = -A_{a^{**}} X - s_{a^{**}}(X)\xi + \sum_{b=1}^{q} \{ & s_{a^{**}b}(X)\xi_b \\ & + s_{a^{**}b^*}(X)\xi_{b^*} + s_{a^{**}b^{**}}(X)\xi_{b^{**}} + s_{a^{**}b^{***}}(X)\xi_{b^{***}} \}, \end{aligned} \tag{2.15$_4$}$$

$$(2.15)_5 \quad \overline{\nabla}_X \xi_{a^{***}} = -A_{a^{***}}X - s_{a^{***}}(X)\xi + \sum_{b=1}^{q}\{s_{a^{***}b}(X)\xi_b + s_{a^{***}b^*}(X)\xi_{b^*} + s_{a^{***}b^{**}}(X)\xi_{b^{**}} + s_{a^{***}b^{***}}(X)\xi_{b^{***}}\}$$

for vector fields X and Y tangent to M, where $s's$ are the coefficients of the normal connection $\nabla^{\perp}$. Here and in the sequel h denotes the second fundamental form and $A, A_a, A_{a^*}, A_{a^{**}}, A_{a^{***}}$ denote the shape operators corresponding to the normals $\xi, \xi_a, \xi_{a^*}, \xi_{a^{**}}, \xi_{a^{***}}$, respectively. They are related by

$$(2.16) \quad h(X,Y) = g(AX,Y)\xi + \sum_{a=1}^{q}\{g(A_aX,Y)\xi_a + g(A_{a^*}X,Y)\xi_{a^*} + g(A_{a^{**}}X,Y)\xi_{a^{**}} + g(A_{a^{***}}X,Y)\xi_{a^{***}}\}.$$

By means of (2.1)-(2.4), (2.13) and $(2.15)_{1-5}$, it can be easily verified that

$$(2.17)_1 \quad A_aX = -\phi A_{a^*}X + s_{a^*}(X)U = -\psi A_{a^{**}}X + s_{a^{**}}(X)V = -\theta A_{a^{***}}X + s_{a^{***}}(X)W,$$

$$(2.17)_2 \quad A_{a^*}X = \phi A_aX - s_a(X)U = \psi A_{a^{***}}X - s_{a^{***}}(X)V = -\theta A_{a^{**}}X + s_{a^{**}}(X)W,$$

$$(2.17)_3 \quad A_{a^{**}}X = -\phi A_{a^{***}}X + s_{a^{***}}(X)U = \psi A_aX - s_a(X)V = \theta A_{a^*}X - s_{a^*}(X)W,$$

$$(2.17)_4 \quad A_{a^{***}}X = \phi A_{a^{**}}X - s_{a^{**}}(X)U = -\psi A_{a^*}X + s_{a^*}(X)V = \theta A_aX - s_a(X)W,$$

$$(2.18)_1 \quad s_a(X) = -u(A_{a^*}X) = -v(A_{a^{**}}X) = -w(A_{a^{***}}X),$$

$$(2.18)_2 \quad s_{a^*}(X) = u(A_aX) = v(A_{a^{***}}X) = -w(A_{a^{**}}X),$$

$$(2.18)_3 \quad s_{a^{**}}(X) = -u(A_{a^{***}}X) = v(A_aX) = w(A_{a^*}X),$$

$$(2.18)_4 \quad s_{a^{***}}(X) = u(A_{a^{**}}X) = -v(A_{a^*}X) = w(A_aX).$$

Moreover, since ϕ, ψ, θ are skew-symmetric and $A_a, A_{a^*}, A_{a^{**}}, A_{a^{***}}$ are symmetric, $(2.17)_{1-4}$ together with (2.6) yield

$$(2.19)_1 \quad \begin{aligned} g((A_a\phi + \phi A_a)X,Y) &= s_a(X)u(Y) - s_a(Y)u(X), \\ g((A_a\psi + \psi A_a)X,Y) &= s_a(X)v(Y) - s_a(Y)v(X), \\ g((A_a\theta + \theta A_a)X,Y) &= s_a(X)w(Y) - s_a(Y)w(X), \end{aligned}$$

$$(2.19)_2 \quad \begin{aligned} g((A_{a^*}\phi+\phi A_{a^*})X,Y) &= s_{a^*}(X)u(Y)-s_{a^*}(Y)u(X),\\ g((A_{a^*}\psi+\psi A_{a^*})X,Y) &= s_{a^*}(X)v(Y)-s_{a^*}(Y)v(X),\\ g((A_{a^*}\theta+\theta A_{a^*})X,Y) &= s_{a^*}(X)w(Y)-s_{a^*}(Y)w(X),\end{aligned}$$

$$(2.19)_3 \quad \begin{aligned} g((A_{a^{**}}\phi+\phi A_{a^{**}})X,Y) &= s_{a^{**}}(X)u(Y)-s_{a^{**}}(Y)u(X),\\ g((A_{a^{**}}\psi+\psi A_{a^{**}})X,Y) &= s_{a^{**}}(X)v(Y)-s_{a^{**}}(Y)v(X),\\ g((A_{a^{**}}\theta+\theta A_{a^{**}})X,Y) &= s_{a^{**}}(X)w(Y)-s_{a^{**}}(Y)w(X),\end{aligned}$$

$$(2.19)_4 \quad \begin{aligned} g((A_{a^{***}}\phi+\phi A_{a^{***}})X,Y) &= s_{a^{***}}(X)u(Y)-s_{a^{***}}(Y)u(X),\\ g((A_{a^{***}}\psi+\psi A_{a^{***}})X,Y) &= s_{a^{***}}(X)v(Y)-s_{a^{***}}(Y)v(X),\\ g((A_{a^{***}}\theta+\theta A_{a^{**}})X,Y) &= s_{a^{***}}(X)w(Y)-s_{a^{***}}(Y)w(X).\end{aligned}$$

On the other side, since the ambient manifold is a quaternionic Kählerian manifold, differentiating the first equation of (2.4) covariantly and making use of (2.2), (2.4) itself, (2.14), $(2.15)_1$ and (2.16), we obtain

$$(2.20) \quad \begin{aligned} (\nabla_Y\phi)X &= r(Y)\psi X - q(Y)\theta X + u(X)AY - g(AY,X)U,\\ (\nabla_Y u)X &= r(Y)v(X) - q(Y)w(X) + g(\phi AY,X).\end{aligned}$$

Similarly, from the second and the third equations of (2.4), we also get respectively

$$(2.21) \quad \begin{aligned} (\nabla_Y\psi)X &= -r(Y)\phi X + p(Y)\theta X + v(X)AY - g(AY,X)V,\\ (\nabla_Y v)X &= -r(Y)u(X) + p(Y)w(X) + g(\psi AY,X),\end{aligned}$$

$$(2.22) \quad \begin{aligned} (\nabla_Y\theta)X &= q(Y)\phi X - p(Y)\psi X + w(X)AY - g(AY,X)W,\\ (\nabla_Y w)X &= q(Y)u(X) - p(Y)v(X) + g(\theta AY,X).\end{aligned}$$

Next, differentiating the first equation of (2.3) covariantly and using (2.2), (2.3) itself, (2.4), (2.14) and $(2.15)_1$, we have

$$(2.23) \quad \nabla_Y U = r(Y)V - q(Y)W + \phi AY.$$

From the second and the third equations of (2.3), similarly we obtain respectively

$$(2.24) \quad \nabla_Y V = -r(Y)U + p(Y)W + \psi AY,$$

$$(2.25) \quad \nabla_Y W = q(Y)U - p(Y)V + \theta AY.$$

Finally if the ambient manifold is a quaternionic space form $\overline{M}(c)$, that is, a quaternionic Kählerian manifold of constant Q-sectional curvature c, its curvature tensor $\overline{R}$ satisfies

$$\begin{aligned}\overline{R}(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y\\ &+ g(FY,Z)FX - g(FX,Z)FY - 2g(FX,Y)FZ\\ &+ g(GY,Z)GX - g(GX,Z)GY - 2g(GX,Y)GZ\\ &+ g(HY,Z)HX - g(HX,Z)HY - 2g(HX,Y)HZ\}\end{aligned}$$

for X, Y, Z tangent to $\overline{M}$ (cf. [5] and [6]). Hence the equations of Gauss and Codazzi imply

$$\begin{aligned}R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y\\ &+ g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\\ &+ g(\psi Y,Z)\psi X - g(\psi X,Z)\psi Y - 2g(\psi X,Y)\psi Z\\ &+ g(\theta Y,Z)\theta X - g(\theta X,Z)\theta Y - 2g(\theta X,Y)\theta Z\}\\ &+ g(AY,Z)AX - g(AX,Z)AY\\ + \sum_{a=1}^{q}\{&g(A_aY,Z)A_aX - g(A_aX,Z)A_aY\\ &+ g(A_{a^*}Y,Z)A_{a^*}X - g(A_{a^*}X,Z)A_{a^*}Y\\ &+ g(A_{a^{**}}Y,Z)A_{a^{**}}X - g(A_{a^{**}}X,Z)A_{a^{**}}Y\\ &+ g(A_{a^{***}}Y,Z)A_{a^{***}}X - g(A_{a^{***}}X,Z)A_{a^{***}}Y\},\end{aligned} \tag{2.26}$$

$$\begin{aligned}&g((\nabla_X A)Y - (\nabla_Y A)X, Z)\\ &= \frac{c}{4}\{g(\phi Y,Z)u(X) - g(\phi X,Z)u(Y) - 2g(\phi X,Y)u(Z)\\ &\quad + g(\psi Y,Z)v(X) - g(\psi X,Z)v(Y) - 2g(\psi X,Y)v(Z)\\ &\quad + g(\theta Y,Z)w(X) - g(\theta X,Z)w(Y) - 2g(\theta X,Y)w(Z)\}\\ &\quad + \sum_{a=1}^{q}\{g(A_aX,Z)s_a(Y) - g(A_aY,Z)s_a(X)\\ &\quad + g(A_{a^*}X,Z)s_{a^*}(Y) - g(A_{a^*}Y,Z)s_{a^*}(X)\\ &\quad + g(A_{a^{**}}X,Z)s_{a^{**}}(Y) - g(A_{a^{**}}Y,Z)s_{a^{**}}(X)\\ &\quad + g(A_{a^{***}}X,Z)s_{a^{***}}(Y) - g(A_{a^{***}}Y,Z)s_{a^{***}}(X)\},\end{aligned} \tag{2.27}$$

for any X, Y, Z tangent to M, where R denotes the curvature tensor of ∇ (cf. [1] and [2]).

3. Fundamental aspects concerning with the conditions (3.1)

In this section we study QR-submanifolds M of maximal QR-dimension in a quaternionic space form $\overline{M}(c)$ which satisfy the conditions

$$\begin{aligned} &h(\phi X,Y)-h(X,\phi Y)=2g(\phi X,Y)\eta,\\ &h(\psi X,Y)-h(X,\psi Y)=2g(\psi X,Y)\eta,\\ &h(\theta X,Y)-h(X,\theta Y)=2g(\theta X,Y)\eta,\quad \eta\in TM^{\perp}\end{aligned}\tag{3.1}$$

for all $X,Y\in TM$.

From now on, we use the orthonormal basis (2.13) of normal vectors to M and set

$$\eta=\rho\xi+\sum_{a=1}^{q}(\rho_a\xi_a+\rho_{a^*}\xi_{a^*}+\rho_{a^{**}}\xi_{a^{**}}+\rho_{a^{***}}\xi_{a^{***}}\}.$$

Then the conditions (3.1) are equivalent to

$$\begin{aligned} &A\phi X+\phi AX=2\rho\phi X,\quad A\psi X+\psi AX=2\rho\psi X,\\ &A\theta X+\theta AX=2\rho\theta X,\end{aligned}\tag{3.2$_1$}$$

$$\begin{aligned} &A_a\phi X+\phi A_aX=2\rho_a\phi X,\quad A_a\psi X+\psi A_aX=2\rho_a\psi X,\\ &A_a\theta X+\theta A_aX=2\rho_a\theta X,\end{aligned}\tag{3.2$_2$}$$

$$\begin{aligned} &A_{a^*}\phi X+\phi A_{a^*}X=2\rho_{a*}\phi X,\quad A_{a^*}\psi X+\psi A_{a^*}X=2\rho_{a^*}\psi X,\\ &A_{a^*}\theta X+\theta A_{a^*}X=2\rho_{a^*}\theta X,\end{aligned}\tag{3.2$_3$}$$

$$\begin{aligned} &A_{a^{**}}\phi X+\phi A_{a^{**}}X=2\rho_{a^{**}}\phi X,\quad A_{a^{**}}\psi X+\psi A_{a^{**}}X=2\rho_{a^{**}}\psi X,\\ &A_{a^{**}}\theta X+\theta A_{a^{**}}X=2\rho_{a^{**}}\theta X,\end{aligned}\tag{3.2$_4$}$$

$$\begin{aligned} &A_{a^{***}}\phi X+\phi A_{a^{***}}X=2\rho_{a^{***}}\phi X,\\ &A_{a^{***}}\psi X+\psi A_{a^{***}}X=2\rho_{a^{***}}\psi X,\\ &A_{a^{***}}\theta X+\theta A_{a^{***}}X=2\rho_{a^{***}}\theta X\end{aligned}\tag{3.2$_5$}$$

for all $a=1,\dots,q:=(p-1)/4$.

Lemma 3.1. *Let M be an n-dimensional QR-submanifold of maximal QR-dimension in a quaternionic Kählerian manifold. If the conditions (3.1) hold on M, then $A=\rho I$, where I denotes the identity transformation.*

Proof. Putting $X = U$ in the first equation of $(3.2)_1$ and using (2.6), we have $\phi AU = 0$, which together with (2.7) implies

$$AU = u(AU)U = g(AU, U)U. \tag{3.4}$$

Similarly, from the other equations of $(3.2)_1$ it follows that

$$\begin{aligned} AV &= v(AV)V = g(AV, V)V, \\ AW &= w(AW)W = g(AW, W)W. \end{aligned} \tag{3.5}$$

Next, inserting ϕX into the first equation of $(3.2)_1$ instead of X and making use of (2.7) and (3.4), we have

$$-AX + u(AU)u(X)U + \phi A\phi X = 2\rho\{-X + u(X)U\},$$

thus putting $X = V$ in the last equation and using (2.12) and (3.5), we obtain

$$v(AV) + w(AW) = 2\rho. \tag{3.6}$$

Similarly, inserting ψX into the second equation of $(3.2)_1$ instead of X and making use of (2.7) and (3.5), we have

$$-AX + v(AV)v(X)V + \psi A\psi X = 2\rho\{-X + v(X)V\},$$

thus also putting $X = U$ in the last equation and taking account of (2.12), (3.4) and (3.5), it turns out to be

$$u(AU) + w(AW) = 2\rho. \tag{3.7}$$

The equation (3.6) coupled with (3.7) gives

$$u(AU) = v(AV). \tag{3.8}$$

Similarly, the third equation of $(3.2)_1$ yields

$$-AX + w(AW)w(X)W + \theta A\theta X = 2\rho\{-X + w(X)W\},$$

thus putting $X = U$ in the last equation and making use of (3.8), we can see that $u(AU) = \rho$, which combined with (3.6) and (3.7) implies

$$u(AU) = v(AV) = w(AW) = \rho. \tag{3.9}$$

Finally, inserting ψX into the first equation of $(3.2)_1$ instead of X and using (2.10), (3.4) and (3.9), we have

$$A\theta X + v(X)AU + \phi A\psi X = 2\rho(\theta X + v(X)U),$$

which joined with (2.10), the second equation of $(3.2)_1$, (3.4), (3.5) and (3.9) reduces to

$$\theta(A - \rho I)X = 0.$$

Hence we get $A = \rho I$. □

Lemma 3.2. *Let M be an n-dimensional QR-submanifold of maximal QR-dimension in a quaternionic Kählerian manifold. If the conditions* (3.1) *hold on M, then*

$$s_a = 0, \quad s_{a^*} = 0, \quad s_{a^{**}} = 0, \quad s_{a^{***}} = 0, \quad a = 1, \dots, q. \tag{3.10}$$

Namely, the distinguished normal vector field ξ is parallel with respect to the normal connection $\nabla^{\perp}$. Moreover, for all $a = 1, \dots, q$

$$\rho_a = \rho_{a^*} = \rho_{a^{**}} = \rho_{a^{***}} = 0, \quad a = 1, \dots, q,$$

and consequently

$$A_a = 0, \quad A_{a^*} = 0, \quad A_{a^{**}} = 0, \quad A_{a^{***}} = 0. \tag{3.11}$$

Proof. From $(2.19)_{1-4}$ and $(3.2)_2 - (3.2)_5$, it is clear that

$$2\rho_a g(\phi X, Y) = s_a(X)u(Y) - s_a(Y)u(X), \tag{3.12_1}$$

$$2\rho_{a^*} g(\phi X, Y) = s_{a^*}(X)u(Y) - s_{a^*}(Y)u(X), \tag{3.12_2}$$

$$2\rho_{a^{**}} g(\phi X, Y) = s_{a^{**}}(X)u(Y) - s_{a^{**}}(Y)u(X), \tag{3.12_3}$$

$$2\rho_{a^{***}} g(\phi X, Y) = s_{a^{***}}(X)u(Y) - s_{a^{***}}(Y)u(X), \tag{3.12_4}$$

$$2\rho_a g(\psi X, Y) = s_a(X)u(Y) - s_a(Y)u(X), \tag{3.13_1}$$

$$2\rho_{a^*} g(\psi X, Y) = s_{a^*}(X)u(Y) - s_{a^*}(Y)u(X), \tag{3.13_2}$$

$$2\rho_{a^{**}} g(\psi X, Y) = s_{a^{**}}(X)u(Y) - s_{a^{**}}(Y)u(X), \tag{3.13_3}$$

$$2\rho_{a^{***}} g(\psi X, Y) = s_{a^{***}}(X)u(Y) - s_{a^{***}}(Y)u(X) \tag{3.13_4}$$

for all $a = 1, \dots, q$.

Putting $Y = U$ in $(3.12)_1 - (3.12)_4$, respectively, and taking account of (2.6), we have

$$\begin{aligned} &s_a(X) = s_a(U)u(X), \quad s_{a^*}(X) = s_{a^*}(U)u(X), \\ &s_{a^{**}}(X) = s_{a^{**}}(U)u(X), \quad s_{a^{***}}(X) = s_{a^{***}}(U)u(X). \end{aligned} \tag{3.14}$$

Similarly, putting $Y = V$ in $(3.13)_1$-$(3.13)_4$, respectively, and making use of (2.6), we get

$$\begin{aligned} &s_a(X) = s_a(V)v(X), \quad s_{a^*}(X) = s_{a^*}(V)v(X), \\ &s_{a^{**}}(X) = s_{a^{**}}(V)v(X), \quad s_{a^{***}}(X) = s_{a^{***}}(V)v(X). \end{aligned} \tag{3.15}$$

Hence the equations (3.14) and (3.15) give rise to

$$s_a(U) = s_a(V) = 0, \quad s_{a^*}(U) = s_{a^*}(V) = 0,$$

$$s_{a^{**}}(U) = s_{a^{**}}(V) = 0, \quad s_{a^{***}}(U) = s_{a^{***}}$$

which coupled with (3.14) imply

$$(3.16) \qquad s_a = 0, \quad s_{a^*} = 0, \quad s_{a^{**}} = 0, \quad s_{a^{***}} = 0,$$

or equivalently, from $(2.18)_{1-4}$, we obtain

$$(3.17)_1 \qquad A_aU = 0, \quad A_{a^*}U = 0, \quad A_{a^{**}}U = 0, \quad A_{a^{***}}U = 0,$$

$$(3.17)_2 \qquad A_aV = 0, \quad A_{a^*}V = 0, \quad A_{a^{**}}V = 0, \quad A_{a^{***}}V = 0,$$

$$(3.17)_3 \qquad A_aW = 0, \quad A_{a^*}W = 0, \quad A_{a^{**}}W = 0, \quad A_{a^{***}}W = 0.$$

Moreover, inserting those equations back into $(3.12)_1$-$(3.12)_4$ it turns out to be

$$\rho_a = \rho_{a^*} = \rho_{a^{**}} = \rho_{a^{***}} = 0,$$

from which combined with $(3.2)_2$-$(3.2)_5$, it follows that

$$(3.18)_1 \qquad A_a\phi + \phi A_a = 0,\ A_a\psi + \psi A_aX = 0,\ A_a\theta + \theta A_a = 0,$$

$$(3.18)_2 \qquad A_{a^*}\phi + \phi A_{a^*}X = 0,\ A_{a^*}\psi + \psi A_{a^*} = 0,\ A_{a^*}\theta + \theta A_{a^*} = 0,$$

$$(3.18)_3 \qquad A_{a^{**}}\phi + \phi A_{a^{**}} = 0,\ A_{a^{**}}\psi + \psi A_{a^{**}} = 0,\ A_{a^{**}}\theta + \theta A_{a^{**}} = 0,$$

$$(3.18)_4 \qquad A_{a^{***}}\phi + \phi A_{a^{***}} = 0,\ A_{a^{***}}\psi + \psi A_{a^{***}} = 0,\ A_{a^{***}}\theta + \theta A_{a^{***}} = 0$$

for all $a = 1, \ldots, q$.

Now, substituting ψX into the both side of the first equation of $(3.18)_1$ and by means of (2.10), we have

$$A_a\theta X + \phi A_a\psi X = 0,$$

which together with (2.10) and the second and third equations of $(3.18)_1$ implies $\theta A_a = 0$ and consequently we get $A_a = 0$. Similarly, from $(3.18)_{2-4}$, we can obtain (3.11). □

Theorem 1. Let M be an $n(> 3)$-dimensional QR-submanifold of maximal QR-dimension in a quaternionic space form $\overline{M}(c)$. If the conditions (3.1) hold on M, then ρ is locally constant. Moreover, $c = 0$.

Proof. We first notice that, under our assumptions, Lemma 3.2 yields that the Codazzi equation (2.27) reduces to

$$(3.19) \qquad \begin{aligned} &g((\nabla_X A)Y - (\nabla_Y A)X, Z) \\ &= \frac{c}{4}\{g(\phi Y, Z)u(X) - g(\phi X, Z)u(Y) - 2g(\phi X, Y)u(Z) \\ &\quad + g(\psi Y, Z)v(X) - g(\psi X, Z)v(Y) - 2g(\psi X, Y)v(Z) \\ &\quad + g(\theta Y, Z)w(X) - g(\theta X, Z)w(Y) - 2g(\theta X, Y)w(Z)\}. \end{aligned}$$

On the other hand, owing to Lemma 3.1, $A = \rho I$, which joined with (3.19) of $Z = U$ gives forth

$$\begin{aligned}\frac{c}{2}\{v(X)w(Y) - w(X)v(Y) - g(\phi X, Y)\} \\ = (X\rho)u(Y) - (Y\rho)u(X).\end{aligned} \tag{3.20}$$

Putting $Y = U$ in (3.20) and making use of (2.6) and (2.12), we obtain

$$X\rho = (U\rho)u(X). \tag{3.21}$$

According to similar method as the above, the equation (3.19) with $Z = V$ yields

$$X\rho = (V\rho)v(X),$$

which together with (3.21) implies $X\rho = 0$, that is, ρ is locally constant. Therefore, (3.20) reduces to

$$c\{v(X)w(Y) - w(X)v(Y) - g(\phi X, Y)\} = 0. \tag{3.22}$$

Inserting ϕX into (3.22) instead of X and taking account of (2.7) and (2.9), we have

$$c\{g(X, Y) - u(X)u(Y) - v(X)v(Y) - w(X)w(Y)\} = 0,$$

and consequently we get $c = 0$. □

4. Main result

In this section we consider QR-submanifolds of maximal QR-dimension in a quaternionic space form $\overline{M}(c)$ which satisfies the conditions (3.1). But, as already shown in Theorem 1, it is enough to consider the case of $c = 0$. Thus, from now on, we let M be an $n(> 3)$-dimensional QR-submanifold of maximal QR-dimension satisfying conditions (3.1) in a quaternionic number space $Q^{(n+p)/4}$ identified with Euclidean $(n + p)$-space R^{n+p}.

On the other hand, in this case we can easily see from Lemma 3.2 that the first normal space of M is contained in Span$\{\xi\}$ which is invariant under parallel translation with respect to the normal connection $\nabla^{\perp}$ from our assumption. Thus we may apply Erbacher's reduction theorem ([3, p.339]) to M and can verify that there exists a totally geodesic Euclidean $(n + 1)$-space R^{n+1} such that $M \subset R^{n+1}$. We notice that $n+1 = 4(m+1)$ for some integer m. Moreover, since the tangent space T_xR^{n+1} of the totally geodesic submanifold R^{n+1} at $x \in M$ is $T_xM \oplus$ Span$\{\xi\}$, R^{n+1} is an invariant submanifold of $R^{(n+p)/4}$ with respect to $\{F, G, H\}$ (for definition, see [1]) because of (2.3) and (2.4). Hence M

can be regarded as a real hypersurface of R^{n+1} which is a totally geodesic invariant submanifold of $R^{(n+p)/4}$.

Tentatively we denote by i_1 the immersion of M into R^{n+1} and i_2 the totally geodesic immersion of R^{n+1} onto $R^{(n+p)/4}$. Then, from the Gauss equation (2.14), it follows that

$$\nabla'_{i_1X} i_1 Y = i_1 \nabla_X Y + h'(X,Y) = i_1 \nabla_X Y + g(A'X,Y)\xi',$$

where it is denoted by h' the second fundamental form of M in R^{n+1}, ξ' a unit normal vector field to M in R^{n+1} and A' the shape operator corresponding to ξ'. Since $i = i_2 \circ i_1$ and R^{n+1} is totally geodesic in $R^{(n+p)/4}$, we have

$$\begin{aligned} \overline{\nabla}_{i_2 \circ i_1 X} i_2 \circ i_1 Y &= i_2 \nabla'_{i_1 X} i_1 Y + \overline{h}(i_1 X, i_1 Y) \\ &= i_2(i_1 \nabla_X Y + g(A'X,Y)\xi'), \end{aligned} \tag{4.1}$$

where $\bar{h}$ denotes the second fundamental form of R^{n+1}.

Comparing (4.1) with (2.14), we easily see that

$$\xi = i_2 \xi', \quad A = A'. \tag{4.2}$$

Since R^{n+1} is an invariant submanifold of $R^{(n+p)/4}$, for any $X' \in TR^{n+1}$,

$$F i_2 X' = i_2 F' X', \quad G i_2 X' = i_2 G' X', \quad H i_2 X' = i_2 H' X'$$

is valid, where $\{F', G', H'\}$ is the induced quaterninic Kähler structure of R^{n+1}. Thus it follows from (2.4) that

$$\begin{aligned} FiX &= Fi_2 \circ i_1 X = i_2 F' i_1 X = i_2(i_1 \phi' X + u'(X)\xi') \\ &= i\phi' X + u'(X) i_2 \xi' = i\phi' X + u'(X)\xi. \end{aligned}$$

Comparing this equation with (2.4), we have $\phi = \phi'$, $u' = u$. Similarly, we also get

$$\phi = \phi', u' = u \quad \psi = \psi', v' = v \quad \theta = \theta', w' = w.$$

Hence M is a real hypersurface of R^{n+1} which satisfies the conditions (3.1) and, moreover it is seen that $A' = \rho I$. Thus we have

Theorem 2. Let M be a complete $n(>3)$-dimensional QR-submanifold of maximal QR-dimension in a quaternionic space form $\overline{M}(c)$. If the conditions (3.1) hold on M, then M is congruent to

$$R^n \quad or \quad S^n.$$

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