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ON BOUNDED SOLUTIONS OF PEXIDER-EXPONENTIAL FUNCTIONAL INEQUALITY

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Abstract. Let G be a commutative group which is 2-divisible, $\mathbb{R}$ the set of real numbers and $f, g : G \to \mathbb{R}$. In this article, we investigate bounded solutions of the Pexider-exponential functional inequality $|f(x+y) - f(x)g(y)| \le \epsilon$ for all $x, y \in G$.

1. Introduction

Let G be a commutative group which is 2-divisible, $\mathbb{R}$ the set of real numbers and $\epsilon, \delta \ge 0$. It is well known that if $f : G \to \mathbb{R}$ satisfies the exponential functional inequality

$$|f(x+y) - f(x)f(y)| \le \epsilon \tag{1.1}$$

for all $x, y \in G$, then f is an unbounded function satisfying

$$f(x+y) = f(x)f(y)$$

for all $x, y \in G$, or a bounded function satisfying

$$|f(x)| \le \frac{1 + \sqrt{1 + 4\epsilon}}{2}$$

for all $x \in G$ (see Baker[3], Baker-Lawrence-Zorzitto[4]). In particular, if $G = V$, where V is a vector space over the filed $\mathbb{Q}$ of rational numbers

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and $f : V \to \mathbb{R}$ is a bounded function satisfying (1.1) for $0 \le \epsilon \le \frac{1}{4}$, then f satisfies either

$$\frac{1+\sqrt{1-4\epsilon}}{2} \le f(x) \le \frac{1+\sqrt{1+4\epsilon}}{2}$$

for all $x \in V$, or else

$$-\epsilon \le f(x) \le \frac{1-\sqrt{1-4\epsilon}}{2}$$

for all $x \in V$ (see Albert and Baker[2]). It is not easy to describe behavior of bounded solutions when two or more unknown functions are involved in a functional inequality. In this paper, we generalize the result of Albert and Baker[2] and investigate bounded solutions $f, g : G \to \mathbb{R}$ of the inequality

$$|f(x+y) - f(x)g(y)| \le \epsilon \tag{1.2}$$

for all $x, y \in G$ under a natural assumption. As a result, we prove the following.

Theorem 1.1. *Let (f, g) be a pair of functions satisfying* (1.2). *Then either (f, g) satisfies*

$$f(x+y) = f(x)g(y) \tag{1.3}$$

for all $x, y \in G$, or else (f, g) satisfies

$$|g(x+y) - g(x)g(y)| \le k\epsilon, \tag{1.4}$$

$$|f(x) - f(0)g(x)| \le \epsilon \tag{1.5}$$

for all $x, y \in G$, where

$$k = \frac{2+M_g}{M_f}, \quad M_f = \sup_{y\in G} |f(y)|, \quad M_g = \sup_{y \in G} |g(y)|.$$

In particular, if $k\epsilon \le \frac{1}{4}$, then we have

$$|g(x) - 1| \le \frac{\epsilon}{M_f}, \tag{1.6}$$

$$|f(x) - f(0)| \le 2\epsilon \tag{1.7}$$

for all $x \in G$.

2. Proof of Theorem 1.1

As a generalization of the result of Albert and Baker[2] we first investigate bounded solutions of the exponential functional inequality

$$|g(x+y) - g(x)g(y)| \le \delta \tag{2.1}$$

for all x, $y \in G$.

Lemma 2.1. *Suppose* $0 \le \delta \le \frac{1}{4}$ *and* $g : G \to \mathbb{R}$ *is a bounded function satisfying* (2.1). *Then,* g *satisfies either*

$$|g(x)| \le 2\delta \tag{2.2}$$

for all $x \in G$, *or*

$$|g(x) - 1| \le 2\delta \tag{2.3}$$

for all $x \in G$.

Proof. Replacing x and y by $\frac{x}{2}$ in (2.1) we have

$$g(x) \ge g\left(\frac{x}{2}\right)^2 - \delta \ge -\delta \tag{2.4}$$

for all $x \in G$. Let $g(0) = \beta$. Putting $x = y = 0$ in (2.1) we have

$$|\beta - \beta^2| \le \delta.$$

Thus, we have either

$$\frac{1-\sqrt{1+4\delta}}{2} \le \beta \le \frac{1-\sqrt{1-4\delta}}{2}, \tag{2.5}$$

or

$$\frac{1+\sqrt{1-4\delta}}{2} \le \beta \le \frac{1+\sqrt{1+4\delta}}{2}. \tag{2.6}$$

If (2.5) holds, putting $y = 0$ in (2.1) and dividing the result by $1 - \beta$ we have

$$|g(x)| \le \frac{\delta}{1-\beta} \le \frac{1-\sqrt{1-4\delta}}{2} \le 2\delta.$$

This gives (2.2). Now, assume that (2.6) holds. If $g(x_0) \le 0$ for some $x_0 \in G$, putting $x = x_0$, $y = -x_0$ in (2.1) we have

$$\frac{1}{4} = \frac{1}{2} - \frac{1}{4} \le \beta - \delta \le g(x_0)g(-x_0),$$

which implies $g(-x_0) \le 0$. Thus, from (2.4) we have

$$-\delta \le g(x_0) \le 0, \quad -\delta \le g(-x_0) \le 0.$$

Thus, we have the contradiction

$$\frac{1}{4} \le g(x_0)g(-x_0) \le \delta^2 \le \frac{1}{16}.$$

Thus, we have $g(x) > 0$ for all $x \in G$. Let $M_g := \sup_{x\in G} g(x)$. Replacing y by $y - x$ in (2.1) and using the result we have

$$g(x) \ge \frac{g(y)-\delta}{g(y-x)} \ge \frac{g(y)-\delta}{M_g} \tag{2.7}$$

for all $x, y \in G$ such that $g(y) - \delta \ge 0$. Let $K = \{y \in G : g(y) \ge \delta\}$. Then, from (2.7) we have

$$g(x) \ge \sup_{y\in K}\left(\frac{g(y)-\delta}{M_g}\right) = \sup_{y\in G}\left(\frac{g(y)-\delta}{M_g}\right) = \frac{M_g-\delta}{M_g} \tag{2.8}$$

for all $x \in G$. Taking the supremum of the left hand side of (2.8) and multiplying both sides of the result by M_g we have

$$M_g^2 - M_g + \delta \ge 0. \tag{2.9}$$

Since $M_g \ge \beta \ge \frac{1}{2}$, it follows from (2.9) that

$$M_g \ge \frac{1+\sqrt{1-4\delta}}{2}. \tag{2.10}$$

From (2.8) and (2.10) we have

$$g(x) \ge 1 - \frac{\delta}{M_g} \ge \frac{1+\sqrt{1-4\delta}}{2} \tag{2.11}$$

for all $x \in G$. Finally, it is well known in [4] that every bounded solution of (2.1) satisfies the inequality

$$g(x) \le \frac{1+\sqrt{1+4\delta}}{2} \tag{2.12}$$

for all $x \in G$. Thus, from (2.11) and (2.12) we have

$$-2\delta \le \frac{-1+\sqrt{1-4\delta}}{2} \le g(x) - 1 \le \frac{-1+\sqrt{1+4\delta}}{2} \le \delta$$

for all $x \in G$. This gives (2.3). Thus, we complete the proof. □

We also use the following result [5].

Lemma 2.2. *Let $f, g : G \to \mathbb{R}$ be bounded functions satisfying the functional inequality*

$$|f(x+y) - f(x)g(y)| \le \epsilon \tag{2.13}$$

for all $x, y \in G$*. Then* f, g *satisfy*

$$|f(x)(1-|g(y)|)| \leq \epsilon \tag{2.14}$$

for all $x, y \in G$.

Proof of Theorem 1.1. First, we assume that g is unbounded. Let y_n, $n = 1, 2, 3, \dots$, be a sequence in G such that $|g(y_n)| \to \infty$. Putting $y = y_n$ in (1.2), dividing the result by $|g(y_n)|$ and letting $n \to \infty$ we have

$$f(x) = \lim_{n\to\infty} \frac{f(x+y_n)}{g(y_n)} \tag{2.15}$$

for all $x \in G$. Multiplying both sides of (2.15) by $g(y)$ and using (1.2) and (2.15) we have

$$\begin{aligned} f(x)g(y) &= \lim_{n\to\infty} \frac{f(x+y_n)g(y)}{g(y_n)} \\ &= \lim_{n\to\infty} \frac{f(x+y+y_n)}{g(y_n)} \\ &= f(x+y) \end{aligned}$$

for all $x, y \in G$. This gives (1.3). Now, we assume that g is bounded. From inequality (1.2) we have

$$\begin{aligned} |f(z)||g(x+y)-g(x)g(y)| \leq &|f(z)g(x+y) - f(x+y+z)| \\ &+ |f(z+x+y) - f(z+x)g(y)| \\ &+ |f(z+x)g(y) - f(z)g(x)g(y))| \\ \leq &(2+|g(y)|)\epsilon \end{aligned} \tag{2.16}$$

for all $x, y, z \in G$. It follows from (2.16) that

$$|g(x+y)-g(x)g(y)| \leq \left(\frac{2+M_g}{|f(z)|}\right)\epsilon \tag{2.17}$$

for all $x, y, z \in G$. Taking the infimum of the right hand side of (2.17) with respect to z we have

$$|g(x+y)-g(x)g(y)| \leq \left(\frac{2+M_g}{M_f}\right)\epsilon \tag{2.18}$$

for all $x, y \in G$. Thus, we have (1.4). Now, putting $x = 0$ in (1.2) we have (1.5). In particular, if

$$\delta := \left(\frac{2+M_g}{M_f}\right)\epsilon \leq \frac{1}{4}, \tag{2.19}$$

then by Lemma 2.1, g satisfies either

$$|g(x)| \le 2\delta \tag{2.20}$$

for all $x \in G$, or

$$|g(x) - 1| \le 2\delta \tag{2.21}$$

for all $x \in G$. Assume that g satisfies (2.20). Replacing y by $y - x$ in (1.2) and using the triangle inequality we have

$$\begin{aligned} |f(y)| &\le |f(x)g(y-x)| + \epsilon \\ &\le 2\delta|f(x)| + \epsilon \\ &\le \frac{1}{2}M_f + \epsilon \end{aligned} \tag{2.22}$$

for all $y \in G$. Taking the supremum of the left hand side of (2.22) we have

$$M_f \le 2\epsilon, \tag{2.23}$$

which contradicts (2.19). Therefore, g satisfies (2.21). From (2.19) and (2.21) we have

$$|g(x)| \le |1 - g(x)| + 1 \le 2\delta + 1 \le \frac{3}{2} \tag{2.24}$$

for all $x \in G$. Thus, from (2.21) and (2.24) we have

$$|g(x) - 1| \le 2\delta \le 2\left(\frac{2 + \frac{3}{2}}{M_f}\right)\epsilon = \frac{7\epsilon}{M_f} \tag{2.25}$$

for all $x \in G$. From (2.19) we have $\frac{7\epsilon}{M_f} < 1$. Thus, from (2.25) we have $g(x) > 0$ for all $x \in G$. Now, by Lemma 2.2 we have

$$|f(x)||1 - g(y)| \le \epsilon$$

for all $x, y \in G$. Thus, we have

$$|g(x) - 1| \le \frac{\epsilon}{M_f} \tag{2.26}$$

for all $x \in G$. This gives (1.6). Multiplying (2.26) by $|f(0)|$ we have

$$|f(0)g(x) - f(0)| \le \frac{|f(0)|}{M_f}\epsilon \le \epsilon \tag{2.27}$$

for all $x \in G$. Putting $x = 0$ in (1.2) we have

$$|f(y) - f(0)g(y)| \le \epsilon \tag{2.28}$$

for all $y \in G$. Using the triangle inequality with (2.27) and (2.28) we have

$$|f(x) - f(0)| \le 2\epsilon \tag{2.29}$$

for all $x \in G$. This gives (1.7). Thus, we complete the proof.

Remark 2.3. If f is not extremely small and g is not extremely large as $\epsilon \to 0$, then (f, g) satisfies the condition (2.19). The opposite case when f is sufficiently small as $\epsilon \to 0$ can be treated as a trivial case. For example, if M_f/ϵ is bounded as $\epsilon \to 0$, i.e.,

$$M_f \le k\epsilon \tag{2.30}$$

for some $k > 0$ as $\epsilon \to 0$, then we can easily describe the behavior of g satisfying (1.2). Indeed, using the triangle inequality we have

$$|f(x)g(y)| \le |f(x+y)| + \epsilon \le (1+k)\epsilon \tag{2.31}$$

for all $x, y \in G$. Thus, from (2.31) we have

$$|g(y)| \le \left(\frac{1+k}{M_f}\right)\epsilon \tag{2.32}$$

for all $y \in G$. If f, g satisfies (2.30) and (2.32) respectively, then we have

$$\begin{aligned} |f(x+y) - f(x)g(y)| &\le |f(x+y) + |f(x)g(y)| \\ &\le k\epsilon + (1+k)\epsilon = (1+2k)\epsilon \end{aligned} \tag{2.33}$$

for all $x, y \in G$. Thus, we have

$$|f(x+y) - f(x)g(y)| = O(\epsilon)$$

as $\epsilon \to 0$.

Remark 2.4. We can also find the behavior of f when g is near 0. Assume that g satisfies

$$|g(x)| \le r < 1 \tag{2.34}$$

for all $x \in G$, then replacing y by $y - x$ in (1.2) and using the triangle inequality we have

$$\begin{aligned} |f(y)| &\le |f(x)g(y-x)| + \epsilon \\ &\le r|f(x)| + \epsilon \\ &\le rM_f + \epsilon \end{aligned} \tag{2.35}$$

for all $y \in G$. Taking the supremum in the left hand side of (2.35) we have

$$M_f \leq \frac{\epsilon}{1-r}. \tag{2.36}$$

If g and f satisfy (2.34) and (2.36) respectively, then we have

$$\begin{aligned} |f(x+y) - f(x)g(y)| &\leq |f(x+y)| + |f(x)||g(y)| \\ &\leq \left(\frac{1+r}{1-r}\right)\epsilon \end{aligned}$$

for all $x, y \in G$.

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