J. of the Chosun Natural Science
Vol. 6, No. 1 (2013) pp. 53 – 56

Rate of Convergence in Inviscid Limit for 2D Navier-Stokes Equations with Navier Fricition Condition for Nonsmooth Initial Data

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Abstract

We are interested in the rate of convergence of solutions of 2D Navier-Stokes equations in a smooth bounded domain as the viscosity tends to zero under Navier friction condition. If the initial velocity is smooth enough($u \in W^{2,p}, p > 2$), it is known that the rate of convergence is linearly propotional to the viscosity. Here, we consider the rate of convergence for nonsmooth velocity fields when the gradient of the corresponding solution of the Euler equations belongs to certain Orlicz spaces. As a corollary, if the initial vorticity is bounded and small enough, we obtain a sublinear rate of convergence.

Key words : Navier-Stokes, Inviscid Limit

1. Introduction

The incompressible Navier-Ntokes equations are equations of motion of viscous fluid, while the incompressible Euler equations are that of nonviscous (ideal) fluid. The Cauchy problems for these equations are known to be well-defined if the initial velocity is smooth enough[1,2]. As a subsequent fundamental question, the relation between these two equations has been sought in the literature. That is, whether the solution of the Navier-Stokes equations converges to that of the Euler equations in a nice sense as the viscosity vanishes. This question of vanishing viscosity limit is relatively easy and answered affirmatively if there is no spatial boundary[3]. Meanwhile, the same question in a bounded domain seems extremely difficult under the no-slip boundary condition. This difficulty seems to be mainly due to the discrepancy of the boundary conditions we impose on the Navier-Stokes and the Euler equations.

When studying the Euler equations in a bounded domain, it is natural to impose the zero-flux condition, which means that the system we consider is isolated. Thus, the Euler has been studied mostly under the zero-flux condition in the literature[1,2]. Meanwhile, for the Navier-Stokes equations in a bounded domain, one should impose an additional condition together with the zero-flux condition so that the system becomes complete. One of the typical condition is the no-slip boundary condition which was firstly introduced by Stokes. An alternative boundary condition is the Navier friction condition, which was firstly introduced by Navier and has been justified as a physically meaningful one in various ways by many authors[4-6]. Navier friction condition is roughly that the rate of strain at the boundary is linearly proportional to the tangential velocity in the opposite direction. With the Navier friction condition, it is easier to handle the inviscid limit in a bounded domain. Indeed, they[7-9] showed that, under the Navier friction condition with a given friction coefficient, the solutions of the Navier-Stokes converge to those of the Euler equations as the viscosity tends to zero for various classes of initial velocity. Also, if the initial velocity is smooth enough, it is known that the rate of convergence is linearly proportional to the viscosity[10,11]. But, for the non-Lipschitz initial velocity, the rate of convergence has not been calculated explicitly.

In this work, we give a rate of convergence in this case. That is, given a Euler velocity field, $v_E \in L^2$, with $\| \nabla v_E \|_{L^p} \le Cp^a, \ \forall p > p_0$ for some $a < 1$, we give a rate of convergence of the difference of the two solutions for the same initial data in $L^\infty(0,\infty;L^2)$.

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(Received : February 27, 2013, Revised : March 20, 2013,
Accepted : March 25, 2013)

2. Background

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and n, τ be the unit outward normal and the unit counterclockwise tangential vector to $\partial\Omega$. The (incompressible) Navier-Stokes equations in Ω are as follows:

$$\frac{\partial v}{\partial t}+(v\cdot\nabla)v=-\nabla p+\nu\Delta v, \quad (1)$$

$$\nabla\cdot v=0, \quad (2)$$

with the initial condition $v(\cdot,0)=v_0$. Here, $v:\Omega\times\mathbb{R}^+\mapsto\mathbb{R}^2$ and $p:\Omega\times\mathbb{R}^+\mapsto\mathbb{R}$ are the velocity field and the pressure, respectively, and the constant $\nu>0$ is the viscosity. v_0 is the initial data, which we always assume to satisfy (2) in this paper. We consider (1) under the following Navier friction boundary condition in this paper:

$$v(x,t)\cdot n=0, \ \ x\in\partial\Omega, \quad (3)$$

$$2n\cdot Dv\cdot\tau=-\alpha v\cdot\tau, \ \ x\in\partial\Omega. \quad (4)$$

Here, $Dv|_{ij}=\dfrac{\partial_i v_j+\partial_j v_i}{2}$, and $\alpha>0$ is a constant called the (Navier) friction coefficient. The condition (3) is also called the no-flux condition and is consistent with (2) and the condition (4) is called the Navier friction condition. It is known that (4) can be written as follows with the aid of (3):

$$\omega=(2\kappa-\alpha)v\cdot\tau, \ \ x\in\partial\Omega.$$

Here, $\kappa=\kappa(x)$ is the curvature of the boundary at $x\in\partial\Omega$[7].

Under the Navier friction boundary condition, the existence of solutions of the Navier-Stokes was established[9], when the initial vorticity belongs to $L^p, p>2$. Namely, they showed the following theorem.

Theorem 1.

Given a divergence free initial velocity $v_0\in L^2$, with its vorticity $\omega_0\in L^p$, $p>2$, there exists a unique solution v_F in $L^\infty([0,\infty);L^2\cap W^{1,p})$ for the Navier-Stokes equations with the Navier friction condition.

Now we consider the Euler equations;

$$\frac{\partial v}{\partial t}+(v\cdot\nabla)v=-\nabla p. \quad (5)$$

For the Euler equations, we impose the no-flux condition (3) as boundary condition. The existence and uniqueness for the initial boundary value problem of (5) under (3) in a 2D domain has been known when the initial vorticity $\omega_0=\nabla\times v_0\in L^\infty$[1,2].

Further, such solutions satisfies

$$\|\omega\|_{L^p}(t)\le\|\omega_0\|_{L^p}, \ \ 1\le p\le\infty. \quad (6)$$

For convenience, we introduce the following Orlicz space.

$EL^a=\{v\in L^p(\Omega;\mathbb{R}^2)\mid \|v\|_{L^p}\le Cp^a\}, \ a<1$ and $\|f\|_{EL^a}=\lim\limits_{p\to\infty}\dfrac{\|f\|_{L^p}}{p^a}$.

3. Convergence Rate

We first present static estimate for certain convective term.

Lemma 1.

For any $u\in H^1$, $v\in L^2$ with $\nabla v\in EL^a$, $a<1$, and $p>2$

$$\left|\int u\cdot\nabla v_E\cdot u\right|\le Cp^a\|\nabla v_E\|_{EL^a}\|\nabla u\|_{L^2}^{\frac{2}{p}}\|u\|_{L^2}^{2\frac{p-1}{p}}. \quad (7)$$

proof)

For $p>2$, writing $|u|^2=|u|^{(2p-4)/p+4/p}$ and applying the Holder inequality, we have

$$\left|\int u\cdot\nabla v_E\cdot u\right|\le\|\nabla v_E\|_{L^p}\|u\|_{L^2}^{\frac{2p-4}{p}}\|u\|_{L^4}^{\frac{4}{p}}$$
$$\le Cp^a\frac{\|\nabla v_E\|_{L^p}}{p^a}\|u\|_{L^2}^{\frac{2(p-2)+2}{p}}\|\nabla u\|_{L^2}^{\frac{2}{p}}.$$

This gives (7).

Theorem 2.

Given initial data $v_0 \in L^2$, let v_F, v_E be the solution of the Navier-Stokes equations and the Euler equations in a two dimensional smooth domain Ω. Assume further that $\nabla v_E \in EL^a$ for some $a < 1$ uniformly up to $t = T$, Then,

$$\sup_{t<T} \| v_F - v_E \|^2_{L^2}(t) \le C(\nu + \nu\alpha) \exp(CT \|\nabla v_E\|_{EL^a} |\ln(\nu+\nu\alpha)|^a). \tag{8}$$

proof)

We denote $v_F - v_E$ by u and $\nabla \times v_E$ by ω_E. Subtracting (8) from (1), we have

$$\partial_t u + v_F \cdot \nabla u + u \cdot \nabla v_E = -\nabla q + 2\nu \nabla \cdot (Dv_F).$$

Here, q is the difference of the two pressure. Multiplying by u and integrating, we have

$$\frac{1}{2}\partial_t \int u^2 \le \|u \cdot \nabla v_E u\|_{L^1} - 2\nu \int (Dv_F, Du) - \nu\alpha \int_{\partial\Omega} v_{F,\tau} u_\tau.$$

Note that

$$\begin{aligned} 2\int (Dv_F, Du) &= 2\int |Du|^2 + 2\int (Dv_E, Du) \\ &\ge \int |Du|^2 - \int |Dv_E|^2, \\ 2\int_{\partial\Omega} v_{F,\tau} u_\tau &\ge 2\int_{\partial\Omega} |u_\tau|^2 + 2\int_{\partial\Omega} v_{E,\tau} u_\tau \\ &\ge \int_{\partial\Omega} |u_\tau|^2 - \int_{\partial\Omega} |v_{E,\tau}|^2 \\ &\ge \int_{\partial\Omega} |u_\tau|^2 - C\|v_E\|^2_{H^1}. \end{aligned}$$

Therefore, by (7) and the above inequalities, we have

$$\partial_t \int u^2 \le Cp^a \|\nabla v_E\|_{EL^a} \|\nabla u\|^{1/p}_{L^2} \|u\|^{2(p-1)/p}_{L^2} + C(\nu+\nu\alpha)\|v_E\|^2_{H^1}.$$

Here, we assume $p > 2$ as before. Since $\|\nabla u\|_{L^2}$ and $\|v_E\|_{H^1}$ are uniformly bounded, we obtain

$$\partial_t \int u^2 \le Cp^a \|\nabla v_E\|_{EL^a} \|u\|_{L^2}^{\frac{2(p-1)}{p}} + C(\nu+\nu\alpha).$$

Now, we consider the set $S = \{0 < t < T \,|\, \|u\|^2_{L^2}(t) > \nu+\nu\alpha\}$. If S is empty, then we are done. On that set, denoting $G = \|u\|^{1/p}_{L^2}$, we have $\partial_t G \le C\|\nabla v_E\|_{EL^a} p^{a-1} + Cp^{-1}G$.

Then, applying the Gronwall inequality, we deduce

$$G(t) \le G(t_0) e^{C\frac{t}{p}} + Ce^{C\frac{t}{p}} p^{a-1} \|\nabla v_E\|_{EL^a} t \; .$$

Therefore,

$$G^p \le e^{CT}(\nu+\nu\alpha)\left(1 + CT\|\nabla v_E\|_{EL^a} \frac{p^{a-1}}{(\nu+\nu\alpha)^{1/p}}\right)^p$$

Finally, we take $p = |\ln(\nu+\nu\alpha)|$ for small ν. Then, $(\nu+\nu\alpha)^{1/p} = e^{-1}$ and $\left(1 + CT\|\nabla v_E\| \frac{p^{a-1}}{(\nu+\nu\alpha)^{1/p}}\right)^p \le \exp(CT\|\nabla v_E\|_{EL^a} p^{a-1} p)$, which gives (8).

If the initial vorticity $\omega_0 \in L^\infty$, then the vorticity of the solution of the Euler equations remains bounded for all time and thus $\|\nabla v_E\|_{EL^1} \le C\|\omega_0\|_\infty$ by (6). Therefore, as a corollary, we have that up to the time $T_0 < \frac{\delta}{C\|\omega_0\|_\infty}$, $\delta > 0$, $\sup_{t<T_0} \| v_F - v_E \|^2_{L^2}(t) \le C(\nu+\nu\alpha)^{1-\delta}$.

References

[1] P. -L. Lions, “Mathematical topics in fluid mechanics”, Vol. 1, “Incompressible models”, Oxford University Press, New York, 1996.

[2] C. Marchioro and M. Pulvirenti, “Mathematical theory of incompressible nonviscous fluids”, Springer-Verlag, New York, 1994.

[3] J. Y. Chemin, “A remark on the inviscid limit for two-dimensional incompressible fluids”, Commun. Part. Diff. Eq., Vol. 21, pp. 1771-1779, 1996.

[4] W. Jager and A. Mikelic, “On the roughness-induced effective boundary conditions for an incompressible

viscous flow", J. Differ. Equations, Vol. 170, pp. 96-122, 2001.

[5] W. Jager and A. Mikelic, "Couette flows over a rough boundary and drag reduction", Commun. Math. Phys., Vol. 232, pp. 429-455, 2003.

[6] J. C. Maxwell, "On stresses in rarified gases arising from inequalities of temperature", Philos. T. R. Soc. A ., Vol. 170, pp. 231-256, 1879.

[7] T. Clopeau, A. Mikelic, and R. Robert, "On the vanishing viscosity limit for the 2D incompressible Navier-Stokes equations with the friction type boundary conditions", Nonlinearity, Vol. 11, pp. 1625-1636, 1998.

[8] J. Kelliher, "Navier-Stokes equations with Navier boundary conditions for a bounded domain in the plane", SIAM J. Math. Anal., Vol. 38, pp. 210-232, 2006.

[9] M. C. Lopes Filho, H. J. Nussenzveig Lopes, and G. Planas, "On the inviscid limit for two-dimensional incompressible flow with Navier friction condition", SIAM J. Math. Anal., Vol. 30, pp. 1130-1141, 2005.

[10] D. Iftimie and G. Planas, "Inviscid limits for the Navier-Stokes equations with Navier friction boundary conditions", Nonlinearity, Vol. 19, pp. 899-918, 2006.

[11] N. Kim, "Corrigendum: Large friction limit and the inviscid limit of 2D Navier-Stokes equations under Navier friction condition", to appear in SIAM J. Math. Anal., 2013.